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MOHAMED S. BAOUENDI
E. C. ZACHMANOGLOU

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UNIQUE CONTINUATION THEOREMS FOR SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS AND INEQUALITIES

M. S. BAOUENDI E. C. ZACHMANOGLOU

In this paper we present a new general unique continuation theorem for solutions of linear P.D.E.'s with analytic coefficients. The solutions are assumed to vanish of infinite order on manifolds of codimension ≥ 1. Some new unique continuation results are also given for certain hyperbolic equations (and inequalities) with nonanalytic coefficients.

We give here only the main ideas of the proofs. The complete proofs will appear in [1].

§1. Unique continuation for PDE's with analytic coefficients.

Let Ω be an open set of $\mathbb{R}^n$ and $P(x,D)$ be a linear partial differential operator of order m with analytic coefficients in Ω. We denote by P_m the principal symbol of P and by $\Sigma(P)$ the characteristic set of P contained in $T^*(\Omega)\backslash 0$, i.e.

$$\Sigma(P) = \{(x,\xi);\ x \in \Omega,\ \xi \in \mathbb{R}^n\backslash\{0\},\ p_m(x,\xi) = 0\} .$$

If M and N are two differentiable manifolds contained in Ω, $M \subset N$, and if u is a continuous function defined in N, we say that u vanishes of infinite order on M if, for all $\alpha \in \mathbb{R}$, the function

$$x \to d(x,M)^{\alpha}u(x)$$

is bounded in any compact set of N. Here $d(x,M)$ denotes the distance of x from M.

We say that the manifold M is <u>P-noncharacteristic</u> if the normal bundle of M (in $T^*(\Omega)\setminus 0$) does not intersect $\Sigma(P)$, i.e. for all $x \in M$ and $\xi \in \mathbb{R}^n\setminus\{0\}$ normal to M at x, $p_m(x,\xi) \neq 0$.

We now are ready to state our first result.

<u>Theorem 1.1</u>. <u>Let</u> M <u>and</u> N <u>be two analytic manifolds in</u> Ω, $M \subset N$, <u>and assume that</u> M <u>is</u> P-<u>noncharacteristic</u>. <u>There is a neighborhood</u> V <u>of</u> M <u>in</u> N <u>such that if</u> u <u>is a continuous function in</u> Ω <u>satisfying</u>:

(i) $Pu = 0$ in Ω, <u>and</u>

(ii) <u>the restriction of</u> u <u>to</u> N <u>vanishes of infinite order on</u> M,

<u>then</u> u <u>must vanish in</u> V.

Condition (ii) in Theorem 1.1 may be replaced by a weaker condition which is easier to formulate in local coordinates (see [1]). In particular, if M divides N into two sides (in which case $\dim N = \dim M + 1$), then it is enough to assume that the restriction of u to N vanishes of infinite order on M from one side only. When $\dim M = n - 1$, the result is Holmgren's uniqueness theorem for continuous solutions.

Taking $N = \Omega$ in Theorem 1.1 we obtain the following corollary.

<u>Corollary 1.1</u>. <u>Let</u> M <u>be an analytic manifold in</u> Ω <u>and assume that</u> M <u>is</u> P-<u>noncharacteristic</u>. <u>There is a neighborhood</u> V <u>of</u> M <u>in</u> Ω <u>such that if</u> u <u>is a continuous function in</u> Ω <u>satisfying</u>

(i) $Pu = 0$ <u>in</u> Ω, <u>and</u>

(ii) u <u>vanishes of infinite order on</u> M,

<u>then</u> u <u>must vanish in</u> V.

<u>Idea of proof of Theorem 1.1</u>

Since we are looking for a local result, we can assume that

(1.1) $$\Omega = \{x \in \mathbb{R}^n;\ \sum_{i=1}^{r} x_i^2 < 2,\ \sum_{i=r+1}^{n} x_i^2 < 1\}$$

(1.2) $$M = \{x \in \Omega;\ x_i = 0 \quad \text{for} \quad r+1 \le i \le n\}$$

(1.3) $$N = \{x \in \Omega;\ x_i = 0 \quad \text{for} \quad r+p+1 \le i \le n\}$$

with $0 < r < r + p \le n$.

We set for $\rho \in (0,1]$

(1.4) $$\Omega_\rho = \{x \in \mathbb{R}^n;\ \sum_{i=1}^{r} x_i^2 < 2;\ \sum_{i=r+1}^{n} x_i^2 < \rho\}$$

We first state an auxiliary result which is a special case of Theorem 1.1.

<u>Lemma 1.1</u>: <u>Let</u> Ω, M, N <u>and</u> Ω_ρ <u>be given by</u> (1.1), (1.2), (1.3) <u>and</u> (1.4) <u>respectively</u>. <u>There is</u> $\rho \in (0,1]$ <u>such that every</u> $u \in C(\Omega)$ <u>satisfying (i) and (ii) of Theorem 1.1 and vanishing for</u> $\sum_{i=1}^{r} x_i^2 \ge 1$ <u>must vanish in</u> $N \cap \Omega_\rho$.

In order to show that Theorem 1.1 follows from Lemma 1.1, we consider the following change of variables $x = \theta(y)$ defined by

$$\begin{cases} x_i = y_i & \text{for} \quad 1 \le i \le r \\ x_i = (1 - \sum_{k=1}^{r} y_k^2) y_i & r+1 \le i \le n\ , \end{cases}$$

and the new partial differential operator Q defined by

$$(1 - \sum_{i=1}^{r} x_i^2)^m P(x,D_x)u(x)\big|_{x=\theta(y)} = Q(y,D_y)u(\theta(y)).$$

If $u \in C(\Omega)$ and satisfies (i) and (ii) of Theorem 1.1, one can check that the function v defined by

$$v(y) = \begin{cases} u(\theta(y)) & \text{for } y \in \Omega \text{ and } 1 - \sum_{k=1}^{r} y_k^2 > 0 \\ 0 & \text{for } y \in \Omega \text{ and } 1 - \sum_{k=1}^{r} y_k^2 \le 0 \end{cases}$$

satisfies assumptions of Lemma 1.1 (for Q instead of P). Since M is Q-noncharacteristic v must vanish in Ω_ρ for some $\rho \in (0,1]$ and hence $u = 0$ in a neighborhood of 0 in N.

Idea of proof of Lemma 1.1.

The proof of this lemma is based on Theorem 4.1 in [3]. We prove that if g is an analytic function defined on M, and if u satisfies assumptions of Lemma 1.1, then the function

$$F(t_1,\dots,t_p) = \int g(x_1,\dots,x_n)u(x_1,\dots,x_r,t_1,\dots,t_p,0,\dots,0)dx_1 \dots dx_r$$

is analytic for $\sum_{i=1}^{p} t_i^2$ sufficiently small and vanishes of infinite order at 0; therefore F is identically zero. Using a density argument, we conclude that $u(x) = 0$ in $\Omega_\rho \cap N$ for some $\rho \in (0,1]$.

It should be noted that if u is assumed to be of class C^m in Theorem 1.1 then the proof can be based on the result of [4].

§2. Unique continuation for second order hyperbolic equations and inequalities.

It is well known from examples of Cohen and of Pliš that Theorem 1 cannot be generally true if the coefficients of P are assumed to be C^∞ (even when $N = \Omega$ and M is a hyperplane). However it is known that, for second order elliptic equations with nonanalytic coefficients, Theorem 1 holds with $N = \Omega$ and M a point in Ω. (See for example [2], [5] and their references).

For hyperbolic (or ultrahyperbolic) second order equations with nonanalytic coefficients we can obtain unique continuation results from noncharacteristic manifolds. Here we restrict ourselves to equations with principal part the wave operator $\Delta_x - \partial_t^2$ (with $\Delta_x = \sum_1^n \partial_{x_i}^2$) and the manifold M being a line segment of the t-axis. Actually the method of proof allows us to state our results for inequalities.

For $x \in \mathbb{R}^n$ we set $r = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ and $x = r\theta$ where θ varies over the unit sphere S_{n-1}. For $R > 0$, let

$$D_R = \{(x,t) \in \mathbb{R}^{n+1};\ \ r + |t| < R\} .$$

<u>Theorem 2.1</u>. <u>Let</u> $u \in C^2(D_R)$ <u>and assume that there exist positive constants</u> C_1 <u>and</u> C_2 <u>such that</u>

$$|\Delta_x u - \partial_t^2 u| \le C_1(r^{-1}|\text{grad } u| + r^{-2}|u|) \quad \text{in } D_R , \quad \text{and} \tag{2.1}$$

$$\int_{S_{n-1}} |\text{grad}_\theta u(r\theta,t)|^2 d\theta \le C_2 \int_{S_{n-1}} |u(r\theta,t)|^2 d\theta \tag{2.2}$$

<u>for</u> $r + |t| < R$. <u>If</u> u <u>and</u> grad u <u>vanish of infinite order on the line segment</u> $\{x = 0,\ |t| < R\}$ <u>then</u> u <u>must vanish in</u> D_R.

We have been unable to prove Theorem 2.1 without condition (2.2). It should be noted that the class of functions satisfying condition (2.2) includes: (a) Functions which have finite expansions with respect to an orthonormal basis of spherical harmonics in $L^2(S_{n-1})$; in particular, functions which depend only on r and t. (b) Functions of the form $u(x,t) = h(r,t)v(r\theta,t)$ where $h \in C^2$ for $r + |t| \le R$ and v is analytic in r and t for $r + |t| \le R$.

Inequality (2.1) is satisfied for example by solutions of the nonlinear equation

$$\Delta_x u - \partial_t^2 u = F(x,t,u,\text{grad } u)$$

provided that

$$|F(x,t,u,\text{grad } u)| \le C_3(r^{-1}|\text{grad } u| + r^{-2}|u|)$$

in D_R, for some positive constant C_3. In particular, inequality (2.1) is satisfied by solutions of the linear equation

$$(2.3) \qquad \Delta_x u - \partial_t^2 u + \frac{1}{r} L(x,t,D_x,D_t)u + \frac{c(x,t)}{r^2} u = 0$$

where L is a first order linear differential operator, the coefficients of L and c being bounded functions in D_R.

When the lower order terms in (2.3) depend only on r and t condition (2.2) can be dropped. We have

Theorem 2.2. *Let* $u \in C^2(D_R)$ *and satisfy*

$$\Delta_x u - \partial_t^2 u + \frac{a(r,t)}{r} \partial_r u + \frac{b(r,t)}{r} \partial_t u + \frac{c(r,t)}{r^2} u = 0$$

in D_R, *where* a,b,c *are bounded functions for* $r + |t| < R$. *If* u *and* grad u *vanish of infinite order on the line segment* $\{x = 0, |t| < R\}$, *then* u *must vanish in* D_R.

Idea of proof of Theorems 2.1 and 2.2.

Theorem 2.1 is a consequence of the following weighted L^2 inequality:

For every $u \in C^2(\bar{D}_R)$ satisfying condition (2.2) and such that u and its first and second order derivatives vanish of infinite order on $\{x = 0, |t| \le R\}$, there is $C > 0$ such that, for all sufficiently large γ,

$$\int_{D_R} (r^{-2}|\text{grad } u|^2 + r^{-4}|u|^2)r^{-\gamma}dxdt \le \frac{C}{\gamma} \int_{D_R} |\Delta_x u - \partial_t^2 u|^2 r^{-\gamma}dxdt.$$

This inequality is proved by the method of multipliers applied to equations with coefficients which are singular in r.

Theorem 2.2 is proved by expanding u in spherical harmonics and applying Theorem 2.1 to each component.

REFERENCES

1. M. S. Baouendi and E. C. Zachmanoglou, Unique Continuation of solutions of partial differential equations and inequalities from manifolds of any dimension, to appear. Duke Journal.

2. H. Cordes, Über die Besstimmtheit der Lösungen elliptischer Differentialgleichungen durch Anfangsvorgaben, Nachr. Akad. Wiss. Göttingen IIa (1956), 230-258.

3. L. Hörmander, Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients, Comm. Pure Appl. Math. 24 (1971), 617-704.

4. F. John, On linear partial differential equations with analytic coefficients, Comm. Pure Appl. Math. 2 (1949), 209-253.

5. M. H. Protter, Unique continuation for elliptic equations, Trans. AMS 95 (1960), 81-91.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY
WEST LAFAYETTE, INDIANA 47907