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CUTWIDTH OF THE r-DIMENSIONAL MESH OF d-ARY TREES *

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Abstract. We prove that the cutwidth of the r-dimensional mesh of d-ary trees is of order $\Theta(d^{(r-1)n+1})$, which improves and generalizes previous results.

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1. INTRODUCTION

The cutwidth is a fundamental parameter of interconnection networks which plays an important role in the VLSI design [7]. Informally, the cutwidth problem is to find a linear layout of vertices of a graph and a drawing of its edges as semiarcs above the line so that the maximum number of cuts of a vertical line separating consecutive vertices with edges is minimized. The corresponding decision problem is NP-complete in general but is solvable in polynomial time for trees [10]. Very little is known on the exact or even approximate values of the cutwidth of specific graphs, see *e.g.* [6,8,9]. We study the cutwidth of the r-dimensional mesh of d-ary trees $MT(r,d,n)$, denoted by $cw(MT(r,d,n))$. This graph is defined as follows. For $d \geq 2, n \geq 1$, let $T(d,n)$ denote the complete d-ary tree of depth n. For $r \geq 1$, consider an r-dimensional d^n-sided array of d^{rn} vertices. Each vertex corresponds to a d^n-ary vector $(i_1, i_2, ..., i_r)$ where $1 \leq i_j \leq d^n$, for $1 \leq j \leq r$. For any fixed j, call a row the set of any d^n vertices of the array such that the corresponding vectors differ in the j-th position only. We say that the row is of dimension j. On each row of the array, put $T(d,n)$ such that the vertices of the row are the leaves of the tree, in a natural way. The resulting graph generalizes both the well known r-dimensional mesh of binary trees [4,5], *i.e.* $MT(r,2,n)$ as

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well as the 2-dimensional mesh of d-ary trees [2], *i.e.* $MT(2,d,n)$. Those graphs were proposed as possible interconnection networks of parallel computers [1, 3–5] for they combine together the mesh and tree structure. The graph $MT(r,d,n)$ has $d^{(r-1)n}(d^{n+1}+(r-1)d^n-r)/(d-1)$ vertices. Barth [2] proved an upper and lower bound for the cutwidth of $MT(2,d,n)$ of orders $O(d^{n+2})$ and $\Omega(d^n)$, respectively. In this paper we show that $cw(MT(r,d,n)) = \Theta(d^{(r-1)n+1})$, where the upper and the lower bound differ by a small multiplicative factor. The upper bound is obtained by a recursive linear layout while the lower bound is derived using refinements of standard methods in the field.

2. Preliminaries

The cutwidth problem is defined as follows. For a graph $G=(V,E), |V|=n$, let $\pi : V \rightarrow \{1,2,...,n\}$ be a 1-1 labeling of vertices of G. Denote

$$cw(G,\pi) = \max_i\{|\{uv \in E : \pi(u) \le i < \pi(v)\}|\}\cdot$$

Then cutwidth of G is defined as

$$cw(G) = \min_\pi\{cw(G,\pi)\}\cdot$$

The problem can be viewed as a placing of vertices of G into integer points 1, 2, 3, ..., n of the x-axis and a drawing of edges above the line x. That is why we will often speak about a linear layout.

Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be graphs such that $|V_1| \le |V_2|$. Let $X \subset V_2, |X| = |V_1|$. An embedding of G_1 in G_2 with respect to X is a couple of mappings (ϕ,ψ) satisfying

$$\phi : V_1 \rightarrow X \quad \text{is an injection,} \quad \psi : E_1 \rightarrow \{\text{set of all paths in } G_2\},$$

such that if $uv \in E_1$ then $\psi(uv)$ is a path between $\phi(u)$ and $\phi(v)$. Define the congestion G_1 in G_2 with respect to X

$$cg_X(G_1,G_2) = \min_{(\phi,\psi)} \max_{e\in E_2}\{|\{f \in E_1 : e \in \psi(f)\}|\}\cdot$$

The bisection width of the graph $G=(V,E)$, with respect to $X \subset V$, denoted by $bw_X(G)$, is the minimum number of edges in G whose removal divides G into $G_1 = (V_1,E_1)$ and $G_2=(V_2,E_2)$ such that $||X \cap V_1| - |X \cap V_2|| \le 1$. If $X=V$ then we use $bw(G)$ only.

If K_m denotes the complete graph on m vertices let $\mathcal{K}(r,d^n)$ denote the Cartesian product of r copies of K_{d^n}.

3. Optimal bounds

In this section we prove asymptotically optimal upper and lower bounds on the cutwidth of the r-dimensional mesh of d-ary trees.

Theorem 3.1. *For any $d \geq 2, n \geq 1$ and $r \geq 1$*

$$\frac{1}{4}d^{(r-1)n+1} \leq cw(MT(r,d,n)) \leq \frac{1}{2}d^{(r-1)n+1} + \frac{5}{2}d^{(r-1)n}.$$

Proof. Upper Bound. We construct a linear layout of $MT(r,d,n)$ recursively. For the sake of clarity we assume that d is even. For odd d the proof is similar.

Firstly, consider the case $n = 1$. We claim that there exists a linear layout $\pi_{r,1}$ of $MT(r,d,1)$ with

$$cw(MT(r,d,1),\pi_{r,1}) \leq \frac{d(d^r-1)}{2(d-1)}. \tag{3.1}$$

The claim is trivial for $r = 1$. Let $r > 1$ and assume that we have constructed a layout $\pi_{r-1,1}$ of $MT(r-1,d,1)$ with

$$cw(MT(r-1,d,1),\pi_{r-1,1}) \leq \frac{d(d^{r-1}-1)}{2(d-1)}.$$

We say that the $T(d,1)$ is of j-th dimension if the corresponding row of its leaves is of j-th dimension. Deleting the d^{r-1} roots of all $T(d,1)$'s of the dimension r we get d^{r-1} copies of $MT(r-1,d,1)$. Place these graphs consecutively on a line using $\pi_{r-1,1}$. Then insert the deleted roots with incident edges in such a way that each inserted star increases the current cutwidth by $d/2$. Hence we have by the inductive assumption

$$cw(MT(r,d,1),\pi_{r,1}) \leq cw(MT(r-1,d,1),\pi_{r-1,1}) + \frac{d^r}{2} \leq \frac{d(d^r-1)}{2(d-1)}.$$

Secondly, let $n > 1$. Consider $MT(r,d,n)$. Assume we have a linear layout $\pi_{r,n-1}$ of $MT(r,d,n-1)$. Deleting all $rd^{(r-1)n}$ roots of the trees $T(d,n)$ in $MT(r,d,n)$ we get d^r graphs isomorphic to $MT(r,d,n-1)$. To imagine this fact one can first restrict to the case $d = 2$ and $r = 2,3$. The extension for $d > 2$ and $r > 3$ is straighforward. For each $MT(r,d,n-1)$ take its "array" vertex with the smallest corresponding vector, where we assume the lexicographic order, the leftmost position is the least significant. We get d^r representatives of all graphs $MT(r,d,n-1)$. Sort the representatives lexicographically and place the graphs $MT(r,d,n-1)$ on a line consecutively using $\pi_{r,n-1}$, in the order given by the representatives. Insert the deleted roots with incident edges, such that the cutwidth of each single star is $d/2$. We claim that the inserted roots of all trees of the j-th dimension increase the current cutwidth by $d^{(r-1)(n-1)+j}/2$. In fact observe that for $j = 1$

the number of roots of all trees of the 1st dimension, whose incident edges can overlap is $d^{(r-1)(n-1)}$. One such root contribute to the current cutwidth by $d/2$. So the contribution of the above roots of the 1st dimension to the current cutwidth is $d^{(r-1)(n-1)} \times d/2$. Let $j = 2$. The ordering of $MT(r, d, n-1)$'s on the line implies that the number of roots of all trees of the 2nd dimension, whose incident edges can overlap, is d times more than in the case $j = 1$. This gives an increase of the current cutwidth by $d \times d^{(r-1)(n-1)} \times d/2$, and so on. Finally, if $j = r$, the number of roots of all trees of the r-th dimension, whose incident edges can overlap is $d^{r-1} \times d^{(r-1)(n-1)}$, *i.e.* all $d^{(r-1)n}$ root vertices of the r-th dimension, and their contribution to the current cutwidth is $d^{(r-1)n+1}/2$. The case $d = 2, r = 3$ is very instructive to imagine this claim.

Hence we have a layout $\pi_{r,n}$ of $MT(r, d, n)$, with

$$\begin{aligned} cw(MT(r,d,n),\pi_{r,n}) &\leq cw(MT(r,d,n-1),\pi_{r,n-1}) + \frac{1}{2}\sum_{j=1}^{r} d^{(r-1)(n-1)+j} \\ &\leq cw(MT(r,d,n-1),\pi_{r,n-1}) + \frac{(d^r-1)d^{(r-1)(n-1)+1}}{2(d-1)}. \end{aligned}$$

The solution of this recurrent relation with the initial condition (3.1) is

$$cw(MT(r,d,n),\pi_{r,n}) \leq \frac{d(d^r-1)(d^{(r-1)n)}-1)}{2(d-1)(d^{r-1}-1)} \leq \frac{1}{2}d^{(r-1)n+1} + \frac{5}{2}d^{(r-1)n}.$$

Lower Bound. We use a simple observation that for any graph $G = (V, E)$ and any $X \subset V$

$$cw(G) \geq bw_X(G). \tag{3.2}$$

We apply the following lower bound formula

$$bw_X(G_2) \geq \frac{bw(G_1)}{cg_X(G_1, G_2)}. \tag{3.3}$$

It was implicitely proved by Leighton [4] with $G_1 = K_{|V_1|}$, $|V_1| = |V_2|$ and $X = V_2$. Our generalization is straightforward.

Let X denote the set of leaves of all $T(d, n)$'s in $MT(r, d, n)$. Thus $|X| = d^{rn}$. Put $G_1 = \mathcal{K}(r, d^n)$ and $G_2 = MT(r, d, n)$. If the vertices of K_{d^n} are labelled by $1, 2, ..., d^n$ then the vertices of $\mathcal{K}(r, d^n)$ coincides with the vertices of the r-dimensional d^n-sided array, *i.e.* the set X. Consider and embedding of $\mathcal{K}(r, d^n)$ into $MT(r, d, n)$ with respect to X, s.t. the mapping ϕ is the identical mapping and the mapping ψ is defined by shortest paths. The embedding implies that

$$cg_X(\mathcal{K}(r, d^n), MT(r, d, n)) = cg_{X'}(K_{d^n}, T(d, n)), \tag{3.4}$$

where X' denotes the set of leaves of $T(d,n)$, and the embedding of K_{d^n} into $T(d,n)$ with respect to X' is the restriction of the original embedding. In this new embedding, observe that an edge incident to the root of $T(d,n)$ belongs to $d^{n-1}(d^n - d^{n-1})$ shortest paths defined by this embedding and that this is the maximum over all edges. Hence

$$cg_{X'}(K_{d^n}), T(d,n)) \leq d^{n-1}(d^n - d^{n-1}). \tag{3.5}$$

Moreover, a result of Nakano [8] implies

$$bw(\mathcal{K}(r,d^n)) \geq \frac{d^{(r+1)n}}{4} \cdot \tag{3.6}$$

Finally, combining (3.2–3.5)and (3.6), we get the result. □

References

[1] D. Barth, *Réseaux d'Interconnexion: Structures et Communications.* PhD. Thesis. LABRI, Université Bordeaux I, France (1994).

[2] D. Barth, Bandwidth and cutwidth of the mesh of d-ary trees, in *Proc. 2nd Intl. Euro-Par Conference*, edited by L. Bougé *et al.* Springer Verlag, Berlin, *Lecture Notes in Comput. Sci.* **1123** (1996) 243-246.

[3] M.M. Eshagian and V.K. Prasanna, Parallel geometric algorithms for digital pictures on mesh of trees, in *Proc. 27th Annual IEEE Symposium on Foundation of Computer Science.* IEEE Computer Society Press, Los Alamitos (1986) 270-273.

[4] F.T. Leighton, *Complexity Issues in VLSI.* MIT Press, Cambridge (1983).

[5] F.T. Leighton, *Introduction to Parallel Algorithms and Architectures: Arrays, Trees, and Hypercubes.* Morgan Kaufmann Publishers, San Mateo (1992).

[6] T. Lengauer, Upper and Lower Bounds for the Min Cut Linear Arrangenents Problem on Trees. *SIAM J. Algebraic Discrete Methods* **3** (1982) 99-113.

[7] A.D. Lopez and H.F.S. Law, A Dense Gate Matrix Layout Method for MOS VLSI. *IEEE Trans. Electr. Dev.* **27** (1980) 1671-1675.

[8] K. Nakano, Linear layout of generalized hypercubes, in *Proc. 19th Intl. Workshop on Graph-Theoretic Concepts in Computer Science.* Springer Verlag, Berlin, *Lecture Notes in Comput. Sci.* **790** (1994) 364-375.

[9] A. Raspaud, O. Sýkora and I. Vrťo, Cutwidth of the de Bruijn Graph. *RAIRO Theoret. Informatics Appl.* **26** (1996) 509-514.

[10] M. Yannakakis, A Polynomial Algorithm for the Min Cut Linear Arrangement of Trees. *J. ACM* **32** (1985) 950-988.

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