INFORMATIQUE THÉORIQUE ET APPLICATIONS

PIERRE-CYRILLE HÉAM

# A lower bound for reversible automata

*Informatique théorique et applications*, tome 34, nº 5 (2000), p. 331-341

<http://www.numdam.org/item?id=ITA_2000__34_5_331_0>

**Theoretical Informatics and Applications**
Theoret. Informatics Appl. **34** (2000) 331–341

# A LOWER BOUND FOR REVERSIBLE AUTOMATA

PIERRE-CYRILLE HÉAM[1]

**Abstract.** A reversible automaton is a finite automaton in which each letter induces a partial one-to-one map from the set of states into itself. We solve the following problem proposed by Pin. Given an alphabet $A$, does there exist a sequence of languages $K_n$ on $A$ which can be accepted by a reversible automaton, and such that the number of states of the minimal automaton of $K_n$ is in $O(n)$, while the minimal number of states of a reversible automaton accepting $K_n$ is in $O(\rho^n)$ for some $\rho > 1$? We give such an example with $\rho = \left(\frac{9}{8}\right)^{\frac{1}{12}}$.

**Résumé.** Un automate réversible est un automate fini dans lequel chaque lettre réalise une fonction injective de l'ensemble des états dans lui-même. On résout dans cet article le problème suivant posé par Pin : étant donné un alphabet $A$, existe-t-il une suite de langages $K_n$ sur $A$ qui peuvent être reconnus par un automate réversible, et tels que le nombre d'états de l'automate minimal de $K_n$ soit en $O(n)$ alors que le nombre minimal d'états d'un automate réversible reconnaissant $K_n$ soit en $O(\rho^n)$ avec $\rho > 1$ ? On donne un tel exemple avec $\rho = \left(\frac{9}{8}\right)^{\frac{1}{12}}$.

**AMS Subject Classification.** 68Q45, 68Q70.

## INTRODUCTION

In this paper we answer an open question on reversible automata proposed by Pin in [10]. Reversible automata form a natural class of automata with links to artificial intelligence [1], and to biprefix codes [4]. Moreover reversible automata are used to study inverse monoids, inverse automata [12,13] and certain topological problems [7,10,12,13]. We prove that computing a reversible automaton which

*Keywords and phrases:* Automata, formal languages, reversible automata.

[1] LIAFA, Université Paris 7, 2 place Jussieu, 75251 Paris Cedex 05, France;
e-mail: heam@liafa.jussieu.fr

recognizes a language given by its minimal automaton can not be done, in some cases, faster than in exponential time (if the alphabet contains at least two letters).

We denote by $|K|$ the cardinality of a set $K$.

We assume that the reader is familiar with the basic definitions on words and formal languages. For more information we refer the reader to [2]. Let us denote by $|u|$ the length of a word and by $u^{\sim}$ its mirror.

Let us recall that a finite automaton is a 5-tuple $\mathcal{A} = (Q, A, E, I, F)$ where $Q$ is a finite set of states, $A$ is the alphabet, $E \subseteq Q \times A \times Q$ is the set of edges (or transitions), $I \subseteq Q$ is the set of initial states and $F \subseteq Q$ is the set of final states. A path in $\mathcal{A}$ is a finite sequence of consecutive edges:

$$p = (q_0, a_0, q_1), (q_1, a_1, q_2), \cdots, (q_{n-1}, a_n, q_n).$$

The label of the path $p$ is the word $a_1 a_2 \cdots a_n$, its origin is $q_0$ and its end is $q_n$. A word is accepted by $\mathcal{A}$ if it is the label of a path in $\mathcal{A}$ having its origin in $I$ and its end in $F$. The set of words accepted by $\mathcal{A}$ is denoted by $L(\mathcal{A})$. For every state $q$ and language $K$, we denote by $q.K$ the subset of $Q$ of all the states which are the end of a path having its origin in $q$ and its label in $K$. An automaton is said to be *trim* if for each state $q$ there exists a path from an initial state to $q$ and a path from $q$ to a final state. An automaton is *deterministic* if it has a unique initial state and does not contain any pair of edges of the form $(q, a, q_1)$ and $(q, a, q_2)$ with $q_1 \neq q_2$. An important result of automata theory states that for an automaton $\mathcal{A}$ there exists exactly one (up to isomorphism) deterministic automaton with a minimal number of states which accepts the same language as $\mathcal{A}$. It is called the *minimal automaton* of $L(\mathcal{A})$. A *reversible* automaton is a finite automaton in which each letter induces a partial one-to-one map from the set of states into itself. It also is an automaton which does not contain any pair of edges of the form $(q, a, q_1)$ and $(q, a, q_2)$ with $q_1 \neq q_2$ or $(q_1, a, q)$ and $(q_2, a, q)$ with $q_1 \neq q_2$. It may happen that a reversible automaton is not deterministic because it may have several initial states. Reversible automata were studied by Pin [10] and by Silva [11]. For an automaton $\mathcal{A} = (Q, A, E, I, F)$, let us introduce the following languages, for $p$ and $q$ in $Q$:

$$L_{p,q}(\mathcal{A}) = \{u \in A^* \mid p.u = q\}\cdot$$

If $I = \{i\}$ we write $L_p(\mathcal{A})$ for $L_{i,p}(\mathcal{A})$, and if $F = \{f\}$ we write $R_q(\mathcal{A})$ for $L_{q,f}(\mathcal{A})$. If there is no ambiguity we just write $L_{p,q}$, $L_p$ and $R_q$.

For a language $L$ which is recognized by a reversible automaton we introduce two invariants: $m(L)$, the number of states of its minimal automaton and $c(L)$ the minimal number of states of a reversible automaton accepting $L$. We denote by $r(n)$ and $R(n)$ the following functions:

$$r(n) = \min\{c(L) \mid m(L) = n\} \text{ and } R(n) = \max\{c(L) \mid m(L) \leq n\}\cdot$$

It is proved in [10] that $r(n) = O\left(\frac{\ln n}{\ln \ln n}\right)^2$. Let us remark that $R(n)$ is an increasing function of $n$. We prove here that $R(n) = \Omega(\rho^n)$ for some $\rho > 1$.

## 1. Main theorem

**Theorem 1.1.** *The relation $R(n) = \Omega(\rho^n)$ holds with $\rho = \left(\frac{3\sqrt{2}}{4}\right)^{\frac{1}{6}} \simeq 1.001$ if the alphabet contains at least two letters.*

For the proof, let us consider the language $L = \{aa, ab, ba\}$ on the alphabet $A = \{a, b\}$. Before we proceed, let us note the following technical result.

**Lemma 1.2.** *Let $\mathcal{A} = (Q, A, E, i, F)$ be a reversible trim automaton accepting a non-empty subset of $L = \{aa, ab, ba\}$. For each state $s \in Q$, let $L_s$ be the set of words which can be read from state $i$ to $s$. Then one of the following holds:*

(a) $|F| = 3$ *and* $|L_s| = 1$ *for each state* $s \in F$;
(b) $|F| = 2$, $F = \{s_1, s_2\}$, $|L_{s_1}| = 2$ *and* $|L_{s_2}| = 1$;
(c) $|F| = 2$ *and* $|L_s| = 1$ *for each state* $s \in F$;
(d) $|F| = 1$, $F = \{s\}$ *and* $|L_s| = 2$;
(e) $|F| = 1$, $F = \{s\}$ *and* $|L_s| = 1$.

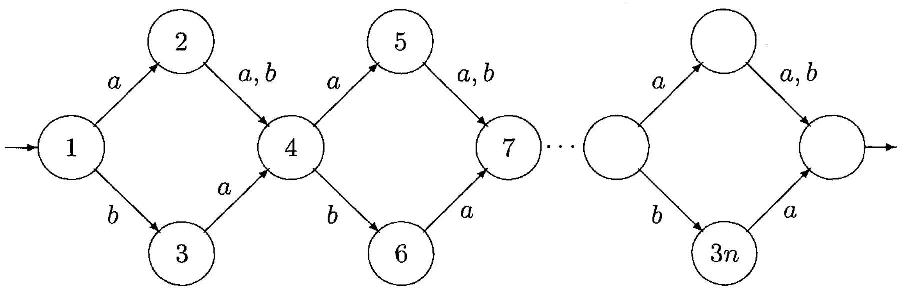
*Proof.* First we observe that in a reversible automaton on 2 letters, each state is the origin (resp. the end) of at most 2 transitions. Since $|L| = 3$ and $\mathcal{A}$ is trim, $F$ has at most 3 elements, and if $|F| = 3$, then $|L_s| = 1$ for each final state $s$, that is *(a)* holds.

If $|F| = 2$, say $F = \{s_1, s_2\}$, then $|L_{s_1}| + |L_{s_2}| \leq 3$. Since $\mathcal{A}$ is trim $|L_s| > 0$ for every state, so *(b)* or *(c)* holds.

Finally, let us assume that $F = \{s\}$ is a singleton. We need to verify that $|L_s| \neq 3$, that is, $\mathcal{A}$ does not accept all of $L$. If there is no $a$-labeled transition out of $i$, then $L_s = L(\mathcal{A}) = \{ba\}$ and *(e)* holds. If there is no $b$-labeled transition out of $i$, then $L_s \subseteq \{ab, aa\}$ and *(d)* or *(e)* holds. We now consider the case where both an $a$-labeled transition and a $b$-labeled transition start in state $i$. If $i.a = i.b$, then there is no $b$-labeled transition from $i.a$ to the final state $s$ since $bb$ cannot be accepted, and hence $L_s = \{aa, ba\}$ and *(d)* holds. On the other hand, suppose that $i.a \neq i.b$. Since $\mathcal{A}$ is trim, there is an $a$-labeled transition from state $i.b$ to $s$. But $\mathcal{A}$ is reversible, so there is no $a$-labeled transition from state $i.a$ to $s$, and the word $aa$ is not accepted. In particular, $L_s = \{ab, ba\}$ and *(d)* holds. □

For every integer $n$, $L^n$ is a finite language which can be recognized by a reversible automaton (it is easy to check that every finite language can be recognized by a reversible automaton). We prove that $c(L^{2n}) \geq \rho^n$, while $m(L^{2n}) = O(n)$.

First observe that, for every positive integer $n$, $|L^n| = 3^n$, $L^n = (L^n)^{\sim}$ and every word of $L^n$ has length $2n$. Furthermore it is easy to check that the minimal automaton recognizing $L^n$ is the following one:




Thus the minimal automaton of $L^n$ has $m(L^n) = 3n + 1$ states. Moreover, if $\mathcal{A}$ is a trim automaton which recognizes a sublanguage of $L^n$, then for all pairs of states $p$ and $q$ of $\mathcal{A}$, all the words of $L_{p,q}$ have the same length.

Let $\mathcal{A} = (Q, A, E, \{i\}, \{f\})$ be a reversible automaton with a unique initial state and a unique final state recognizing a sublanguage of $L^{2n}$. Let $M(\mathcal{A}) = i.A^{2n}$ be the set of states of $\mathcal{A}$ we can reach from $i$ by reading a word of length $2n$ (half of the length of the words of $L^{2n}$).

**Proposition 1.3.** *With the above notation, the following relation holds*

$$|L(\mathcal{A})| \leq \frac{1}{2}\left(\sum_{q\in M(\mathcal{A})} |L_q|^2 + |R_q^{\sim}|^2\right) \cdot \tag{1}$$

*Proof.* By definition we have:

$$L(\mathcal{A}) = \bigcup_{q\in M(\mathcal{A})} L_q R_q.$$

But $\mathcal{A}$ is deterministic and hence, if $p \neq q$ then $L_p \cap L_q = \emptyset$. Consequently, if $p, q \in M(\mathcal{A})$ then $L_p R_p \cap L_q R_q = \emptyset$. Indeed the prefix of length $2n$ of a word of $L_p R_p \cap L_q R_q$ would have to be in $L_p \cap L_q$. Thus we get:

$$|L(\mathcal{A})| = \sum_{q\in M(\mathcal{A})} |L_q||R_q|.$$

Since $R_q$ is finite we have $|R_q| = |R_q^{\sim}|$. But for every pair of real numbers $(x, y)$ we have $xy \leq \frac{1}{2}(x^2 + y^2)$. From this we deduce (1). $\square$

For the next proposition and the next three lemmas we consider a reversible automaton $\mathcal{A} = (Q, A, E, \{i\}, F)$ recognizing a sublanguage of $L^n$ (where $n$ is a strictly positive integer), and having a unique initial state $i$. For every state $q$ of $Q$ let us denote by $n_q$ the cardinal of $L_q$.

**Proposition 1.4.** *The following equality holds:*

$$\sum_{q\in F} n_q^2 \leq 8^n. \tag{2}$$

To prove Proposition 1.4 we proced by induction on $n$ and we need some technical lemmas. For the proof we can assume without loss of generality that $\mathcal{A}$ is trim.

**Lemma 1.5.** *If $n = 1$, Proposition 1.4 is true.*

*Proof.* Applying Lemma 1.2 to $\mathcal{A}$, we get:
In case *(a)*, $\sum_{q\in F} n_q^2 = |L_{i,s_1}|^2 + |L_{i,s_2}|^2 + |L_{i,s_3}|^2 = 3$.
In case *(b)*, $\sum_{q\in F} n_q^2 = |L_{i,s_1}|^2 + |L_{i,s_2}|^2 = 4 + 1 = 5$.
In case *(c)*, $\sum_{q\in F} n_q^2 = |L_{i,s_1}|^2 + |L_{i,s_2}|^2 = 1 + 1 = 2$.
In case *(d)*, $\sum_{q\in F} n_q^2 = |L_{i,s_1}|^2 = 4$.
In case *(e)*, $\sum_{q\in F} n_q^2 = |L_{i,s_1}|^2 = 1$.
This completes the proof. □

We denote by $S$ the set of states which we can reach by reading a word of length $2n-2$ (every word of $L^n$ has length $2n$).

**Lemma 1.6.** *For every $q$ in $F$ we have*

$$n_q = \sum_{s\in S} n_s |L_{s,q}|. \tag{3}$$

*Proof.* Since every word of $L(\mathcal{A})$ has length $2n$ we have

$$L_q = \bigcup_{s\in S} L_s L_{s,q}. \tag{4}$$

But $\mathcal{A}$ is deterministic, so as in the proof of Proposition 1.3 the union in (4) is disjoint. Since $n_q = |L_q|$ for every state $q$ in $Q$, we can now deduce formula (1.6) from formula (4). □

We now prove that:

**Lemma 1.7.** *For all $s \in S$ we have*

$$\sum_{q\in F} \sum_{t\in S} |L_{s,q}||L_{t,q}| \leq 8. \tag{5}$$

*Proof.* Let $s \in S$ and let $F_s$ be the set of states in $F$, which we can reach from $s$. If $q \in F$, we denote by $I_q$ the set of states of $S$ from which $q$ can be reached. The sum in formula (5) is in fact equal to

$$\sum_{q \in F_s} \sum_{t \in I_q} |L_{s,q}||L_{t,q}|.$$

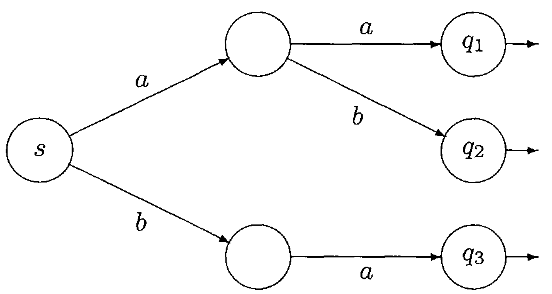
Since $\mathcal{A}$ is trim, the words which can be read from a state in $S$ to a state in $F$, are all in $L$. Moreover, as $\mathcal{A}$ is reversible, if $q \in F$, then the languages $L_{t,q}$ $(t \in I_q)$ are pairwise disjoint and hence

$$\sum_{t \in I_q} |L_{t,q}| \leq 3.$$

Consider the automaton obtained from $\mathcal{A}$ by first making $s$ the initial state and then trimming. This automaton satisfies the hypothesis of Lemma 1.2 and one of the following five cases arises.

(a) $|F_s| = 3$ and $|L_{s,q}| = 1$ for each $q \in F_s$.

We have, say, the following configuration




Note that there is no state $t \in I_{q_1}$ such that $t.b = s.a$: if there was, we would have $t.bb = q_2$, so $bb \in L_{t,q_2} \subseteq L$, a contradiction. Since $\mathcal{A}$ is trim, it follows that $ba \notin \bigcup_{t \in I_{q_1}} L_{t,q_1}$ and hence

$$\sum_{t \in I_{q_1}} |L_{t,q_1}| \leq 2.$$

We already observed that

$$\sum_{t \in I_{q_2}} |L_{t,q_2}| \leq 3 \text{ and } \sum_{t \in I_{q_3}} |L_{t,q_3}| \leq 3$$

so

$$\sum_{q \in F_s} \sum_{t \in I_q} |L_{s,q}||L_{t,q}| = \sum_{q \in F_s} \sum_{t \in I_q} |L_{t,q}| \leq 2 + 3 + 3 = 8.$$

(b) $F_s = \{q_1, q_2\}$ with $|L_{s,q_1}| = 2$ and $|L_{s,q_2}| = 1$. Since $\mathcal{A}$ is reversible we have either $I_{q_1} = \{s\}$ or $I_{q_1} = \{s, u\}$ with $|L_{u,q_1}| = 1$.

(b1) If $I_{q_1} = \{s\}$ we have

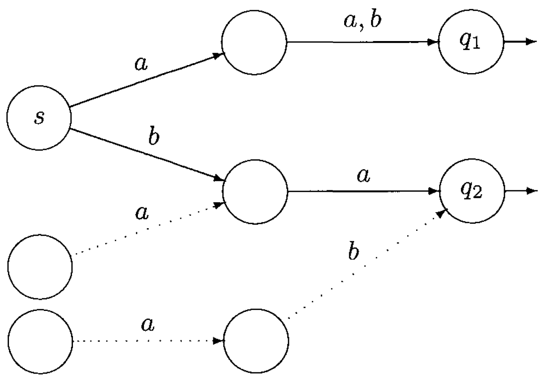
$$\sum_{t \in I_{q_1}} |L_{t,q_1}||L_{s,q_1}| = 4$$

and
$$\sum_{t \in I_{q_2}} |L_{t,q_2}||L_{s,q_2}| = \sum_{t \in I_{q_2}} |L_{t,q_2}| \leq 3$$

so

$$\sum_{q \in F_s} \sum_{t \in I_q} |L_{s,q}||L_{t,q}| \leq 4 + 3 = 7.$$

This case can be illustrated in the following figure:



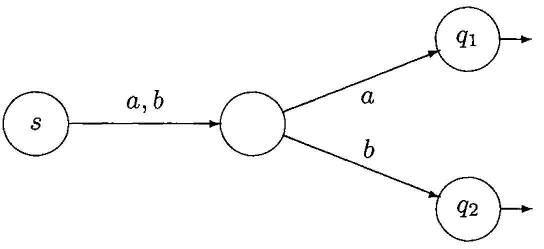

(b2) Assume that $I_{q_1} = \{s, u\}$. By reversibility, we have $L_{u,q_1} = L \setminus L_{s,q_1}$. Since $|L_{s,q_1}| = 2$, it follows that $|L_{u,q_1}| = 1$ and

$$\sum_{t \in I_{q_1}} |L_{s,q_1}||L_{t,q_1}| = 2\,(|L_{s,q_1}| + |L_{u,q_1}|) = 6.$$

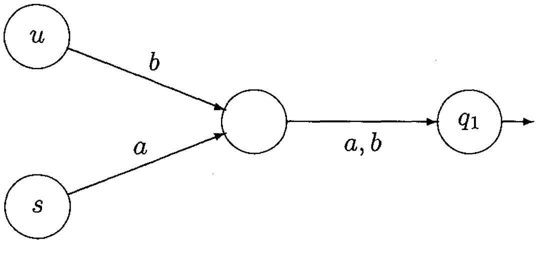
If $I_{q_2} = \{s\}$, we are finished:

$$\sum_{q \in F_s} \sum_{t \in I_q} |L_{s,q}||L_{t,q}| = 6 + 1 = 7.$$

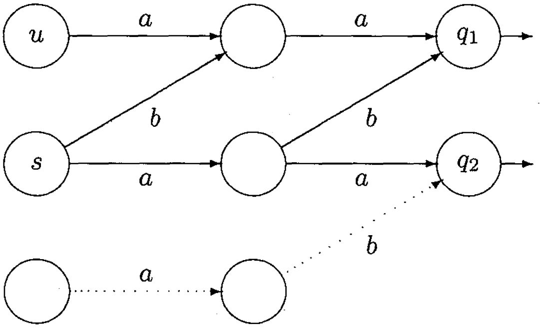
In order to go further, let us analyze this situation in more detail. If $L_{s,q_1} = \{aa, ba\}$, then by reversibility we have $s.b = s.a$. Moreover in that case $L_{s,q_2} = \{ab\}$, so that $(s.a).b = q_2$. This implies in turn that $s.bb = q_2$, $bb \in L_{s,q_2}$, a contradiction.




Now if $L_{s,q_1} = \{aa, ab\}$ then by determinism both the $a$-labeled and the $b$-labeled transitions into $q_1$ come from $s.a$. Moreover $L_{u,q_1} = \{ba\}$, so $u.b = s.a$ and we now have $u.bb = q_1$, $bb \in L_{u,q_1}$, again a contradiction.




Thus $aa \notin L_{s,q_1}$ and we have $L_{s,q_1} = \{ab, ba\}$, $L_{s,q_2} = L_{u,q_1} = \{aa\}$ and the following configuration




Suppose that $v \in I_{q_2}$, $v \neq s$. Then $aa \notin L_{v,q_2}$ by reversibility. Also $ba \notin L_{v,q_2}$: otherwise $v.b = s.a$ and $v.bb = q_1$. This implies $bb \in L_{v,q_1} \subseteq L$, a contradiction. Thus if $v \neq s$ lies in $I_{q_2}$, then $L_{v,q_2} = \{ab\}$. This shows that

$$\sum_{t \in I_{q_2}} |L_{s,q_2}||L_{t,q_2}| = \sum_{t \in I_{q_2}} |L_{t,q_2}| \leq 2$$

and hence

$$\sum_{q\in F_s}\sum_{t\in I_q}|L_{s,q}||L_{t,q}| \leq 8.$$

(c) $|F_s| = 2$ and $|L_{s,q}| = 1$ for each $q \in F_s$. Since $\sum_{t\in I_q}|L_{t,q}| \leq 3$ for each $q \in F$, we have

$$\sum_{q\in F_s}\sum_{t\in I_q}|L_{s,q}||L_{t,q}| = \sum_{q\in F_s}\sum_{t\in I_q}|L_{t,q}| \leq 2\times 3 = 6.$$

(d–e) $F_s = \{q\}$ and $|L_{s,q}| = 2$ (resp. 1). Again using that $\sum_{t\in I_q}|L_{t,q}| \leq 3$, we find that

$$\sum_{q\in F_s}\sum_{t\in I_q}|L_{s,q}||L_{t,q}| \leq 6 \text{ (resp. 3)}.$$

□

Now we can prove Proposition 1.4.

*Proof.* We proceed by induction on $n$. The proposition is true for $n = 1$ by Lemma 1.5.

Assume now that it is true for $n$. Let $\mathcal{A} = (Q, A, E, \{i\}, F)$ be a reversible trim automaton with a unique initial state $i$ recognizing a subset of $L^{n+1}$. According to Lemma 1.6

$$n_q^2 = \sum_{(s,t)\in S^2} n_s n_t |L_{s,q}||L_{t,q}|. \tag{6}$$

Thus we have:

$$\begin{aligned}\sum_{q\in F} n_q^2 &= \sum_{q\in F}\sum_{(s,t)\in S^2} n_s n_t |L_{s,q}||L_{t,q}| = \sum_{(s,t)\in S^2}\left(\sum_{q\in F}|L_{s,q}||L_{t,q}|\right) n_s n_t \\ &\leq \frac{1}{2}\sum_{(s,t)\in S^2}\left(\sum_{q\in F}|L_{s,q}||L_{t,q}|\right) n_s^2 + \frac{1}{2}\sum_{(s,t)\in S^2}\left(\sum_{q\in F}|L_{s,q}||L_{t,q}|\right) n_t^2 \\ &\leq \sum_{(s,t)\in S^2}\left(\sum_{q\in F}|L_{s,q}||L_{t,q}|\right) n_s^2.\end{aligned}$$

From this, by Lemma 1.7, we have

$$\sum_{q\in F} n_q^2 \leq 8\sum_{s\in S} n_s^2.$$

Now we can apply the induction hypothesis to the automaton deduced from $\mathcal{A}$ by taking $S$ for the set of final states rather than $F$, and by making it trim. We obtain:

$$\sum_{s\in S} n_s^2 \leq 8^n.$$

This establishes the proposition. □

Now we prove the main theorem.

*Proof.* Let $n$ be a positive integer and let $\mathcal{A} = (Q, A, E, I, F)$ be a reversible automaton recognizing $L^{2n}$. For each $i \in I$ and $f \in F$ let us denote by $\mathcal{A}_{if}$ the automaton $(Q, A, E, \{i\}, \{f\})$.

It is clear that

$$L(\mathcal{A}) = \bigcup_{i\in I, f\in F} L(\mathcal{A}_{if}). \tag{7}$$

According to formula (1) we have

$$|L(\mathcal{A}_{if})| \leq \frac{1}{2}\left(\sum_{q\in M(\mathcal{A}_{if})} |L_q(\mathcal{A}_{if})|^2 + |R_q(\mathcal{A}_{if})^{\sim}|^2\right). \tag{8}$$

Now the reversible automata $(Q, A, E, \{i\}, M(\mathcal{A}_{if}))$ recognizes a sublanguage of $L^n$. By formula (2), we have

$$\sum_{q\in M(\mathcal{A}_{if})} |L_q(\mathcal{A}_{if})|^2 \leq 8^n.$$

Moreover, let us denote by $E' = \{(p, a_i, q) \mid (q, a_i, p) \in E\}$. The reversible automata $(Q, A, E', \{f\}, M(\mathcal{A}_{if}))$ also recognizes a sublanguage of $L^n$, and by (2), we have

$$\sum_{q\in M(\mathcal{A}_{if})} |R_q(\mathcal{A}_{if})^{\sim}|^2 \leq 8^n.$$

Thus, $|L(\mathcal{A}_{if})| \leq 8^n$ by formula 8.

By formula (7), it comes $|L(\mathcal{A})| \leq |I||F|8^n$. But $|L(\mathcal{A})| = 9^n$, so

$$|I||F| \geq \left(\frac{9}{8}\right)^n.$$

It follows that $I$ or $F$ contain a least $r^n$ elements with $r = \frac{3\sqrt{2}}{4}$. Since $I$ and $F$ are subsets of $Q$ we have $|Q| \geq r^n$.

We have proved that $c(L^{2n}) \geq r^n$. But $m(L^{2n}) = 6n + 1$. Hence $R(6n + 1) = \Omega(r^n)$ and, since $R(n)$ is an increasing function of $n$, this proves the theorem with $\rho = r^{\frac{1}{6}}$. □

The author wishes to express his thanks to J.-E Pin for suggesting the problem, to C. Nicaud for helpful comments, and to P. Weil for his active interest in the preparation of this paper.

## References


[1] D. Angluin, Inference of reversible languages. *J. ACM* **29** (1982) 741-765.

[2] J. Berstel, *Transductions and Context-Free Languages*. B.G. Teubner, Stuttgart (1979).

[3] M. Hall, A topology for free groups and related groups. *Ann. of Math.* **52** (1950).

[4] T. Hall, Biprefix codes, inverse semigroups and syntactic monoids of injective automata. *Theoret. Comput. Sci.* **32** (1984) 201-213.

[5] B. Herwig and D. Lascar, Extending partial automorphisms and the profinite topology on free groups. *Trans. Amer. Math. Soc.* **352** (2000) 1985-2021.

[6] S. Margolis and J. Meakin, Inverse monoids, trees, and context-free languages. *Trans. Amer. Math. Soc.* **335** (1993) 259-276.

[7] S. Margolis, M. Sapir and P. Weil, *Closed subgroups in pro-***V** *topologies and the extension problem for inverse automata*. Preprint (1998).

[8] S.W Margolis and J.-E. Pin, Languages and inverse semigroups, edited by J. Paredaens, *Automata, Languages and Programming, 11th Colloquium*. Antwerp, Belgium. Springer-Verlag, *Lecture Notes in Comput. Sci.* **172** (1984) 337-346.

[9] J.-E. Pin, Topologies for the free monoid. *J. Algebra* **137** (1991).

[10] J.-E. Pin, On reversible automata, edited by I. Simon, in *Proc. of Latin American Symposium on Theoretical Informatics (LATIN '92)*. Springer, Berlin, *Lecture Notes in Comput. Sci.* **583** (1992) 401-416; A preliminary version appeared in the Proceedings of ICALP'87, *Lecture Notes in Comput. Sci.* **267**.

[11] P.V. Silva, On free inverse monoid languages. *RAIRO: Theoret. Informatics Appl.* **30** (1996) 349-378.

[12] B. Steinberg, *Finite state automata: A geometric approach*. Technical Report, Univ. of Porto (1999).

[13] B. Steinberg, *Inverse automata and profinite topologies on a free group*. Preprint (1999).


Communicated by J.-E. Pin.
Received February 9, 2000. Accepted September 29, 2000.