MASATAKE KURANISHI

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<http://www.numdam.org/item?id=JEDP_1983___A11_0>
THE NASH-MOSER INVERSE MAPPING THEOREM

by M. KURANISHI

To prove a local embedding theorem for strongly pseudo-convex CR structures (of dimension $\mathfrak{p}q$) (cf. [2]) we used a variant of Nash-Moser inverse mapping theorem. We try to explain in general terms how it was done, without bothering too much about technical details.

For a $\varepsilon_0 > 0$ we define $\varepsilon_\nu > 0$ inductively by

$$\varepsilon_{\nu+1} = \varepsilon_\nu^a, \quad a = 3/2.$$  \hfill (1)

The Nash-Moser inverse mapping theorem (cf. [3]) is based on the following:

**Lemma:**

Let $s, t > 0$ be given. Pick $\lambda, \mu > 0$ so large that

$$s + (a-2) \leq 0$$ \hfill (2)

$$t + a^2 \mu + (1-a)\lambda \leq -a.$$  

Let $p_\nu > 0$ be a sequence. Assume that for a constant $C^* > 0$

$$p_\nu \leq C^* (\varepsilon_\nu^{-s} (p_{\nu-1})^2 + \varepsilon_\nu^\lambda \varepsilon_{\nu-1}^{-\lambda-t}).$$ \hfill (3)

Then

$$p_\nu \leq (e_{\nu+1})^{\mu/2} C^*.$$ \hfill (4)

Proof goes as follows: we set $g_\nu = e_{\nu+1}^{-\mu} p_\nu$. Then

$$g_\nu \leq C^* (\varepsilon_\nu^{-s+(a-2)\mu} g_{\nu-1}^2 + e_{\nu-1}^{-t+a^2 \mu + (1-a)\lambda}).$$
Hence \( g_\nu < C ( (g_{\nu-1})^2 + \epsilon_\nu) \). We now prove \( g_\nu < \frac{1}{2} C_\ast \) by induction on \( \nu \).

We apply the above lemma in the following setting: we consider open sets \( F', G' \) in Frechet spaces \( F, G \) and a map

\[ \phi : F' \to G' \]

Each of these Frechet spaces is assumed to be endowed with an increasing sequence of semi-norms \( || | \) which defines its topology. In practice, we consider the Frechet spaces of \( C^\infty \) sections of vector bundles over a manifold \( M \). \( || | \) is defined by measuring the partial derivatives up to degree \( k \) of sections. \( \phi \) is given by a non-linear partial differential operator involving partial derivatives up to order, say \( r \). This is translated into an assumption

\[
||\phi(f)||_k \leq C_k (1 + ||f||_{k+r})
\]

For \( k \) sufficiently large any map with the above assumption is called tame (cf. R. Hamilton [1] for more details). We assume that \( \phi \) is infinitely differentiable and all partial derivatives are tame. In particular there is for each \( f \in F' \) a continuous linear map.

\[ d_f \phi : F \to G \]

such that with \( R_f(h) = \phi(f+h) - \phi(f) - d_f \phi(h) \)

\[
||R_f(h)||_k \leq C_k (||f||_{k+r} ||h||^{2}_{k_0} + ||h||^{2}_{n+r})
\]

for \( k > k_1 \). We also assume that there is a mollifier \( M_\epsilon (\epsilon > 0) \) with the standard properties: for \( s > 0 \)

\[
||M_\epsilon f||_{k+s} \leq C_{k,s} \epsilon^{-s} ||f||_k
\]

\[
||f - M_\epsilon f||_k \leq C_{k,s} \epsilon^s ||f||_{k+s}
\]
We now wish to show that an element \( g \in G^* \) is in the image of \( \phi \). We may assume that \( g = 0 \). We solve the problem by a successive approximation. Namely, for a \( \nu \)-th approximation \( f_\nu \) we define \( f_{\nu+1} \) as follows: note that \( \phi(f_\nu + h) \) is very close to \( \phi(f_\nu) + d_{f_\nu} \phi(h) \). Hence we solve the equation:

\[
(8) \quad \phi(f_\nu) + d_{f_\nu} \phi(h) = 0
\]

However, in the process we usually lose derivatives. We compensate this by setting

\[
(9) \quad f_{\nu+1} = f_\nu + M_{\epsilon_{\nu+1}} h_\nu
\]

where \( h_\nu \) is a solution of (8) and where \( \epsilon_\nu \) is given in (1). In fact, we assume that we can find \( h_\nu \) with

\[
(10) \quad \| h_\nu \|_{k-r_1} \leq C_k \| \phi(f_\nu) \|_k
\]

This estimate is essential for this method to work. In order to show that \( f_\nu \) converge to a solution \( f \) of our problem, it is enough to show that \( p_\nu = \| \phi(f_\nu) \|_k \) satisfy (3) in the lemma. If this is the case, \( p_\nu \) has estimate (4). In view of (9) and (10) it then follows that \( f_\nu \) will also converge. Now:

\[
\phi(f_\nu + M_{\epsilon_{\nu+1}} h_\nu) = \phi(f_\nu) + d_{f_\nu} \phi(M_{\epsilon_{\nu+1}} h_\nu) + R_{f_\nu} (M_{\epsilon_{\nu+1}} h_\nu)
\]

\[
= R_{f_\nu} (M_{\epsilon_{\nu+1}} h_\nu) - d_{f_\nu} \phi(h_\nu) - M_{\epsilon_{\nu+1}} h_\nu
\]

Note (7) and (6). From the first term (resp. the second term) we obtain terms \( C^\#_{\epsilon_{\nu+1}} \| p_\nu \|^{2-s} \) (resp. \( C^\#_{\epsilon_{\nu+1}} \| \epsilon_\nu \|^{\lambda-t} \)) for a choice of \( s \) and \( t \).

The above shows that we can solve the equation \( \phi(f) = g \) for a given \( g \) provided we find a very good approximation \( f_0 \) so that the last inequality in (30) is satisfied. In particular, we find that a small neighborhood of \( f_0 \) is covered by \( \phi \).

For a local embedding theorem mentioned in the beginning we have a following more general setting. Namely, we have a manifold \( M \) and for each open \( U \subset M \) we have:
\[ \Phi \rightarrow F'(U) \rightarrow G'(U) \rightarrow H'(U) \]

with \( \Psi, \Phi = 0 \). They are related by compatible restriction maps. We are given \( g \in G'(M) \) with \( \Psi(g) = 0 \) and a reference point \( p_o \) in \( M \). We wish to show that the restriction of \( g \) to a suitable open neighborhood \( U \) of \( p_o \) is in the image of \( \Phi \). We may assume that \( g = 0 \). The existence of \( \Psi \) means that we may not be able to solve the equation (8). We have to replace \( \Phi(f_v) \) by its projection to the image of \( d_v \Phi \). Moreover, we could find such projection only for \( U \) satisfying certain conditions which also depend on \( f_v \). Namely, for each \( f \in F(U_1) \), where \( p_o \in U_1 \), we have a way to define \( r_f > 0 \) and a distance function \( t_f \) to \( p_o \) with the following properties: for \( 0 < r < r_f \) set

\[
U(f, r) = \{ p \in U_1 ; t_f(p) < r \}.
\]

Let \( f' \) be the restriction of \( f \) to \( F(U(f, r)) \), \( f' = f|U(f, r) \).

Then there is

\[
V_{f'} : G(U(f, r)) \rightarrow F(U(f, r))
\]

such that with \( h' = V_{f'}(\Phi(f')) \)

\[
-\Phi(f') = (d_f \Phi)(h') + A(\Phi(f'))
\]

where \( A(\Psi) \) is given by a composition.

\[
A(\Psi) = A_1 \circ A_2(\Psi)
\]

\( A_1 \) is a linear map, \( A_2 \) is a non-linear partial differential operator starting with quadratic terms. Since our error term \( A(\Phi(f')) \) is of quadratic nature as \( R_f \) in (6) we may try to solve our problem by the same method as in the standard case.

We first find \( f_o \in F(U_o) \) such that:

\[
||\Phi(f_o)|U(f_o, r)||_k \leq O(r^N)
\]
for all $N$. This is achieved by solving the differential equation $\phi(f) = 0$ as a formal power series centered at $p_o$ whose Taylor series agree with the solution formal power series will satisfy our requirement. For $\alpha > \beta$ we set for

$$0 < r_o < r_{f_o}$$

$$(16) \quad \varepsilon_o = r_o^{\alpha}, \quad \delta_o = r_o^{\beta}$$

and define $\varepsilon_v$ and $\delta_v$ as in (1). We then set

$$(17) \quad r_{v+1} = r_v - 3 \delta_v$$

Starting from $f_o | U(f_o, r_o)$ we construct $f_1$ as in the standard case replacing $h$ in (8) by $h^1$ in (13). We then show that, if $r_{f_o}$ is properly chosen, $r_1 + \delta_1 \leq r_{f_1}$ and $U(f_1, r_1 + \delta_1) \subseteq U(f_o, r_o - \delta_o)$. We then consider $f_1 | U(f_1, r_1)$ and proceed inductively. We do this construction for all $r_o$ in $[0, r_{f_o}]$.

Thus the success of our method depends essentially on the nature of $V_{f_1}^1$ in (12) which solves the equations (13) and how we could handle the new error term $A(\phi(f^1))$. In our case $V_{f_1}^1$ is obtained by using the solution mapping $N_{f_1}$ of generalized Neumann boundary value problem on $U(f, r)$ associated with $d_{f, \phi}$. $N_{f_1}$ also enters in the construction of $A_1$ in (14). The fact is we could only find $N_{f_1}$ for $U(f, r)$ as in (11), where $t_{f_1}$ satisfies certain conditions. This is the reason why we had to change $U$ as each step of the successive approximation. On the other hand, since $U(f_{v+1}, r_{v+1}) \subseteq U(f_v, r_v)$, we could use the interior estimate. In such estimate a factor $(\delta_v)^{-\ell}$ (cf. (17)) will come in the constant of the inequality. However, we can admit such factor in view of (3). Using estimates for $N_{f_1}$ on $U(f, r)$ as well as interior estimate, we prove the first inequality in (3) for all $r$ in $[0, r_{f_o}]$, provided $p_0, \ldots, p_{v-1}$ is sufficiently small. We now need the second and the third inequality in (3). In view of (16) the second is satisfied for sufficiently small $r_o$. Similarly the third is satisfied in view of (15).
REFERENCES:

