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A GENERAL CLASS OF GEVREY-TYPE
PSEUDO DIFFERENTIAL OPERATORS

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Recently much attention has been paid to the study of new classes of analytic and Gevrey-type pseudo differential operators; see for example Matsuzawa [8], Iftimie [5], Bolley -Camus-Métivier [2].

We shall consider here symbols $a(x, \xi)$ of general Gevrey type for which

$$(1) \quad |D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C^{|\alpha|+|\beta|+1} \alpha! \beta! \varphi(\xi)^{m-|\alpha|} \psi(\xi)^{m'+|\alpha|-|\beta|}$$

if $C^{|\beta|} \leq \varphi(\xi)$.

The weight functions φ, ψ are continuous in \mathbb{R}^n and satisfy for suitable positive constants $\varepsilon, \varepsilon'$ independent of $\xi, \eta \in \mathbb{R}^n$:

$$(2) \quad \varepsilon(1 + |\xi|)^\varepsilon \leq \varphi(\xi) \leq \varepsilon' \psi(\xi)$$

$$(3) \quad \varepsilon \leq \varphi(\xi) \varphi(\eta)^{-1} \leq \varepsilon', \quad \varepsilon \leq \psi(\xi) \psi(\eta)^{-1} \leq \varepsilon'$$

if $|\xi - \eta| \leq \varepsilon \psi(\xi)$.

To these conditions, which are quite common for general pseudo differential operators, we add the technical assumption:

$$(4) \quad \text{for every } \delta \text{ there exists } \delta' \text{ such that } \psi(\eta) \leq \delta |\xi - \eta| \\ \text{implies } \varphi(\eta) \psi(\xi) \leq \delta' |\xi - \eta| \varphi(\xi).$$

From the C^∞ point of view our symbols can be regarded as elements of a class of Beals [1] $S_{\mathbb{H}, \vartheta}^\lambda$, with $\mathbb{H} = \psi$, $\vartheta = \varphi/\psi$. The reason why we prefer here to refer to the function $\varphi = \vartheta\psi$ for the estimates in (1) is that a peculiar property of the pseudo differential operator

$$(5) \quad a(x, D) f(x) = (2\pi)^{-n} \int e^{ix\xi} a(x, \xi) \hat{f}(\xi) d\xi$$

associated with $a(x, \xi)$ turns out to be the continuity from

G_ψ to G_φ , where G_ψ, G_φ are the inhomogeneous Gevrey classes related to the weight functions ψ, φ , respectively.

Let us begin by giving a general definition of such classes in terms of Fourier transform. Let φ (or ψ) be a weight function as in (2), (3). More generally, let $\lambda: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be Lipschitzian, in the sense that $|\lambda(\xi) - \lambda(\eta)| \leq C|\xi - \eta|$ for some constant C independent of ξ, η , and assume also $\varepsilon(1 + |\xi|)^\varepsilon \leq \lambda(\xi)$ for some $\varepsilon > 0$. Let X be open in \mathbb{R}^n .

Definition 1. We say that $f \in \mathcal{D}'(X)$ is of class G_λ at $x_0 \in X$ if there is a neighborhood U of x_0 , $U \subset X$, and a bounded sequence $f_j \in \mathcal{E}'(X)$ such that $f = f_j$ in U and

$$(6) \quad |\hat{f}_j(\xi)| < c(cj/\lambda(\xi))^j, \quad j = 1, 2, \dots$$

We denote by $G_\lambda(X)$ the set of all $f \in \mathcal{D}'(X)$ which are of class G_λ at every $x_0 \in X$.

When $\lambda(\xi) = (1 + |\xi|)^\rho$, $0 < \rho \leq 1$, $G_\lambda(X)$ is the standard class $G^{1/\rho}(X)$ of all the functions $f \in C^\infty(X)$ which satisfy in every $K \subset X$ the estimates

$$(7) \quad |D^\alpha f(x)| < C^{|\alpha|+1} (\alpha!)^{1/\rho}$$

(cf. Hörmander [4], Proposition 2.4).

In particular for $\lambda(\xi) = 1 + |\xi|$ we have $G_\lambda(X) = \mathcal{A}(X)$, the set of all the real analytic functions in X .

Classes $G_\lambda(X)$ with inhomogeneous λ have been considered by several authors under different definitions; see for example Liess [6] and the references there. The advantage of the present definition is that it can be microlocalized in a natural

way, adapting the procedure used by Rodino [10] in the C^∞ framework. Fix $\Gamma \subset \mathbb{R}_\xi^n$ and set for $\varepsilon > 0$

$$(8) \quad \Gamma_{\varepsilon\lambda} = \{\xi \in \mathbb{R}^n, \text{dist}(\xi, \Gamma) < \varepsilon \lambda(\xi)\}.$$

Definition 2. We shall say that f is G_λ -smooth at $\{x_0\} \times \Gamma$ and we shall write formally $\text{WF}_\lambda f \cap (\{x_0\} \times \Gamma) = \emptyset$ if the estimates (6) are satisfied in $\Gamma_{\varepsilon\lambda}$, for a sufficiently small $\varepsilon > 0$.

It is natural then to introduce the space of the "microfunctions" at $\{x_0\} \times \Gamma$.

Definition 3. We denote by $C_{x_0, \Gamma, \lambda}^\infty$ the factor space $C_{x_0}^\infty / \sim$, where $C_{x_0}^\infty$ is the set of the germs of C^∞ functions defined near x_0 and $f \sim g$ in $C_{x_0}^\infty$ iff $\text{WF}_\lambda(f-g) \cap (\{x_0\} \times \Gamma) = \emptyset$.

It is convenient in certain applications to use also a different kind of microlocalization. Precisely, set for $\varepsilon > 0$

$$(8)' \quad \Gamma_{[\varepsilon\lambda]} = \{\xi \in \mathbb{R}^n, \lambda(\xi - \eta) < \varepsilon \lambda(\xi) \text{ for some } \eta \in \Gamma\}.$$

Definition 2'. We shall say that f is strongly G_λ -smooth at $\{x_0\} \times \Gamma$ and we shall write formally $\text{WF}_\lambda^* f \cap (\{x_0\} \times \Gamma) = \emptyset$ if the estimates (6) are satisfied in $\Gamma_{[\varepsilon\lambda]}$, for a sufficiently small $\varepsilon > 0$.

For example, if Γ is the half-ray generated by $\xi_0 \neq 0$ and $\lambda(\xi) = (1 + |\xi|)^\rho$, $0 < \rho \leq 1$, then $\text{WF}_\lambda^* f \cap (\{x_0\} \times \Gamma) = \emptyset$ means that (x_0, ξ_0) is not in the Gevrey wave front set $\text{WF}_{1/\rho} f$ of Hörmander [4].

Note that strong G_λ -smoothness at $\{x_0\} \times \Gamma$ implies G_λ -smoothness there, but the converse is not true in general.

Let us now return to pseudo differential operators and give a precise definition of our classes from the microlocal point of view.

Assume φ and ψ satisfy the conditions (2), (3), (4). Let X be open in \mathbb{R}_x^n and fix $\Gamma \subset \mathbb{R}_\xi^n$.

Definition 4. We define $S_{\varphi, \psi}^{m, m'}(X, \Gamma)$ to be the set of all $a(x, \xi) \in C^\infty(X \times \Gamma)$ which can be extended for some $\varepsilon > 0$ to functions in $C^\infty(X \times \Gamma_{\varepsilon\psi})$ such that (1) is satisfied with suitable positive constants C, C' independent of $x \in X, \xi \in \Gamma_{\varepsilon\psi}$.

A symbol $a(x, \xi) \in S_{\varphi, \psi}^{m, m'}(X, \Gamma)$ can be further extended to a function $\tilde{a}(x, \xi) \in C^\infty(X \times \mathbb{R}^n)$, by cutting off in the ξ variables, and $\tilde{a}(x, D)$ from (5) is then defined as a map from $C_0^\infty(X)$ to $C^\infty(X)$. The continuity property can now be expressed in the following microlocal form.

Theorem 5. Let $a(x, \xi)$ be in $S_{\varphi, \psi}^{m, m'}(X, \Gamma)$, and take $x_0 \in X, \Lambda \subset \Gamma$. Then $\tilde{a}(x, D)$ defines by factorization an operator

$$(9) \quad a(x, D) : C_{x_0, \Lambda, \psi}^\infty \rightarrow C_{x_0, \Lambda, \varphi}^\infty$$

which depends only on a and not also on the extensions \tilde{a} of a .

The symbolic calculus for the operators $a(x, D)$ in (9) follows the lines of the calculus of the C^∞ -general pseudo differential operators (cf. Beals [1]), with some evident complications in the estimates due to the factor

$C^{|\alpha|+|\beta|+1} \alpha! \beta!$ which we expect in (1). From Theorem 5 and from symbolic calculus one deduces by means of a standard argument the following result on existence of parametrices.

Theorem 6. Consider $a(x, \xi) \in S_\psi^m(X, \Gamma) = S_{\psi, \psi}^{0, m}(X, \Gamma) \subset S_{\varphi, \psi}^{0, m}(X, \Gamma)$ and fix $x_0 \in X, \Lambda \subset \Gamma$. Assume there exist a neighborhood U of $x_0, U \subset X$, real numbers m_1, m_1' and positive constants c, c', ε, C

such that

$$(10) \quad |a(x, \xi)| \geq c \varphi(\xi)^{m_1} \psi(\xi)^{m'_1} \quad \text{for } x \in U, \xi \in \Lambda_{\varepsilon\psi} \quad \text{and} \\ |\xi| \geq c$$

$$(11) \quad |D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C^{|\alpha|+|\beta|} \alpha! \beta! |a(x, \xi)| \varphi(\xi)^{-|\alpha|} \psi(\xi)^{|\alpha|-|\beta|}$$

for all α and all x, ξ, β with $x \in U, \xi \in \Lambda_{\varepsilon\psi}, c'|\beta| \leq \varphi(\xi)$
and $|\xi| \geq c$.

Then there is $b \in S_{\varphi, \psi}^{-m_1, -m'_1}(U, \Lambda)$ such that $b(x, D) \circ a(x, D)$:

$C_{x_0, \Lambda, \psi}^\infty \rightarrow C_{x_0, \Lambda, \varphi}^\infty$ is the natural inclusion. In particular,

for any fixed extension \tilde{a} of a , we have that $WF_\psi \tilde{a}(x, D) f \cap$
 $(\{x_0\} \times \Lambda) = \emptyset$ implies $WF_\varphi f \cap (\{x_0\} \times \Lambda) = \emptyset$.

When $A = a(x, D)$ is a linear partial differential operator with analytic coefficients in X there are some obvious simplifications in the statement; namely, if for every $K \subset X$ we have for large $|\xi|$ and suitable constants $|a(x, \xi)| \geq c|\xi|^r$ and

$$(11)' \quad |D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C^{|\alpha|+|\beta|} \alpha! \beta! |a(x, \xi)| \varphi(\xi)^{-|\alpha|} \psi(\xi)^{|\alpha|-|\beta|}$$

then $Af \in G_\psi(X)$ implies $f \in G_\varphi(X)$ for every $f \in \mathcal{D}'(X)$; in particular all solutions of $Af = 0$ are in $G_\varphi(X)$.

A simple example is given by the hypoelliptic operators with constant coefficients $P = p(D)$. Let $\delta(\xi)$ be the distance from $\xi \in \mathbb{R}^n$ to the surface $\{\zeta \in \mathbb{C}^n, p(\zeta) = 0\}$, and set $\psi(\xi) = 1 + \delta(\xi)$. It is well known that

$$(12) \quad |D_\xi^\beta p(\xi)| \leq C |p(\xi)| \psi(\xi)^{-|\beta|}$$

and (11)' is then satisfied with $\psi = \varphi$. We conclude that

$Pf \in G_\psi(X)$ implies $f \in G_\varphi(X)$ for any $X \subset \mathbb{R}^n$ and all $f \in \mathcal{D}'(\mathbb{R}^n)$.

An example of operator for which $\varphi \neq \psi$ (that means a loss of Gevrey regularity for the solutions) is given by

$$(13) \quad A = 1 + |x|^{2k} p(D) ,$$

where $p(D)$ is hypoelliptic and $p(\xi) \geq 0$; the estimates (11)' are satisfied for $\psi(\xi)$ as in preceding example and any $\varphi(\xi)$ for which $p(\xi) < (\psi(\xi)/\varphi(\xi))^{2k}$.

Theorem 6, as well as Theorem 5, can be restated in terms of strong G_λ -smoothness, according to Definition 2'. A relevant application is given by the choice $\psi(\xi) = (1+|\xi|)^\rho$, $\varphi(\xi) = (1+|\xi|)^{\rho-\delta}$, $0 \leq \delta < \rho \leq 1$, which corresponds to the operators in [2], [5], [8]. Since the related Gevrey wave front sets are invariant under canonical transformations, geometric invariant statements are possible in this case; for example, let us consider a classical analytic symbol:

$a(x, \xi) \sim \sum_{j=0}^{\infty} a_{m-j}(x, \xi)$ and assume the principal part $a_m(x, \xi)$ vanishes exactly of order k , $k \geq 2$, on an involutive manifold $\Sigma \subset T^*X \setminus 0$. Noting a'_{m-1} the subprincipal symbol, set for any $\gamma \in \Sigma$ and for any C^∞ vector field Y defined in a neighborhood of γ

$$(14) \quad I_a(\gamma, Y) = (k!)^{-1} (Y^k a_m)(\gamma) + a'_{m-1}(\gamma).$$

Theorem 7. Assume $I_a(\gamma, Y) \neq 0$ for every γ and Y . Then, writing $s = k/(k-1)$, we have $WF_s a(x, D)f = WF_s f$ for all $f \in \mathcal{E}'(X)$. In particular $a(x, D)f \in G^S(X)$ implies $f \in G^S(X)$.

In fact, after conjugation by a Fourier integral operator, $a(x, D)$ becomes an operator to which Theorem 6 applies with $\psi(\xi) = \varphi(\xi) = (1+|\xi|)^{1/s}$ (cf. Parenti-Rodino [9], where C^∞ -

hypocoellipticity was proved under the same assumptions). Similarly we can prove a G^2 -hypocoellipticity result for the operators in the classes of Boutet de Monvel-Grigis-Helffer [2].

Another application of Theorem 6 refers the choice $\psi(\xi) = \sum_{j=1}^n |\xi_j|^{1/M_j}$, where $M = (M_1, \dots, M_n)$ is a n -tuple of rational numbers ≥ 1 ; the related hypocoellipticity results can be expressed in terms of the anisotropic Gevrey wave front set WF_M of Zanghirati [12], Rodino [11]. Details and proofs of the results announced here will be found in Liess-Rodino [7].

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