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Polynomial bound on the distribution of poles in scattering by an obstacle.

by Richard Melrose.

Summary. Let $\mathcal{O} \subset R^n$, n 3 odd be a smooth compact obstacle. In the Lax-Phillips scattering theory(|1|) the scattering matrix for $\mathcal{O}$ with Dirichlet, Neumann or Robin boundary condition is méromorphic in the complex plane. Let $\{\mu_j\}$ be the sequence of these poles repeated according to multiplicity and arranged to have $|\mu_j|$ non-decreasing. In this note it is shown that there is a constant C such that

(*) $N(r) = \max\{j ; |\mu_j| \leq \} r < C\ r^n + C$

The proof is similar to that in |2| for scattering by a potential with compact support by sufficiently simplified that this rather precise grouth is obtained, of the same as for the interior problem (after assistance from D.Jerison).

1. Poles of the resolvent

Let $\mathcal{O} \subset R^n$, $n \geq 3$ odd, be a smooth compact obstacle. That is, $\mathcal{O}$ is compact and is the closure of an open set with C^∞ boundary such that $\Omega = R^n \setminus \mathcal{O}$ is connected. Set $S = \partial\mathcal{O}$, a compact C^∞ oriented (towards Ω) embedded hypersurface. For a given choice of boundary condition :

$(1.1)_{D,R}$ $u_{S} = 0$ (Dirichlet) or $\partial_\nu u_S = \gamma . u_S$ (Robin)

where $\gamma \in C^\infty(S)$ and ∂_ν is the outward unit normal ; consider

$$D_B(\Delta) = \{u \in H^2(\Omega) ; \text{u satisfies } (1.1)_B\}$$

With this domain, $\Delta = \Delta_B$ is an unbounded self-adjoint operator on $L^2(\Omega)$.

(1.2) Proposition. $R(\lambda)=(\lambda^2-\Delta_B)^{-1}$ is holomorphic in $\mathrm{Im}\lambda<0$, except for a possible finite number of poles on iR^-, as a bounded operator in $L^2(\Omega)$. As an operator $R(\lambda): C_c^\infty(\bar{\Omega}) \to C^\infty(\bar{\Omega})$, $R(\lambda)$ extends to be meromorphic in the λ-plane with poles and multiplicities, $\{\lambda_j\}$, the same as the scattering matrix of Lax & Phillips.

Proof. Recall from |1| , and for precise notation |2|, that the poles of the scattering matrix of Lax & Phillips as just, with multiplicities , the eigenvalues of the infinitesimal generator, L_B, of the semi-group $Z_B(t)$. Use of the modified Radon transform R of Lax & Phillips allows the resolvent $R(\lambda)$, for $C_0^\infty(\ )$, to be expressed in terms of $(\lambda-L_B)^{-1}$. From this the existence of the meromorphic extension of $R(\lambda)$ and the identification of its poles with those of $(\lambda-L_B)^{-1}$, including multiplicities, is straight forward.

2. Reduction to the boundary

For $\mathrm{Im}\lambda<0$ the boundary problem

(2.1) $(\Delta-\lambda^2)u=0$ in Ω , $u|_S=\varphi\in C^\infty(S)$

has a unique solution in $L^2(\Omega)$. The Neumann operator

$$N(\lambda): C^\infty(S) \to C^\infty(S),\ N(\lambda)\varphi = \partial_\nu u|_S$$

is well-defined and holomorphic in $\mathrm{Im}\,\lambda<0$.

(2.2) Proposition. $N(\lambda)$ extends to a meromorphic family in the λ-plane with poles, including multiplicities, exactly the $\{\lambda_j\}$ and values in the pseudo-differential operators of order 1 on S.

Proof. $N(\lambda)$ can easily be represented in terms of $R(\lambda)$, and conversely. Thus, if $e: C^\infty(S) \to C_c^\infty(\bar{\Omega})$ is a linear extension operator then

$$N(\lambda)\varphi = \partial_\nu\, e(\varphi)|_S + \partial_\nu\, .R(\lambda)\,[(\Delta-\lambda^2)e(\varphi)]$$

holds for $\mathrm{Im}\lambda<0$, and hence proves the meromorphy of $N(\lambda)$, with poles included amongst those of $R(\lambda)$, ie amongst the $\{\lambda_j\}$.

The fact that the free problem $\theta = \emptyset$ has resolvent $R_o(\lambda)$ with entire kernel allows $R(\lambda)$ to be expressed in terms of $N(\lambda)$.

Consider the operators

(2.3) $Q^D(\lambda) = R_o(\lambda)\ (\varphi.\delta_\nu)|$, $Q^N(\lambda) = R_o(\lambda)\ (\varphi.\delta_s)$

where δ_s is the Dirac mass on S and δ_ν its normal derivative. The "jumps formula" shows that

(2.4) $u = Q^D(\lambda)\varphi + Q^N(\lambda)\ N(\lambda)\varphi$

is the solution of $(\lambda^2-\Delta)u = 0$, $u|_S = \varphi$ for $\mathrm{Im}\lambda < 0$. If $f \in C_c^\infty(\Omega)$ set $w(\lambda) = R_o(\lambda)f \in C^\infty(\Omega)$ (by restriction), then

$$R(\lambda)f = R_o(\lambda)f - Q^D(\lambda)\left[R_o(\lambda)f|_S\right] - Q^N(\lambda)\ \left[N(\lambda)R_o(\lambda)f|_S\right]$$

shows that the poles of $N(\lambda)$ and $R(\lambda)$ must be identical.

(2.5) Remark. Similar results can be proved for the Dirichlet operator for a Robin boundary problem in precisely the same way.

3. The determinant

Recall that, for $\mathrm{Im}\,\lambda < 0$, the operators

$$C_{00}(\lambda)V = Q^D(\lambda)\ V\Big|_S\ ,\quad C_{10}(\lambda)V = \partial_\nu Q^D(\lambda)V\Big|_S$$

(3.1)

$$C_{01}(\lambda)V = Q^N(\lambda)\ V\Big|_S,\quad C_{11}(\lambda)V = \partial_\nu Q^N(\lambda)V\Big|_S$$

define the Caldéron projector :

$$C(\lambda) = \begin{pmatrix} C_{00}(\lambda) & C_{01}(\lambda) \\ C_{10}(\lambda) & C_{11}(\lambda) \end{pmatrix}$$

for the operator $\lambda^2-\Delta$ in Ω. The uniqueness of the solution to the Dirichlet problem, for $\mathrm{Im}\,\lambda < 0$, therefore shows that

(3.1) $C_{00}(\lambda) + C_{01}(\lambda) \cdot N(\lambda) = Id$

(3.2) Proposition : If Δ_s is the induced Laplacian on S then

(3.3) $(\Delta_s+1)^{1/2} \; C_{01}(\lambda) = I\lambda + P_D(\lambda)$

where $P_D(\lambda)$ is an entire family of pseudodifferential operators of order -1 such that -1 is an eigenvalue of $P_D(\lambda_j)$ with algebraic multiplicity at least that of λ_j as a pole of $R(\lambda)$.

Proof . The fact that Id- $C_{00}(\lambda)$ is entire shows that λ_j is, with multiplicity, amongst the zeros of $C_{01}(\lambda)$. Thus it suffices to show that $P_D(\lambda)$ defined by (3.3) is a pseudo-differential family of order -1. In fact $\sigma_{-1}(C_0(\lambda)) = {}^1/2|\xi|$ in terms of the induced metric, so the proposition is immediate.

For the Robin problem one obtains similarly

(3.4) $2. \; (\Delta_s+1)^{-1/2} \quad |C_{10}(\lambda) + C_{11}(\lambda)\; \gamma-\gamma\; | = I\lambda + P_R(\lambda)$

where $P_R(\lambda)$ has properties analogous to those of $P_D(\lambda)$.

(3.5) Remark. The operator $I\lambda + P(\lambda)$, which has the poles of the scattering matrix amongst its zeroes, according to Proposition 3.2 also has the squareroots of both signs of the eigenvalues of the interior problem as zeroes. Since the distribution of the spectrum for the interor problem is well-known any result on the distribution of the zeroes of $Id+P(\lambda)$ can be translated to a a result on the distribution of the poles. Since a pseudo-differential operator on S of order less than $l = \dim S$ is trace class, the function

(3.6) $d_B(\lambda) = \det \quad (Id + P_B^n(\lambda)\;) \qquad (n = l+1)$

is a well-defined entire function of λ . Proposition 3.2 shows immediately that

$d_B(\lambda_j) = 0$ (with multiplicity).

4. Eigenvalue estimates.

Using (3.1) and the well-known explicit form of the fundamental solution $R_0(\lambda)$ (see for example |2|) it follows that

(4.1) $P(\lambda) = \sum_{p\ 1} c_p \ \lambda^p \ T_p$

is an entire function of exponential type :

(4.2) $|c_p| \leqslant C^{p+1} /p\,!$

with $T_p = (\Delta_s+1)^{1/2} \,.\, T'_{p+1}$ where T'_p has Schwartz kernel

(4.3) $K'_p = |x-y|^{-l+p}$ $\qquad x,y \quad S,\ p>2.$

(4.4) Proposition (with D.Jerison).The characteristic values of the operators T_p satisfy

$$\chi_j(T_p) \leqslant C^{p+1} \ , \quad \chi_j(T_p) \leqslant \ p!\ C^{p+1}/j^{p/l}$$

for some constant C, independant of p.

Proof. The first estimate follows easily from the uniform bound $\|T_p\| \leqslant C^{p+1}$, as an operator on $L^2(S)$, which in turn follows from (4.3).

In the second estimate only the uniformity in p is at all subtle, since any pseudodifferential operator of order p has characteristic values bounded by $K\, j^{-p/l}$, l = dim S. Let $T''_p : C^\infty(S) \to C^\infty(\Theta)$ be the operator with kernel given by (4.3), where $x \in \Theta, y \in S$. Giving the Sobolev spaces $H^m(\Theta)$ the usual restriction norms :

$$\|\psi\|^{\Theta}_m = \Big(\sum_{|\alpha| \leqslant m} \|D^\alpha \psi\|^2_{L^2(\Theta)}\Big)^{1/2} \qquad m \in N$$

and interpolation for m>0, it follows easily by differentiation that

(4.5) $\|T''_p\|^{\theta}_{p+\frac{1}{2}} \leqslant C^{p+1}\, p!\|\varphi\|_{L^2(S)}$

Thus, if $H^p(S)$ is given the extension norm

(4.6) $\|\psi\|_p = \inf\{\|\psi\|^{\theta}_{p+\frac{1}{2}};\tilde{\psi}|_S=\psi\}$

then $\|T'_p\|_{0,p} \leqslant p!\, C^{p+1}$ follows from (4.5).

A norm uniformily equivalent to (4.6), up to factors C^{p+1}, is realized by the solution of the Dirichlet problem in O. Using this, and suitable "almost analytic" partitions of unity allows T''_p to be compared with a fixed finite sum of the p^{th} powers of the action of transversal vector fiels on the solution of the Dirichlet problem. For such p^{th} powers of a fixed operator the estimates $C^{p+1}\, j^{-p/l}$ on the characteristic values are immediate, proving the Proposition.

5. Proof of (*)

For the Dirichlet problem it has already been remarked that the $\{\mu_j\}$ of (*) are precisely the $\{\lambda_j\}$. For the Robin problem there may be a finite number of eigenvalues amongst the $\{\lambda_j\}$not usually included in the poles $\{\mu_j\}$. Naturally such niceties make no difference to the estimates (*). Standard results on the distribution of zeroes of entire holomorphic functions of exponential type show that (*) follows immediately from

(5.1) $|d_B(\lambda)| < C \exp(C\,|\lambda|^n) \qquad \lambda \quad C.$

Weyl's connexity estimates show that

(5.2) $|d_B(\lambda)| < \prod_{j=1}^{\infty} (1+\chi_j(P(\lambda))^n)$

where $\chi_j(P(\lambda))$ are the characteristic values of $P(\lambda)$. Then (5.1)follows from (5.2) and

(5.3) Proposition $P_B(\ )$ as definied in (3.3.) or (3.4) there exists a constant C such that

$$\chi_j(P(\lambda)) \leq C \exp (C|\lambda|) \qquad j<C|\lambda|^{l}$$

$$\chi_j(P(\lambda)) \leq C\ |\lambda|/j^{1/l} \qquad j> C|\lambda|^{l}.$$

Proof The first estimate follows from (4.1), (4.2) and the first estimate in Proposition 4.4, and is in fact valid for all j. Similarly from the second estimates in Proposition 4.4, (4.2) and (4.1) it follows that , again using Weyl's convexity estimates :

$$\chi_j(P(\lambda)) \leq C. \sum_{p>1} \left(\frac{C\ |\lambda|}{j^{1/l}}\right)^p$$

Thus, if $j^{1/l} > C|\lambda|$, far C>0 large, this geometric series is bounded by twice its first term, giving the second part of the Proposition.

(5.4) Remarks : Note that, just as in |2| , the important part of this estimate is that $P(\lambda)$ has only polynomial many, in $|\lambda|$, characteristic values which are exponentially large in $|\lambda|$. It is quite easy to prove a cruder version of Proposition 4.4., with l replaced by $n = l+1$. This has the effect of replacing n in(*) by n+1. The interest in the more precise estimate (*) is that the two contributions, the "local" terms coming from the pseudodifferential character of P( ) and the "global terms" coming from the exponential behaviour of the kernel are apparently of the same order.

Thus, to improve (*) to a proper asymptotic estimate, with a precise leading term probably of order n, it may be necessary to show either that the global contributions are in fact of lowe order or else to derive an asymptotic expression for them.

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