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THE NASH-MOSER INVERSE MAPPING THEOREM

by M. KURANISHI

To prove a local embedding theorem for strongly pseudo-convex CR structures (of dimension $\geq q$) (cf. [2]) we used a variant of Nash-Moser inverse mapping theorem. We try to explain in general terms how it was done, without bothering too much about technical details.

For a $\varepsilon_0 > 0$ we define $\varepsilon_\nu > 0$ inductively by

$$(1) \quad \varepsilon_{\nu+1} = \varepsilon_\nu^a, \quad a = 3/2.$$

The Nash-Moser inverse mapping theorem (cf. [3]) is based on the following :

LEMMA :

Let $s, t \geq 0$ be given. Pick $\lambda, \mu > 0$ so large that

$$(2) \quad \begin{aligned} s + (a-2) &\leq 0 \\ t + a^2\mu + (1-a)\lambda &\leq -a. \end{aligned}$$

Let $p_\nu > 0$ be a sequence. Assume that for a constant $C^* > 0$

$$(3) \quad \begin{aligned} p_\nu &\leq C^* (\varepsilon_\nu^{-s} (p_{\nu-1})^2 + \varepsilon_\nu^\lambda \varepsilon_{\nu-1}^{-\lambda-t}) \\ \varepsilon_1 &\leq 1/(2C^*)^2, \quad p_0 \leq \varepsilon_1^\mu / 2C^*. \end{aligned}$$

Then

$$(4) \quad p_\nu \leq (\varepsilon_{\nu+1})^{\mu/2} C^*$$

Proof goes as follows : we set $g_\nu = \varepsilon_{\nu+1}^{-\mu} p_\nu$. Then

$$g_\nu \leq C^* (\varepsilon_\nu^{-(s+(a-2)\mu)} g_{\nu-1}^2 + \varepsilon_{\nu-1}^{-(t+a^2\mu+(1-a)\lambda)}).$$

Hence $g_v \leq C^* ((g_{v-1})^2 + \varepsilon_v)$. We now prove $g_v \leq 1/2 C_*$ by induction on v .

We apply the above lemma in the following setting : we consider open sets F', G' in Frechet spaces F, G and a map

$$\Phi : F' \rightarrow G'$$

Each of these Frechet spaces is assumed to be endowed with an increasing sequence of semi-norms $|| \cdot ||_k$ which defines its topology. In practice, we consider the Frechet spaces of C^∞ sections of vector bundles over a manifold M . $|| \cdot ||_k$ is defined by measuring the partial derivatives up to degree k of sections. Φ is given by a non-linear partial differential operator involving partial derivatives up to order, say r . This is translated into an assumption

$$(5) \quad ||\Phi(f)||_k \leq C_k (1 + ||f||_{k+r})$$

For k sufficiently large any map with the above assumption is called tame (cf. R. Hamilton [1] for more details). We assume that Φ is infinitely differentiable and all partial derivatives are tame. In particular there is for each $f \in F'$ a continuous linear map.

$$d_f \Phi : F \rightarrow G$$

such that with $R_f(h) = \Phi(f+h) - \Phi(f) - d_f \Phi(h)$

$$(6) \quad ||R_f(h)||_k \leq C_k (||f||_{k+r} ||h||_{k_0}^2 + ||h||_{n+r}^2)$$

for $k \geq k_1$. We also assume that there is a mollifier M_ε ($\varepsilon > 0$) with the standard properties : for $s \geq 0$

$$(7) \quad \begin{aligned} ||M_\varepsilon f||_{k+s} &\leq C_{k,s} \varepsilon^{-s} ||f||_k \\ ||f - M_\varepsilon f||_k &\leq C_{k,s} \varepsilon^s ||f||_{k+s} \end{aligned}$$

We now wish to show that an element $g \in G'$ is in the image of Φ . We may assume that $g = 0$. We solve the problem by a successive approximation. Namely, for a v -th approximation f_v we define f_{v+1} as follows : note that $\Phi(f_v + h)$ is very closed to $\Phi(f_v) + d_{f_v} \Phi(h)$. Hence we solve the equation :

$$(8) \quad \Phi(f_v) + d_{f_v} \Phi(h) = 0$$

However, in the process we usually lose derivatives. We compensate this by setting

$$(9) \quad f_{v+1} = f_v + M_{\varepsilon_{v+1}} h_v$$

where h_v is a solution of (8) and where ε_v is given in (1). In fact, we assume that we can find h_v with

$$(10) \quad \|h_v\|_{k-r_1} \leq C_k \|\Phi(f_v)\|_k$$

This estimate is essential for this method to work. In order to show that f_v converge to a solution f of our problem, it is enough to show that $p_v = \|\Phi(f_v)\|_k$ satisfy (3) in the lemma. If this is the case, p_v has estimate (4). In view of (9) and (10) it then follows that f_v will also converge. Now :

$$\begin{aligned} \Phi(f_v + M_{\varepsilon_{v+1}} h_v) &= \Phi(f_v) + d_{f_v} \Phi(M_{\varepsilon_{v+1}} h_v) + R_{f_v}(M_{\varepsilon_{v+1}} h_v) \\ &= R_{f_v}(M_{\varepsilon_{v+1}} h_v) - d_{f_v} \Phi(h_v - M_{\varepsilon_{v+1}} h_v) \end{aligned}$$

Note (7) and (6). From the first term (resp. the second term) we obtain terms $C^* \varepsilon_{v+1}^{-s} (p_v)^2$ (resp. $\varepsilon_{v+1}^\lambda \varepsilon_v^{-\lambda-t}$) for a choice of s and t .

The above shows that we can solve the equation $\Phi(f) = g$ for a given g provided we find a very good approximation f_0 so that the last inequality in (30) is satisfied. In particular, we find that a small neighborhood of f_0 is covered by Φ .

For a local embedding theorem mentioned in the beginning we have a following more general setting. Namely, we have a manifold M and for each open $U \subset M$ we have :

$$F'(U) \xrightarrow{\Phi} G'(U) \xrightarrow{\Psi} H'(U)$$

with $\Psi \cdot \Phi = 0$. They are related by compatible restriction maps. We are given $g \in G'(M)$ with $\Psi(g) = 0$ and a reference point p_0 in M . We wish to show that the restriction of g to a suitable open neighborhood U of p_0 is in the image of Φ . We may assume that $g = 0$. The existence of Ψ means that we may not be able to solve the equation (8). We have to replace $\Phi(f_v)$ by its projection to the image of $d_{f_v} \Phi$. Moreover, we could find such projection only for U satisfying certain conditions which also depend on f_v . Namely, for each $f \in F(U_1)$, where $p_0 \in U_1$, we have a way to define $r_f > 0$ and a distance function t_f to p_0 with the following properties : for $0 < r < r_f$ set

$$(11) \quad U(f, r) = \{p \in U_1 ; t_f(p) < r\}.$$

Let f' be the restriction of f to $F(U(f, r))$, $f' = f|_{U(f, r)}$.

Then there is

$$(12) \quad V_{f'} : G(U(f, r)) \rightarrow F(U(f, r))$$

such that with $h' = V_{f'}(\Phi(f'))$

$$(13) \quad -\Phi(f') = (d_{f'}, \Phi)(h') + A(\Phi(f'))$$

where $A(\Psi)$ is given by a composition.

$$(14) \quad A(\Psi) = A_1 \circ A_2(\Psi)$$

A_1 is a linear map. A_2 is a non-linear partial differential operator starting with quadratic terms. Since our error term $A(\Phi(f'))$ is of quadratic nature as R_f in (6) we may try to solve our problem by the same method as in the standard case.

We first find $f_0 \in \mathcal{F}(U_0)$ such that :

$$(15) \quad \|\Phi(f_0)|_{U(f_0, r)}\|_k \leq O(r^N)$$

for all N . This is achieved by solving the differential equation $\Phi(f) = 0$ as a formal power series centered at p_0 whose Taylor series agree with the solution formal power series will satisfy our requirement. For $\alpha > \beta$ we set for

$$0 < r_0 < r_{f_0}$$

$$(16) \quad \varepsilon_0 = r_0^\alpha, \quad \delta_0 = r_0^\beta$$

and define ε_v and δ_v as in (1). We then set

$$(17) \quad r_{v+1} = r_v - 3 \delta_v$$

Starting from $f_0|_{U(f_0, r_0)}$ we construct f_1 as in the standard case replacing h in (8) by h' in (13). We then, show that, if r_{f_0} is properly chosen, $r_1 + \delta_1 \leq r_{f_1}$ and $U(f_1, r_1 + \delta_1) \subseteq U(f_0, r_0 - \delta_0)$. We then consider $f_1|_{U(f_1, r_1)}$ and proceed inductively. We do this construction for all r_0 in $]0, r_{f_0}[$.

Thus the success of our method depends essentially on the nature of V_f , in (12) which solves the equations (13) and how we could handle the new error term $A(\Phi(f'))$. In our case V_f is obtained by using the solution mapping N_f , of generalized Neumann boundary value problem on $U(f, r)$ associated with $d_f \Phi$. N_f also enters in the construction of A_1 in (14). The fact is we could only find N_f for $U(f, r)$ as in (11), where t_f satisfies certain conditions. This is the reason why we had to change U as each step of the successive approximation. On the other hand, since $U(f_{v+1}, r_{v+1}) \subset U(f_v, r_v)$, we could use the interior estimate. In such estimate a factor $(\delta_v)^{-\ell}$ (cf. (17)) will come in the constant of the inequality. However, we can admit such factor in view of (3). Using estimates for N_f on $U(f, r)$ as well as interior estimate, we prove the first inequality in (3) for all r in $]0, r_{f_0}[$, provided $p_0, \dots, p_{v-1}$ is sufficiently small. We now need the second and the third inequality in (3). In view of (16) the second is satisfied for sufficiently small r_0 . Similarly the third is satisfied in view of (15).

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