

# JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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*Journées Équations aux dérivées partielles*, n° 1 (1985), p. 1-4

[http://www.numdam.org/item?id=JEDP\\_1985\\_\\_1\\_A1\\_0](http://www.numdam.org/item?id=JEDP_1985__1_A1_0)

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SEMI-RIGID CR STRUCTURES  
AND HOLOMORPHIC EXTENDABILITY

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Let  $\Omega \subset \mathbb{R}^{2n+\ell}$  be an open set,  $0 \in \Omega$ , and  $\mathbb{C}T\Omega$ , the complexified tangent bundle to  $\Omega$ . Let  $V$  be a subbundle of  $\mathbb{C}T\Omega$  such  $\dim_{\mathbb{C}} V_{\omega} = n$ ,  $\forall \omega \in \Omega$ . We denote by  $\mathbb{L}$  the space of smooth sections of  $V$  defined in  $\Omega$ . We shall assume the Frobenius condition, i.e.

$$[V, V] \subset V,$$

and also

$$V_{\omega} \cap \overline{V_{\omega}} = \{0\}, \quad \forall \omega \in \Omega.$$

With the above assumptions we say that  $\Omega$  is equipped with an abstract CR structure of codimension  $\ell$ .

If in addition for every  $\omega_0 \in \Omega$ , there exist an open set  $\Omega' \subset \Omega$ ,  $\omega_0 \in \Omega'$ , and smooth functions in  $\Omega'$ , with independent differentials,  $Z_1, \dots, Z_{n+\ell}$ , satisfying

$$L Z_j = 0, \quad j=1, \dots, n+\ell, \quad \forall L \in \mathbb{L},$$

we say that  $V$  (or  $\mathbb{L}$ ) is locally integrable. We denote by  $M \subset \mathbb{C}^{n+\ell}$  the image of  $\Omega'$ . It is a (germ of a) generic CR manifold of codimension  $\ell$ .

We shall say that  $V$  is of finite type in  $\Omega$  at  $\omega$  (see Kohn [9] or Bloom-Graham [5]) if for any  $\xi \in T_{\omega}^* \Omega \setminus \{0\}$  there exists a commutator

$$(1) \quad L^{(k)} = [L_1^{(-)}, [L_2^{(-)}, \dots, [L_{k-1}^{(-)}, L_k^{(-)}] \dots]] ,$$

each  $L_j^{(-)} \in \mathbb{L} \otimes \bar{\mathbb{L}}$ , such that the symbol  $\sigma(L^{(k)})$  satisfies

$$(2) \quad \sigma(L^{(k)})(\omega, \xi) \neq 0 .$$

Let  $m(\omega, \xi)$  be the smallest integer  $k$  such that (2) is satisfied. The Hörmander numbers at  $\omega$  are the  $r$  distinct integers  $2 \leq m_1 < m_2 < \dots < m_r$  obtained as  $m(\omega, \xi)$  for some  $\xi \in T_\omega^* \Omega \setminus \{0\}$ ,  $\xi$  characteristic for  $\mathbb{L}$ .

We shall say that a CR structure  $\mathcal{V}$  of finite type is semi-rigid at  $\omega_0$  if for all  $\xi \in T_{\omega_0}^* \Omega$

$$\sigma([L^{(k)}, L^{(p)}])(\omega_0, \xi) = 0$$

for all commutators  $L^{(k)}, L^{(p)}$  of the form (1) with  $k, p \geq 2$  and  $k+p \leq m(\omega_0, \xi)$ .

The associated embedded generic CR manifold  $M$  will also be said to be semi-rigid.

The following result gives local normal forms for such manifolds.

Theorem 1: Let  $M$  be a generic CR manifold of codimension  $\ell$  in  $\mathbb{C}^{n+\ell}$ .

If  $M$  is of finite type at the origin, there are holomorphic coordinates around the origin,  $(z, w) \in \mathbb{C}^{n+\ell}$  such that on  $M$

$$z_j = x_j + i y_j \quad 1 \leq j \leq n ,$$

$$w_k = s_k + i [ p_{m_k}(z, \bar{z}, s_1, \dots, s_{k-1}) + O(m_k + 1) ] \quad 1 \leq k \leq r ,$$

where  $p_{m_k}$  is homogeneous of weight  $m_k$  and  $O(m_k + 1)$  is of weight  $m_k + 1$ . Here the  $x, y \in \mathbb{R}^n$  are given weight 1, while  $s_j \in \mathbb{R}^{\ell_j}$  is given weight  $m_j$ , and  $\ell_1 + \dots + \ell_r = \ell$ . Furthermore, the  $p_{m_k}$  may be chosen independent of all the  $s_j$  if and only if  $M$  is semi-rigid.

The first statement of Theorem 1 is in Bloom-Graham [5]; our proof, as well as the proof of the second statement, uses methods of Helffer-Nourrigat [7].

The following are examples of semi-rigid CR manifolds :

- 1 - Any hypersurface in  $\mathbb{C}^{n+1}$  of finite type.
- 2 - Any generic CR manifold of finite type in  $\mathbb{C}^{n+\ell}$  with Hörmander's numbers  $m_j \leq 3$ , for all  $j$ .
- 3 - Any generic CR manifold of finite type such that there exists  $m \geq 2$  satisfying  $m \leq m_j \leq m+1$  for all  $j$ .

A function  $h$  on  $M$  is said to be CR if it satisfies the equations

$$Lh = 0 \quad \text{for all } L \in \mathbb{L} .$$

We are concerned with the holomorphic extendability of CR functions across a point in  $M$ .

In order to state our main result we shall define the following sets of extendability. If a generic CR manifold in  $\mathbb{C}^{n+\ell}$  is defined by

$$(4) \quad \text{Im } w = \Phi(z, \bar{z}, \text{Re } w) , \quad z \in \mathbb{C}^n , \quad w \in \mathbb{C}^\ell ,$$

$\Phi(0) = 0$ ,  $\Phi'(0) = 0$ , and if  $\Gamma$  is a strictly convex open cone in  $\mathbb{R}^\ell \setminus \{0\}$ , a wedge with edge  $M$  is defined by

$$(5) \quad \mathcal{W}_\Gamma = \{(z, w) \in \mathcal{O} \subset \mathbb{C}^{n+\ell} : \text{Im } w - \Phi(z, \bar{z}, \text{Re } w) \in \Gamma\},$$

where  $\mathcal{O}$  is a neighborhood of 0.

Theorem 2. Let  $M$  be a semi-rigid CR manifold of finite type at the origin. Then any CR function on  $M$  extends holomorphically to a wedge of the form (5).

When the CR manifold  $M$  defined by (4) is real analytic, we have the following nonextendability result :

Theorem 3. Assume that  $M$  is a generic real analytic CR manifold in  $\mathbb{C}^{n+\ell}$  which is not of finite type at the origin. Then there exists a CR function defined near 0 on  $M$  which does not extend to any wedge.

Many extendability results have been proved since the classical work of H. Lewy [8]. Some recent ones are [3], [6], [4], [11]. A weaker version of Theorem 2 is proved in [2].

As in [2], the proof of Theorem 2 is based on the use of a generalized FBI transform (see Sjöstrand [10] and [1]) of the form

$$\int_e i^{(w-s-i\Phi(z, \bar{z}, s))\sigma - |\sigma|} (w-s-i\Phi(z, \bar{z}, s))^2 \chi(s) h(x, y, s) \det(I+i\Phi_s(z, \bar{z}, s)) ds.$$

Details of proofs will appear elsewhere.

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