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Existence of solutions for transversally elliptic left invariant differential operators on nilpotent Lie groups.

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1. Introduction and notation. We describe here some recent results, obtained jointly with Lawrence Corwin [3] on solvability of left invariant differential operators on nilpotent Lie groups. For related results see [2], [8], [9], [10], [11], [14], [15], [16], [17].

We consider first operators on a 2-step nilpotent Lie group G, i.e. we assume that the Lie algebra $\mathfrak{G}$ is a vector space direct sum $\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2$ with $[\mathfrak{G}_1, \mathfrak{G}_1] = \mathfrak{G}_2$ and $[\mathfrak{G}_2, \mathfrak{G}] = (0)$. For $\eta \in \mathfrak{G}_2^*$ let B_η be the bilinear form $B_\eta(X_1, X_2) = \eta([X_1, X_2])$ for $X_1, X_2 \in \mathfrak{G}_1$. B_η assumes its maximal rank on a Zariski open subset in $\mathfrak{G}_2^*$. Recall that to every $\ell \in \mathfrak{G}^*$ we may associate, by the Kirillov theory, an irreducible unitary representation π_ℓ of G, realized on a Hilbert space of the form $L^2(\mathbb{R}^k)$ for some k.

By a transversally elliptic operator on G we shall mean a left invariant differential operator L on G which is an elliptic polynomial on $\mathfrak{G}_1$, i.e.

$$L = L_m + L_{m-1} + \cdots + L_0, \tag{1.1}$$

with L_j homogeneous of degree j, and L_m an elliptic polynomial on $\mathfrak{G}_1$.

2. Necessary conditions for local solvability. We give the following criterion, which generalizes known results [2] for homogeneous operators i.e. those for which $L_j \equiv 0$, $j < m$.

Theorem 1. Let L *be a left invariant operator on* G *which is transversally elliptic. Assume that there is a non-empty open set* $V \subset \mathfrak{G}^*$ *such that*

$$\ker \pi_\ell(L^\tau) \neq 0 \quad \text{for all} \quad \ell \in V, \tag{2.1}$$

or, *equivalently*,

$$\ker L^\tau \cap L^2(G) \neq 0 . \tag{2.2}$$

Then L *is not locally solvable*.

The idea of the proof is as follows. First, if B_η is degenerate for all η, then [1] may be applied to show that the hypothesis is vacuous. So assume B_η nondegenerate for η in a Zariski open set. We show, using microlocal constructions as in [6] that there is a pseudo-differential operator Π not of order $-\infty$ such that $L^\tau \Pi$ is of order $-\infty$. Now for any distribution σ for which $Lv - \sigma = 0$ in an open set U for some distribution v, $\Pi^\tau(Lv - \sigma)$ is smooth, and hence $\Pi^\tau \sigma$ is smooth. Hence σ cannot be arbitrary.

3. *Sufficient conditions for solvability on H-groups*. G is called an H-*group* if B_η is nondegenerate for $\eta \neq 0$. We prove the following converse to Theorem 2 for H-groups. A globally defined differential operator P is *uniformly semi-globally solvable* if there is an integer r such that for every bounded open neighborhood U of 0 there exists a distribution σ_U of order at most r such that $L\sigma_U = \delta$ in U.

Theorem 2. If G *is an* H-*group and* L *a left invariant transversally elliptic operator on* G *then* L *is uniformly semi-globally solvable if*

(2.1) and (2.2) do not hold.

The proof of Theorem 2 is somewhat similar to that of the corresponding result [17] in the case where L is homogeneous. Both rely on the theorem of Lojasiewicz which says that one can divide a distribution by a non-zero analytic function.

Corollary. If L_m is locally solvable, then L is locally solvable.

4. *Existence of global fundamental solutions.* Here we allow G to be any connected Lie group, not necessarily nilpotent.

Theorem 3. Let L be a left invariant differential operator on G which is uniformly semi-globally solvable. Suppose that G is L-convex. Then L has a global fundamental solution; i.e. there is a distribution σ on G for which $L\sigma = \delta$.

The proof of Theorem 3 involves a construction similar to that used in proving that L-convexity implies global solvability. The theorem gives a new result even for homogeneous operators.

Corollary 1. Let L be a homogeneous left invariant differential operator on a nilpotent Lie group G with dilations. If L is locally solvable at 0 then L has a global fundamental solution.

Corollary 2. If L is a transversally elliptic operator on an H-group which satisfies the hypothesis of Theorem 2 then L has a global fundamental solution.

5. Global criteria for hypoellipticity. The various global criteria for local solvability for homogeneous differential operators on nilpotent groups, e.g. $\ker L^{\tau} \cap L^2(G) = (0)$, suggest that the representation-theoretic criterion of Helffer-Nourrigat [5] may be reformulated. Indeed, using a recent Liouville-type theorem of Geller [4] one may obtain the following.

Theorem 4. (Geller, Helffer-Nourrigat). *Let* G *be a stratified nilpotent Lie group and* L *a homogeneous left invariant differential operator on* G. *Then* L *is hypoelliptic if and only if there is no non-constant bounded function* f *on* G *such that* $Lf = 0$.

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