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THE EIGENVALUES OF HYPOELLIPTIC OPERATORS

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Let $P = P(x,D)$ be a self-adjoint pseudo-differential operator of order $m > 0$, with principal symbol $p_m(x,\xi) \geq 0$ on a smooth n-dimensional compact riemannian manifold M without boundary. If P is elliptic then P has a discrete set of eigenvalues bounded from below. Denoting by $N(\lambda)$ the number of eigenvalues $\leq \lambda$ (counting multiplicities) the distribution of eigenvalues of P may be described by the formula

$$(1) \qquad N(\lambda) \sim \frac{\lambda^{n/m}}{(2\pi)^n} \int_{p_m(x,\xi)\leq 1} dx \wedge d\xi \qquad \text{as} \quad \lambda \longrightarrow \infty .$$

This result has a long history. It may be obtained by studying the singularities of one of the functions

$$\operatorname{tr}(P-\lambda I)^{-1}, \quad \operatorname{tr}(e^{-tP}), \quad \operatorname{tr}(P^z), \quad \operatorname{tr}(e^{itP})$$

(see [1], [4] or [8]). Here we would like to consider the same problem for hypoelliptic operators.

A result in this direction has been obtained by Metivier [7], who studied the spectral function of hypoelliptic operators which are the sums of squares of real vector fields. He described the spectral function for operators which have a uniform behavior in the base space, but, for example for the Grušin operator, $D^2_{x''} + |x''|^2\, |D_{x'}|^2$, his results do not give the asymptotic behavior of the eigenvalues. Other results which overlap with ours have been presented at this meeting by Bolley, Camus and Pham [2].

We will discuss the eigenvalues of self-adjoint operators P which are hypoelliptic with the loss of one derivative. Let $\Sigma = \{p_m(x,\xi) = 0\}$ be the characteristic variety of P. We will suppose that Σ is a smooth symplectic submanifold of $T^*(M)$ and that p_m vanishes to exactly second order on Σ. Let $2n' = \dim \Sigma$, $2n'' = \operatorname{codim} \Sigma$ and $\pm i\mu_j$, $j = 1,\dots, n''$, with $\mu_j > 0$ be the eigenvalues of the Hamilton matrix of p_m

(cf. [9]) restricted to the orthogonal space of Σ. Then, P will be hypoelliptic with the loss of one derivative if and only if

(2) $$p'_{m-1}(x,\xi) + \sum_{j=1}^{n''} \mu_j(x,\xi)(1+2\alpha_j) \neq 0$$

for any set of non-negative integers α_j, at every point $(x,\xi)\in\Sigma$. Here p'_{m-1} is the subprincipal symbol of P, (cf. [3] or [9]). In fact, P will have a parametrix $Q \in L^{1-m}_{\frac{1}{2},\frac{1}{2}}$, i.e.

(3) $$QP = I + K$$

where K is a compact operator on $L^2(M)$.

If $m > 1$ and P is hypoelliptic, then P will have only eigenvalues of finite multiplicity whose only limit points can be $\pm\infty$.

We will further suppose that on Σ

(4) $$p'_{m-1} + \sum_{j=1}^{n''} \mu_j > 0.$$

It will then follow from a theorem of Melin [5] that there is a constant C such that

(5) $$(Pu,u) \geq -C\,\|u\|^2$$

and consequently that the spectrum of P is bounded below. Then e^{-tP} is well defined for $t \geq 0$ and our goal will be to show

<u>THEOREM 1</u>. <u>Under the above assumptions</u>

(6) $$\operatorname{tr}(e^{-tP}) \sim \begin{cases} C_1\, t^{-n'/(m-1)} & \underline{\text{if}} \quad n' > n''(m-1) \\ C_2\, t^{-n/m} \log t & \underline{\text{if}} \quad n' = n''(m-1) \\ C_3\, t^{-n/m} & \underline{\text{if}} \quad n' < n''(m-1) \end{cases}$$

<u>as</u> $t \downarrow 0$.

Since $\operatorname{tr}(e^{-tP}) = \sum e^{-\lambda_j t}$ where λ_j are the eigenvalues of P, we may apply Karamata's Tauberian Theorem to conclude.

<u>COROLLARY 2</u>. Denoting the number of eigenvalues $\leq \lambda$ by $N(\lambda)$ we have

$$(7) \qquad N(\lambda) \sim \begin{cases} a_1 \lambda^{n'/(m-1)} & \underline{\text{if}} \quad n' > n''(m-1) \\ a_2 \lambda^{n/m} \log\lambda & \underline{\text{if}} \quad n' = n''(m-1) \\ a_3 \lambda^{n/m} & \underline{\text{if}} \quad n'' < n''(m-1) \end{cases}$$

as $\lambda \to \infty$ (a_3, incidently is the same constant as in formula (1)).

1. THE ELLIPTIC CASE.

We will begin our discussion of Theorem 1 by rederiving formula (1) for the elliptic case in a way amenable to generalization. To approximate $\exp(-tP)$ we will seek a solution of

$$(1.1) \qquad \begin{aligned} &D_t w = i\, P(x,D_x)w \qquad \text{or} \quad \mathbb{R}^+ \times M \\ &w(x,0) = u(x), \end{aligned}$$

micro-locally of the form

$$(1.2) \qquad w(x,t) = A_t u(x) = (2\pi)^{-n} \int e^{i\varphi(t,x,\eta)}\, a(t,x,\eta)\, \hat{u}(\eta)\, d\eta.$$

Applying $D_t - i\,P(x,D_x)$ to (1.2) and grouping terms as if φ were homogenous of degree 1 in η we will get an eikonal equation of the form

$$(1.3) \qquad \varphi_t - i\, p_m(x,\varphi_x') = 0 \quad ; \quad \varphi(0,x,\xi) = x.\eta$$

and various transport equations. Making the change of variables $t = |\eta|^{m-1}s$, (1.3) will become

$$(1.4) \qquad \varphi_s - i\,p'(x,\varphi_x') = 0 \quad \text{where} \quad p' = p_m(x,\varphi_x')/|\eta|^{m-1}$$

for which we will try to find a solution which is homogenous of degree 1 in η. Expanding φ as a power series in s we can find

$$(1.5) \qquad \varphi(s,x,\eta) = \langle x,\eta\rangle + i\,P'(x,\eta)s + \psi_2(x,\eta)s^2 + \dots$$

which satisfies (1.4) modulo an arbitrarily high power of s. From the first transport equation we find that $a = 1 + O(s)$. Since φ leaves the real axis rapidly we may modify φ and a for large s so as to get a solution of (1.1) modulo an operator with C^∞

kernel in x and t.

As a result

$$e^{-tP}u(x) \approx A(t)u(x) = (2\pi)^{-n}\int e^{i<x-y,\eta>-t\,p_m(x,\eta)+\dots} a(t,x,\eta)u(y)dyd\eta$$

and

$$tr(e^{-tP}) \approx (2\pi)^{-n}\iint e^{-tp_m(x,\xi)} dx\wedge d\xi + \dots$$

$$= (2\pi)^{-n}\, t^{-n/m}\, \frac{n}{m}\,\Gamma(\frac{n}{m}) \iint_{p_m(x,\xi)\leq 1} dx\wedge d\xi + \dots$$

modulo a function less singular in t. Applying Karamata's Tauberian Theorem gives (1).

2. THE HYPOELLIPTIC CASE.

We will now attempt to find a solution of (1.1) micro-locally of the form (1.2) when P satisfies the assumption of Theorem 1. The eikonal equation will be of the form

(2.1) $$\varphi'_t = ip_m(x,\varphi'_x)$$

again. We make the same change of variables as before to make (2.1) homogenous. But this time it will be necessary to solve (2.1) as $s \longrightarrow \infty$. This is because the solutions of (2.1) will not leave the real axis everywhere. In fact, bicharacteristics starting in Σ stay in Σ giving a point where Im φ stays 0.

We'll solve (2.1) using Hamilton-Jacobi Theory. We'll make a series of canonical transformations to simplify our problem. To begin with let us choose new canonical coordinates so that $\Sigma = \{x'' = \xi'' = 0\}$ where $(x,\xi) = (x',x'',\xi',\xi'')$, $x' \in \mathbf{R}^{n'}$, $x'' \in \mathbf{R}^{n''}$ etc. Setting $t = s|\eta'|^{m-1}$, (2.1)

becomes

(2.2) $$\varphi'_s = ip_m(x,\varphi'_x)/|\eta'|^{m-1} = i\,p'(x,\varphi'_x).$$

Expanding p' as a Taylor's series in (x'',ξ'') we find

(2.3) $$p'(x,\xi) = \sum_{|\alpha+\beta|=2} a_{\alpha\beta}(x',\xi')x''^{\alpha}\xi''^{\beta} + O(|\xi|^m(|x''|+|\xi''|\,|\xi|))^3).$$

The quadratic terms in (2.3) may be expressed as

$$\sigma((x'',\xi''),H(x'',\xi''))$$

where H, the (transversal) Hamilton matrix of p is skew-symetric with respect to the standard symplectic form σ in $\mathbb{R}^{2n''}$.

Recalling the results of [9], H has eigenvalues of the form $\pm i\mu_j(x',\xi')$ with $\mu_j > 0$ for $j = 1,\dots,n''$, and if V_+ (V_-) denotes the span of the positive (negative) eigenvectors of H in $\mathbb{C}^{2n''}$, then V_+ (V_-) is a positive (negative) definite Lagrangean plane in $\mathbb{C}^{2n''}$, and

$$\mathbb{C}^{n''} \oplus \mathbb{C}^{n''} = V_+ \oplus V_- .$$

Since $V_\pm$ depend smoothly on (x',ξ') we may make a complex canonical change of variables so that $V_- = \{x'' = 0\}$ and $V_+ = \{\xi'' = 0\}$. In terms of these new coordinates

$$H = \frac{i}{2}\begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} \tag{2.4}$$

where A is a matrix with only positive eigenvalues.

Since we have made a complex change of variable the following considerations will be only formal and will required justification.

Equation (2.2) now takes the form

$$\varphi'_s = - \langle A(x',\varphi'_{x'})\,x'',\ \varphi'_{x''}\rangle + \sum_{3<|\alpha+\beta|} b_{\alpha\beta}\, x''^{\alpha} \varphi'^{\beta}_{x''} . \tag{2.5}$$

It is possible to find one more canonical transformation so as to make the higher order term in (2.5) takes the form $0(|x''||\varphi'_{x''}|(|x''| + |\varphi''|))$. Solving (2.5) by using formal power series in (x'',η'') we will get a solution

$$\varphi = \langle x',\eta'\rangle + \langle e^{-sA}x'',\eta''\rangle + \text{cubic term in } (x'',\eta'').$$

The phase function of A_t is

$$\psi = \langle e^{-sA}x'' - y'',\ \eta''\rangle + \langle x' - y',\eta'\rangle + \dots$$

where the other higher order terms converge to 0 exponentially fast.

Denoting by $C_s = \{(x,\varphi'_{x'}\ -\varphi'_{\eta}\ ,\ \eta)\}$ the canonical relation generated by ψ we may note the C_0 is the graph of the identity and $C_\infty = \{(x',x'',\xi',0),(x',0,\xi',\xi'')\}$.

The fist transport equation is

$$\frac{da}{ds} + \left(\tfrac{1}{2}\operatorname{tr} A + p'_{m-1}\right)a = O(|x''| + |\xi''|) \tag{2.7}$$

whose solution is

$$a(s,x,\xi) = e^{-s(\frac{1}{2}\operatorname{tr} A + p'_{m-1})} + O(|x''| + |\xi''|).$$

The leading term of the solution $A_t u$ is

$$(2\pi)^{-n}\int e^{i\langle e^{-sA}x''-y'',\xi''\rangle + \langle x'-y',\xi'\rangle - s(\operatorname{tr}^+ H + p'_{m-1})} u(y)\, dy\, d\xi .$$

The leading term of $\operatorname{tr}(e^{-tP})$ is then

$$(2\pi)^{-n}\int e^{i\langle (e^{-sA}-I)x'',\xi''\rangle}\, e^{-s(\operatorname{tr}^+ H + p'_{m-1})}\, dx d\xi. \tag{2.8}$$

When $n' > n''(m-1)$ we will compute the singular part of (2.8).

Evaluate the integral with respect to (x'',ξ'') in (2.8) by the "method of stationary phases" (thinking of $s^{-1} = |\xi|^{m-1}/t$ as the large parameter). This gives that the leading term of $\operatorname{tr}(\exp(-tP))$ is

$$(2\pi)^{-n'}\int \frac{e^{-s(\operatorname{tr}^+ H + p'_{m-1})}}{\det(I - e^{-sA})}\, dx' \wedge d\xi' . \tag{2.9}$$

It is easily seen that

$$\det(I - e^{-sA})^{-1} = \prod (1 - e^{-2s\mu_j})^{-1}$$

$$= \sum_{0 \le \alpha \in \mathbb{Z}^n} e^{-2(\alpha\cdot\mu)s}$$

where $2\mu_1,\ldots,2\mu_{n''}$ are the eigenvalues of A. When $n' > (m-1)n''$ the integral (2.9) is convergent and equals

$$\frac{t^{-\frac{n'}{m-1}}}{(2\pi)^{n'}}\, \frac{n'}{m-1}\, \Gamma\left(\frac{n'}{m-1}\right) \int_{\Sigma\cap\{F(x',\xi')\ge 1\}} dx' \wedge d\xi' \tag{2.10}$$

where

$$F(x',\xi') = \sum_{0\le\alpha\in\mathbb{Z}^{n''}} \left(p'_{m-1}(x',\xi') + (1+2\alpha_j)\mu_j(x',\xi')\right)^{-n'/m-1} \tag{2.11}$$

(P is hypoelliptic if and only if $F \neq \infty$ for all $(x',\xi') \in \Sigma$).

Applying a Tauberian theorem will yield

$$(2.12) \qquad N(\lambda) \sim \frac{\lambda^{n'/(m-1)}}{(2\pi)^{n'}} \int_{\{F \geq 1\} \cap \Sigma} dx' \wedge d\xi' .$$

This completes a sketch of the proof of Theorem 1. A justification of our formal changes and variable and complete details of the proof will appear in a future publication.

After this conference we learned that Trèves has also constructed exponential e^{-tP} for the same class of operators considered here. Trèves' construction is different from ours. As an application he proves the local analytic hypoellipticity of the $\tilde{\partial}$ - Neuman-problem.

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