INFORMATIQUE THÉORIQUE ET APPLICATIONS

F. BLANCHET-SADRI

On semidirect and two-sided semidirect products of finite $\mathcal{J}$-trivial monoids

Informatique théorique et applications, tome 30, nº 5 (1996), p. 457-482

<http://www.numdam.org/item?id=ITA_1996__30_5_457_0>

Informatique théorique et Applications/Theoretical Informatics and Applications
(vol. 30, n° 5, 1996, pp. 457-482)

ON SEMIDIRECT AND TWO-SIDED SEMIDIRECT PRODUCTS OF FINITE $\mathcal{J}$-TRIVIAL MONOIDS (*) (**)

by F. BLANCHET-SADRI ([1])

Communicated by RYTHER

Abstract. – *In this paper, using results of Almeida and Weil, we give criteria for the semidirect or two-sided semidirect product of two locally finite pseudovarieties* $\mathbf{V}$ *and* $\mathbf{W}$ *to satisfy an identity* $u = v$. *We illustrate these criteria with various semidirect and two-sided semidirect products of pseudovarieties of* $\mathcal{J}$*-trivial monoids. In particular, let* $\mathbf{J}_1$ *denote the class of all finite semilattice monoids and let* $\mathbf{W}_i$ *be the sequence of pseudovarieties of monoids defined by* $\mathbf{W}_1 = \mathbf{J}_1$ *and* $\mathbf{W}_{i+1} = \mathbf{J}_1 \star\star \mathbf{W}_i$ *(the two-sided semidirect product of* $\mathbf{J}_1$ *by* $\mathbf{W}_i$*). Each* $\mathbf{W}_k$ *turns out to be perfectly related to the* k*-move standard Ehrenfeucht-Fraïssé game. The union* $\bigcup_{k \geq 1} \mathbf{W}_k$ *is then the class* $\mathbf{A}$ *of all finite aperiodic monoids.*

Résumé. – *Dans cet article, utilisant des résultats d'Almeida et de Weil, nous donnons des critères pour que le produit semidirect ou semidirect bilatère de deux pseudovariétés localement finies* $\mathbf{V}$ *et* $\mathbf{W}$ *satisfasse une identité* $u = v$. *Nous illustrons ces critères avec plusieurs produits semidirects ou semidirects bilatères de pseudovariétés de monoïdes* $\mathcal{J}$*-triviaux. En particulier, soit* $\mathbf{J}_1$ *la classe des demi-treillis finis et soit* $\mathbf{W}_i$ *la suite de pseudovariétés de monoïdes définie par* $\mathbf{W}_1 = \mathbf{J}_1$ *et* $\mathbf{W}_{i+1} = \mathbf{J}_1 \star\star \mathbf{W}_i$ *(le produit semidirect bilatère de* $\mathbf{J}_1$ *par* $\mathbf{W}_i$*). Chaque* $\mathbf{W}_k$ *devient parfaitement lié au jeu standard de Ehrenfeucht-Fraïssé avec* k *tours. L'union* $\bigcup_{k \geq 1} \mathbf{W}_k$ *est alors la classe* $\mathbf{A}$ *des monoïdes apériodiques finis.*

1. INTRODUCTION

Given two pseudovarieties of semigroups $\mathbf{V}$ and $\mathbf{W}$, their semidirect product $\mathbf{V} \star \mathbf{W}$ (respectively two-sided semidirect product $\mathbf{V} \star\star \mathbf{W}$) is defined to be the pseudovariety of semigroups generated by all semidirect (respectively two-sided semidirect) products of the form $S \star T$ (respectively

(*) Received December 1995.
(**) This material is based upon work supported by the National Science Foundation under Grant No. CCR-9300738. A Research Assignment from the University of North Carolina at Greensboro is gratefully acknowledged. Many thanks to the referees of preliminary versions of this paper for their valuable comments and suggestions.

([1]) Department of Mathematical Sciences, University of North Carolina, Greensboro, NC 27412, USA, E-mail: blanchet@iris.uncg.edu

$S \star \star T$) with $S \in \mathbf{V}$ and $T \in \mathbf{W}$. This paper relates to the following two problems:

1. Does a given finite semigroup belong to $\mathbf{V} \star \mathbf{W}$ (respectively $\mathbf{V} \star\star \mathbf{W}$)?
2. Does $\mathbf{V} \star \mathbf{W}$ (respectively $\mathbf{V} \star\star \mathbf{W}$) satisfy a given identity $u = v$?

The knowledge of identities for $\mathbf{V} \star \mathbf{W}$ (respectively $\mathbf{V} \star \star \mathbf{W}$) may help solve the membership problem (1). For instance, if $\mathbf{V} \star \mathbf{W}$ (respectively $\mathbf{V} \star\star \mathbf{W}$) admits a finite basis of identities (or a finite set of generators), then $\mathbf{V} \star \mathbf{W}$ (respectively $\mathbf{V} \star\star \mathbf{W}$) has a decidable membership problem.

Almeida [2, 4] (respectively Almeida and Weil [5]) proposes a new approach to treat problems that ask for algorithms to decide whether a given finite semigroup belongs to the semidirect product $\mathbf{V} \star \mathbf{W}$ (respectively two-sided semidirect product $\mathbf{V} \star \star \mathbf{W}$) of pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ for which such algorithms are known. We illustrate their methods in this paper (and also in the papers [14, 17]). Here, we are converting bases of identities for pseudovarieties of $\mathcal{J}$-trivial monoids into bases of identities for various semidirect and two-sided semidirect products of such pseudovarieties (if S is a monoid and s, $t \in S$, then s is said to be $\mathcal{J}$-below t, written $s \leq_{\mathcal{J}} t$, if $s = xty$ for some x, $y \in S$, and s, t are said to be $\mathcal{J}$-equivalent, written $s \sim_{\mathcal{J}} t$, if $s \leq_{\mathcal{J}} t$ and $t \leq_{\mathcal{J}} s$; S is said to be $\mathcal{J}$-trivial if this equivalence relation is the identity).

Results related to those above include: A result of Albert, Baldinger and Rhodes which implies that the join of two decidable pseudovarieties of semigroups may be undecidable [1], and the authors mention that an analogous result holds with join replaced by semidirect product. The authors establish the existence of two finitely based pseudovarieties of semigroups whose join does not have a decidable membership problem. A result of Irastorza which implies that the semidirect product of two pseudovarieties of semigroups admitting finite bases of identities may be equational without such a basis [25].

1.1 Preliminaries

The reader is referred to the books of Almeida [4], Burris and Sankappanavar [20], Eilenberg [23] or Pin [27] for terminology not defined in this paper.

1.1.1. *Varieties of finite monoids*

A *Semigroup* is a set S together with an associative binary operation (generally denoted multiplicatively). If there is an element 1 of S such that

$1s = s1 = s$ for each $s \in S$, then S is called a *monoid* and 1 is its unit. A subset of S is a *subsemigroup* (respectively *submonoid*) of S if the induced binary operation makes it a semigroup (respectively monoid).

Let S and T be monoids. A monoid *morphism* $\varphi : S \to T$ is a mapping such that $\varphi(ss') = \varphi(s)\varphi(s')$ for all $s, s' \in S$ and $\varphi(1) = 1$. We say that S *divides* T, and write $S < T$, if S is the image by a morphism of a submonoid of T.

Let A be a finite alphabet and let A^* denote the free monoid on the set A (A^+ will denote the free semigroup on A). A^+ is the set of all finite strings (called words) $a_1, \ldots, a_i$ of elements of A and $A^* = A^+ \cup \{1\}$, where 1 is the empty word. The operation in A^* is the concatenation of these words.

A *variety of finite monoids* or *pseudovariety of monoids* is a class of finite monoids closed under morphic images, submonoids and finite direct products (or closed under division and finite direct products). A *variety of monoids* is a class of monoids closed under morphic images, submonoids and direct products. Given a class C of finite monoids, the intersection of all pseudovarieties containing C is still a pseudovariety, called the *pseudovariety generated by* C.

1.1.2. *Varieties of languages*

Let A be a finite alphabet. A *language* on A is a subset L of A^*. A language L in A^* is said to be *recognizable* if there exists a finite monoid S and a morphism $\varphi : A^* \to S$ such that $L = \varphi^{-1}(\varphi(L))$. In that case, we say that S (or φ) *recognizes* L. The notions of recognizable sets (by finite monoids and by finite automata) are equivalent. To each language L, we associate a congruence $\sim_L$ defined, for $u, v \in A^*$, by $u \sim_L v$ if and only if xuy and xvy are both in L or both in $A^* \backslash L$, for all x, y in A^*. The congruence $\sim_L$ is called the *syntactic congruence* of L and the monoid $M(L) = A^*/\sim_L$ is called the *syntactic monoid* of L. A monoid recognizes L if and only if it is divided by $M(L)$.

A $\star$-variety $\mathcal{V}$ is a family $A^*\mathcal{V}$ of classes of recognizable languages of A^* defined for all finite alphabets A and satisfying the following conditions:

- $A^*\mathcal{V}$ is a boolean algebra, that is, if K and L are in $A^*\mathcal{V}$, then so are $K \cup L$, $K \cap L$ and $A^* \backslash L$.
- If $\varphi : A^* \to B^*$ is a morphism and $L \in B^*\mathcal{V}$, then $\varphi^{-1}(L) \in A^*\mathcal{V}$.
- If $L \in A^*\mathcal{V}$ and $a \in A$, then both $\{u \in A^* | au \in L\}$ and $\{u \in A^* | ua \in L\}$ are in $A^*\mathcal{V}$.

Eilenberg [23] proved that pseudovarieties of monoids and $\star$-varieties are in one-to-one correspondence. If $\mathbf{V}$ is a pseudovariety of monoids, then $A^* \mathcal{V} = \{L \subseteq A^* | M(L) \in \mathbf{V}\}$ defines the corresponding $\star$-variety $\mathcal{V}$. If $\mathcal{V}$ is a $\star$-variety, then the pseudovariety generated by $\{M(L) | L \in A^* \mathcal{V}$ for some $A\}$ defines the corresponding pseudovariety $\mathbf{V}$.

Let $\mathbf{V}$ be a pseudovariety generated by the monoids $S_1, \ldots, S_m$. Thus $\mathbf{V}$ is generated by $S = S_1 \times \cdots \times S_m$. Let $\mathcal{V}$ be the $\star$-variety associated to $\mathbf{V}$. Then $A^* \mathcal{V}$ is the boolean closure of the sets $\varphi^{-1}(s)$ for all $s \in S$ and all morphisms $\varphi : A^* \to S$. Consequently, $A^* \mathcal{V}$ is finite.

1.1.3. *Products of varieties of finite monoids*

Let S and T be monoids. By a *left unitary action* of T and S, we mean a monoid morphism φ from T into the monoid of monoid endomorphisms of S with functions written and composed on the left. If we write S additively and let 0 denote its unit, T multiplicatively and let 1 denote its unit, and abbreviate $\varphi(t)(s)$ by ts, the condition that φ is a monoid morphism mean that

- $(tt')s = t(t's)$
- $1s = s$

for all $s \in S$ and $t, t' \in T$, and the condition that $\varphi(t)$ is a monoid endomorphism of S means that $t(s + s') = ts + ts'$ and $t0 = 0$ for all s, $s' \in S$ and $t \in T$. By a *right unitary action of* T on S, we mean a function

$$T \times S \to S$$
$$(t, s) \mapsto st$$

satisfying the following conditions:

- $s(tt') = (st)t'$
- $s1 = s$
- $(s + s')t = st + s't$
- $0t = 0$

for all $s, s' \in S$ and $t, t' \in T$.

Given a left unitary action, we define the associated *semidirect product* $S \star T$ as the monoid with underlying set the cartesian product $S \times T$ and operation defined by

$$(s, t)(s', t') = (s + ts', tt').$$

An easy calculation shows that $S \star T$ is a monoid with unit (0, 1).

Now, given a left and a right unitary actions in such a way that $t(st') = (ts)t'$ for all $s \in S$ and $t, t' \in T$, we define the associated *two-sided semidirect product* $S \star\star T$ as the monoid with underlying set $S \times T$ and operation defined by

$$(s, t)(s', t') = (st' + ts', tt').$$

An easy calculation shows that $S \star\star T$ is a monoid with unit (0, 1). When the right unitary action of T on S is trivial, then $S \star\star T$ is in fact a semidirect product. Two-sided semidirect products were introduced by Rhodes and Tilson [31].

Neither $\star$ nor $\star\star$ is associative on monoids.

Given two pseudovarieties of monoids $\mathbf{V}$ and $\mathbf{W}$, their *semidirect product* $\mathbf{V} \star \mathbf{W}$ (respectively *two-sided semidirect product* $\mathbf{V} \star\star \mathbf{W}$) is defined to be the pseudovariety of monoids generated by all semidirect (respectively two-sided semidirect) products $S \star T$ (respectively $S \star\star T$) with $S \in \mathbf{V}$ and $T \in \mathbf{W}$. The operation $\star$ on pseudovarieties is associative and commutes with directed unions [4]. The operation $\star\star$ on pseudovarieties is not associative. We will represent by $\mathbf{V}^i$ the semidirect product of i copies of the pseudovariety $\mathbf{V}$.

For a pseudovariety $\mathbf{V}$ of monoids, we will denote by $F_A(\mathbf{V})$ the free object on the set A in the variety generated by $\mathbf{V}$. The following lemmas are representations of $F_A(\mathbf{V} \star \mathbf{W})$ and $F_A(\mathbf{V} \star\star \mathbf{W})$ as submonoids of $F_B(\mathbf{V}) \star F_A(\mathbf{W})$ and $F_B(\mathbf{V}) \star\star F_A(\mathbf{W})$ respectively (where B is an appropriate set) (these lemmas apply more generally [4, 5]).

LEMMA 1.1 (Almeida [2]): *Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of monoids that admit finite free objects on finite sets. Then so does the pseudovariety $\mathbf{V} \star \mathbf{W}$.*

Moreover, for a finite set A, let $T = F_A(\mathbf{W})$ and $S = F_B(\mathbf{V})$ where $B = T \times A$. There is an embedding of $F_A(\mathbf{V} \star \mathbf{W})$ into $S \star T$ defined by $a \mapsto ((1, a), a)$, where the left unitary action of T on S is given by $t(t', a) = (tt', a)$ for $t, t' \in T$ and $a \in A$.

LEMMA 1.2 (Almeida and Weil [5]): *Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of monoids that admit finite free objects on finite sets. Then so does the pseudovariety $\mathbf{V} \star\star \mathbf{W}$.*

Moreover, for a finite set A, let $T = F_A(\mathbf{W})$ and $S = F_B(\mathbf{V})$ where $B = T \times A \times T$. There is an embedding of $F_A(\mathbf{V} \star\star \mathbf{W})$ into $S \star\star T$ defined by $a \mapsto ((1, a, 1), a)$, where the left unitary action of T on S is given by $t(t_1, a, t_2) = (tt_1, a, t_2)$ and the right unitary action by $(t_1, a, t_2)t = (t_1, a, t_2 t)$ for $t, t_1, t_2 \in T$ and $a \in A$.

1.1.4. *Identities and varieties of finite monoids*

We end this section with a few more definitions and notations. Let A be a set. A monoid identity is an expression $u = v$ with $u, v \in A^*$. The identity $u = v$ is said to *hold* in a monoid S (or S *satisfies* $u = v$) and we write $S \models u = v$ if, for every morphism $\varphi : A^* \to S$, we have $\varphi(u) = \varphi(v)$. A monoid S satisfies a set of identities $\mathcal{E}$ ($S \models \mathcal{E}$) if $S \models e$ for every $e \in \mathcal{E}$. We write $\mathbf{V} \models u = v$ if for every $S \in \mathbf{V}$ we have $S \models u = v$. An identity $u = v$ is *deducible* from a set $\mathcal{E}$ of monoid identities and we write $\mathcal{E} \models u = v$ if, there exist words $w_0, w_1, \ldots, w_\ell \in A^*$ with $u = w_0$, $v = w_\ell$, and there exist words $a_i, b_i \in A^*$, $u_i, v_i \in A^*$, and a morphism $\varphi_i : A^* \to A^*$ such that $w_i = a_i \varphi_i(u_i) b_i$, $w_{i+1} = a_i \varphi_i(v_i) b_i$, and $u_i = v_i \in \mathcal{E}$ or $v_i = u_i \in \mathcal{E}$ for every $0 \leq i < \ell$.

Given a set $\mathcal{E}$ of monoid identities, the class of all finite monoids that satisfy every identity in $\mathcal{E}$ is a pseudovariety $\mathbf{V}(\mathcal{E})$ that is said to be *defined by* $\mathcal{E}$. The set $\mathcal{E}$ is also said to be a *basis (of monoid identities)* for $\mathbf{V}(\mathcal{E})$. Pseudovarieties are ultimately defined by sequences of identities (that is, a monoid belongs to the given pseudovariety if and only if it satisfies all but finitely many of the identities in the sequence), and finitely generated pseudovarieties are defined by sequences of identities (that is, a monoid belongs to the given pseudovariety if and only if it satisfies all the identities in the sequence) [24].

1.2. Games and aperiodic monoids

Let A be a finite alphabet. The set $A^* \mathcal{V}_0 = \{\emptyset, A^*\}$ constitutes level 0 of Straubing's hierarchy of star-free languages on A. The set $A^* \mathcal{V}_{k+1}$ which constitutes level $k+1$ of the hierarchy is then defined as the boolean algebra generated by the languages of the form $L_0 a_1 L_1 \ldots a_i L_i$ where $i \geq 0$, $L_0, \ldots, L_i \in A^* \mathcal{V}_k$ and $a_1, \ldots, a_i \in A$. We are led to $\star$-varieties of languages $\mathcal{V}_k$ for every $k \geq 0$. We will denote by $\mathbf{V}_k$ the pseudovariety of monoids corresponding to $\mathcal{V}_k$. In particular, $\mathbf{V}_0$ is the trivial pseudovariety of monoid $\mathbf{I}$. Sraubing's hierarchy which was defined in [35] is related to Brzozowski's dot-depth hierarchy defined in [21]. Straubing's hierarchy is strict [19, 38] and $\bigcup_{k \geq 0} \mathbf{V}_k$ is the pseudovariety of aperiodic monoids $\mathbf{A}$.

Each level of the hierarchy $A^* \mathcal{V}_1, A^* \mathcal{V}_2, \ldots$ contains a subhierarchy that can be defined in the following way. For every $m \geq 1$, we define $A^* \mathcal{V}_{k+1,m}$ as the boolean algebra generated by the languages of the form $L_0 a_1 L_1 \ldots a_i L_i$ where $0 \leq i \leq m$, $L_0, \ldots, L_i \in A^* \mathcal{V}_k$ and $a_1, \ldots, a_i \in A$. We have then $A^* \mathcal{V}_k = \bigcup_{m \geq 1} A^* \mathcal{V}_{k,m}$. We are led to

$\star$-varieties of languages $\mathcal{V}_{k,m}$ for every k, $m \geq 1$. We will denote by $\mathbf{V}_{k,m}$ the pseudovariety of monoids corresponding to $\mathcal{V}_{k,m}$.

The set $A^* \mathcal{V}_1$ is the boolean algebra generated by the languages of the form $A^* a_1 A^* \ldots a_i A^*$ where $i \geq 0$ and $a_1, \ldots, a_i \in A$, and hence $\mathcal{V}_1$ is the $\star$-variety of piecewise testable languages. From a result of Simon [32, 33], $\mathbf{V}_1$ is the pseudovariety of $\mathcal{J}$-trivial monoids $\mathbf{J}$. We then have an algorithm to test whether a recognizable language is of level 1 in Straubing's hierarchy.

For each integer $m \geq 1$ and each $u \in A^*$, we define $\alpha_m(u)$ to be the set of all the subwords of u of length less than or equal to m (a word $a_1 \ldots a_i \in A^*$ is a subword of a word $v \in A^*$ if there exist words $v_0, \ldots, v_i \in A^*$ such that $v = v_0 a_1 v_1 \ldots a_i v_i$). We consider the equivalence relation α_m on A^* defined by $u \, \alpha_m \, v$ if $\alpha_m(u) = \alpha_m(v)$. We will abbreviate $\alpha_1(u)$ by $\alpha(u)$ the set of letters that occur in u. Note that α_m is a congruence of finite index on A^*. By definition, a language is piecewise testable if and only if it is the union of classes modulo α_m for some m. More precisely, a language is in $A^* \mathcal{V}_{1,m}$ if and only if it is the union of classes modulo α_m. We will also denote $\mathbf{V}_{1,m}$ by $\mathbf{J}_m$.

We proceed with a generalization of α_m related to an Ehrenfeucht-Fraïssé game. We identify each $u \in A^*$ with a word model $u = (\{1, \ldots, |u|\}, <^u, (R_a^u)_{a \in A})$where the universe $\{1, \ldots, |u|\}$ represents the set of positions of letters in the word u, $<^u$ denotes the usual order relation on $\{1, \ldots, |u|\}$, and R_a^u are unary relations on $\{1, \ldots, |u|\}$ containing the positions with letter a, for each $a \in A$. The game $G_{\bar{m}}(u, v)$, where $\bar{m} = (m_1, \ldots, m_k)$ is a k-tuple of positive integers and $u, v \in A^*$, is played between two players I and II on the word models u and v. A play of the game consists of k moves. In the ith move, Player I chooses, in u or in v, a sequence of m_i positions; then, Player II chooses, in the remaining word (v or u), also a sequence of m_i positions. Before each move, Player I has to decide whether to choose his next elements from u or from v. After k moves, by concatenating the position sequences chosen from u and from v, two sequences $p_1, \ldots, p_n$ from u and $q_1, \ldots, q_n$ from v have been formed where $n = m_1 + \cdots + m_k$. Player II has won the play if the following two conditions are satisfied:

1. $p_i <^u p_j$ if and only if $q_i <^u q_j$ for all $1 \leq i, j \leq n$.
2. $R_a^u p_i$ if and only if $R_a^u q_i$ for all $1 \leq i, j \leq n$ and $a \in A$.

Equivalently, the two subwords in u and v given by the position sequences $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ should coincide. If there is a winning strategy for Player II in the game to win each play we say that Player II wins

$G_{\bar{m}}(u, v)$ and write $u\,\alpha_{\bar{m}}\,v$. The special case $G_{\bar{1}_k}(u, v)$ where $\bar{1}_k$ denotes the k-tuple of 1's is the standard k-move Ehrenfeucht-Fraïssé game [22]. The equivalence relation $\alpha_{\bar{m}}$ naturally defines a congruence on A^*. For fixed $\bar{m}$, we define the pseudovariety $\mathbf{V}_{\bar{m}}$ as follows: an A-generated monoid S is in $\mathbf{V}_{\bar{m}}$ if and only if S is a morphic image of $A^*/\alpha_{\bar{m}}$. It is known that each $\mathbf{V}_{\bar{m}}$ is decidable [15]. Note that the equalities $\alpha_{(m)} = \alpha_m$ and $\mathbf{V}_{(m)} = \mathbf{J}_m$ hold. The pseudovariety $\mathbf{V}_k$ (respectively $\mathbf{V}_{k,m}$) turns out to be the union $\bigcup_{(m_1, \ldots, m_k)} \mathbf{V}_{(m_1, \ldots, m_k)}$ (respectively $\bigcup_{(m, m_1, \ldots, m_{k-1})} \mathbf{V}_{(m, m_1, \ldots, m_{k-1})}$ [37, 38, 26]. If $\bar{m} = (m_1, \ldots, m_k)$, then $(m, \bar{m})$ will denote $(m, m_1, \ldots, m_k)$.

1.3. Identities and aperiodic monoids

Blanchet-Sadri [11, 12] describes a simple basis of identities $\mathcal{A}_m$ for $\mathbf{J}_m$. Let $m \geq 1$ and let $\mathcal{X}$ be a countable set of variables $x_1, x_2, x_3, \ldots$ Letting $x = x_1$, the basis $\mathcal{A}_m$ consists of the following type of identities on $\mathcal{X}^+$:

$$u_i \ldots u_1\, x v_1 \ldots v_j = u_i \ldots u_1\, v_1 \ldots v_j \tag{1}$$

where $\{x\} \subseteq \alpha(u_1) \subseteq \cdots \subseteq \alpha(u_i)$ and $\{x\} \subseteq \alpha(v_1) \subseteq \cdots \subseteq \alpha(v_j)$, and where $i + j = m$. The basis $\mathcal{A}_1$ is equivalent to the identities $xy = yx$ and $x^2 = x$, $\mathcal{A}_2$ to $(xy)^2 = (yx)^2$ and $xyxzx = xyzx$, and $\mathcal{A}_3$ to $(xy)^3 = (yx)^3$, $xyxzxuxvz = xyzxuxvz$ and $zvxuxzxyx = zvxuxzyx$. The pseudovarieties $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$ are hence finitely based. However, for every $m \geq 4$, the pseudovariety $\mathbf{J}_m$ is not. Also, in [12] we show that $\mathbf{V}_{2,1}$ is ultimately defined by the following two types of identities on $\mathcal{X}^+$ ($x = x_1$ and $y = x_2$):

$$u_i \ldots u_1\, x^2\, v_1 \ldots v_i = u_i \ldots u_1\, x\, v_1 \ldots v_i$$

where $\{x\} \subseteq \alpha(u_1) \subseteq \cdots \subseteq \alpha(u_i)$ and $\{x\} \subseteq \alpha(v_1) \subseteq \cdots \subseteq \alpha(v_i)$ and

$$u_i \ldots u_1\, xy\, v_1 \ldots v_i = u_i \ldots u_1\, yx\, v_1 \ldots v_i$$

where $\{x, y\} \subseteq \alpha(u_1) \subseteq \cdots \subseteq \alpha(u_i)$ and $\{x, y\} \subseteq \alpha(v_1) \subseteq \cdots \subseteq \alpha(v_j)$, and where $i \geq 1$.

Almeida [3] gives a basis of identities $\mathcal{B}_m$ for $\mathbf{J}_1^{m+1}$ which we now describe. Let $m \geq 1$. Letting $x = x_1$ and $y = x_2$, the basis $\mathcal{B}_m$ consists of the two following types of identities on $\mathcal{X}^+$:

$$u_m \ldots u_1\, x^2 = u_m \ldots u_1\, x,$$

$$v_m \ldots v_1\, xy = v_m \ldots v_1\, yx$$

where $\{x\} \subseteq \alpha(u_1) \subseteq \cdots \subseteq \alpha(u_m)$ and $\{x, y\} \subseteq \alpha(v_1) \subseteq \cdots \subseteq \alpha(v_m)$. There, he also shows that for every $m \geq 3$, the pseudovariety $\mathbf{J}_1^m$ is not finitely based. Almeida's basis $\mathcal{B}_1$ is equivalent to the identities $xux^2 = xux$ and $xuyvxy = xuyvyx$ (previously shown in [29] to describe $\mathbf{J}_1^2$). It is known that $\bigcup_{m\geq 1} \mathbf{J}_1^m$ is the pseudovariety $\mathbf{R}$ of all $\mathcal{R}$-trivial monoids [34] and that each $\mathbf{J}_1^m$ is decidable [28].

In this paper, we discuss a technique to produce indentities for the semidirect or two-sided semidirect product of two locally finite pseudovarieties $\mathbf{V}$ and $\mathbf{W}$. In this case both $\mathbf{V}$ and $\mathbf{W}$ have finite free objects on finite alphabets.

The notion of congruence plays a central role in our approach. For any finite alphabet A, we say that a monoid S is *A-generated* if there exists a congruence γ on A^* such that S is isomorphic to A^*/γ. A pseudovariety of monoids $\mathbf{V}$ is *locally finite* if for any A there are finitely many A-generated monoids in $\mathbf{V}$. Equivalently, there exists for each A a congruence γ_A of finite index such that an A-generated monoid S is in $\mathbf{V}$ if and only if S is a morphic image of A^*/γ_A.

Let $\mathbf{V}$ and $\mathbf{W}$ be two locally finite pseudovarieties of monoids. Let γ be the congruence generating $\mathbf{W}$ for the finite alphabet A and let β be the congruence generating $\mathbf{V}$ for the finite alphabet $F_A(\mathbf{W}) \times A(F_A(\mathbf{W}))$ is isomorphic to the quotient A^*/γ). The idea is to associate with $\mathbf{V} \star \mathbf{W}$ a congruence $\sim_{\beta,\gamma}$ on A^*. Section 2 gives a criterion to determine when an identity on A holds in $\mathbf{V}\star\mathbf{W}$ with the help of $\sim_{\beta,\gamma}$. This leads to a proof that such $\mathbf{V} \star \mathbf{W}$ are locally finite and hence decidable. The essential ingredient in our proof is a semidirect product representation of the free objects in $\mathbf{V} \star \mathbf{W}$ due to Almeida [2]. If β denotes instead the congruence generating $\mathbf{V}$ for the finite alphabet $F_A(\mathbf{W}) \times A \times F_A(\mathbf{W})$, we can associate with $\mathbf{V} \star\star \mathbf{W}$ a congruence $\approx_{\beta,\gamma}$ on A^* and obtained similar results by applying a result of Almeida and Weil [5].

In Section 3, further exploration of the basic criteria of Section 2 leads to bases of identities for the products $\mathbf{V} \star \mathbf{J}_m$ (Section 3.1) and $\mathbf{V} \star\star \mathbf{J}_m$ (Section 3.2) where $\mathbf{V}$ denotes a locally finite pseudovariety of monoids whose generating congruence is included in α_1. Case studies are then proposed. We study semidirect products of the form $\mathbf{J}_{m_1} \star \ldots \star \mathbf{J}_{m_k}$ (Section 3.1.1-3.1.2) and $(\mathbf{J}_1 \star \mathbf{J}_{m_1})^{\varrho} \star \mathbf{J}_{m_2}$ (Section 3.1.3) where $\mathbf{V}^{\varrho}$ denotes the reversal of $\mathbf{V}$. A simple basis of identities is described for each of these semidirect products. Our results imply the relations $\mathbf{J}_1 \star \mathbf{J}_m = \mathbf{J}_1^{m+1}$ and $\mathbf{J}_{m_1} \star \ldots \star \mathbf{J}_{m_k} = \mathbf{J}_{m_1} \star \mathbf{J}_{m_2+\cdots+m_k}$. We also study two-sided semidirect products of the form $\mathbf{J}_{m_1} \star\star \mathbf{J}_{m_2}$ (Section 3.2.1) and some iterated two-sided semidirect products of $\mathbf{J}_1$ (Section 3.2.2). We give a basis of identifies for $\mathbf{J}_{m_1} \star\star \mathbf{J}_{m_2}$ and for each $\mathbf{W}_i$ where $\mathbf{W}_1 = \mathbf{J}_1$ and $\mathbf{W}_{i+1} = \mathbf{W}_i \star\star \mathbf{J}_1$. Our results imply that the k-more standard Ehrenfeucht-Fraïssé game is perfectly related to $\mathbf{W}'_k$ where $\mathbf{W}'_1 = \mathbf{J}_1$ and for all $i \geq 1$, $\mathbf{W}'_{i+1} = \mathbf{J}_1 \star\star \mathbf{W}'_i$ (we have $\mathbf{W}'_k = \mathbf{V}_{\bar{1}k}$). Our results also imply the relations $\mathbf{J}_1 \star\star \mathbf{V}_{\bar{m}} = \mathbf{V}_{(1,\bar{m})}$ and $\mathbf{A} = \bigcup_{k \geq 1} \mathbf{W}'_k$.

2. IDENTITY CRITERIA FOR SEMIDIRECT PRODUCTS OF LOCALLY FINITE PSEUDOVARIETIES

In this section, we give criteria to determine when an identity is satisfied in the semidirect or two-sided semidirect product of two locally finite pseudovarieties of monoids.

2.1. Preliminaries on locally finite pseudovarieties

Let A be a finite alphabet. Let $\mathbf{W}$ be a locally finite pseudovariety of monoids and let γ_A be the congruence of finite index on A^* such that an A-generated monoid S belongs to $\mathbf{W}$ if and only if S is a morphic image of A^*/γ_A. The pseudovariety $\mathbf{W}$ admits finite free objects on finite sets. Let π_{γ_A} from A^* into $F_A(\mathbf{W})$ be the canonical projection that maps a to the generator a of $F_A(\mathbf{W})$. If $u, v \in A^*$, then $\pi_{\gamma_A}(u) = \pi_{\gamma_A}(v)$ if and only if $u \,\gamma_A\, v$.

DEFINITION 2.1: *Let A be a finite alphabet. Let $u = a_1 \ldots a_i \in A^*$.* We write $\sigma_{\gamma_A}(u)$ *for the word*

$$(1, a_1)(\pi_{\gamma_A}(a_1), a_2) \ldots (\pi_{\gamma_A}(a_1 \ldots a_{i-1}), a_i)$$

on the alphabet $B = F_A(\mathbf{W}) \times A$. Also, if $w \in A^*$, we write $\sigma^w_{\gamma_A}(u)$ for the word

$$(\pi_{\gamma_A}(w), a_1)(\pi_{\gamma_A}(wa_1), a_2) \ldots (\pi_{\gamma_A}(wa_1 \ldots a_{i-1}), a_i).$$

DEFINITION 2.2: *Let A be a finite alphabet. Let $u = a_1 \ldots a_i \in A^*$. We write $\tau_{\gamma_A}(u)$ for the word*

$$(1, a_1, \pi_{\gamma_A}(a_2 \ldots a_i))(\pi_{\gamma_A}(a_1), a_2, \pi_{\gamma_A}(a_3 \ldots a_i)) \ldots (\pi_{\gamma_A}(a_1 \ldots a_{i-1}), a_i, 1)$$

on the alphabet $B = F_A(\mathbf{W}) \times A \times F_A(\mathbf{W})$. *Also, if* $w, w' \in A^*$, *we write* $\tau_{\gamma_A}^{w, w'}(u)$ *for the word*

$$(\pi_{\gamma_A}(w), a_1, \pi_{\gamma_A}(a_2 \ldots a_i w'))(\pi_{\gamma_A}(wa_1), a_2, \pi_{\gamma_A}(a_3 \ldots a_i w')) \ldots (\pi_{\gamma_A}(wa_1 \ldots a_{i-1}), a_i, \pi_{\gamma_A}(w')).$$

2.2. On semidirect products of two locally finite pseudovarieties V and W

Fix two locally finite pseudovarieties of monoids $\mathbf{V}$ and $\mathbf{W}$. Let β_A (respectively γ_A) be the congruence of finite index generating $\mathbf{V}$ (respectively $\mathbf{W}$) for the finite alphabet A.

2.2.1. *The case* $\mathbf{V} \star \mathbf{W}$

Let A be a finite alphabet and let $B = F_A(\mathbf{W}) \times A$. If $u, v \in A^*$, we write $u \sim_{\beta_B, \gamma_A} v$ for $\sigma_{\gamma_A}(u) \beta_B \sigma_{\gamma_A}(v)$ and $u \gamma_A v$.

LEMMA 2.1: *The equivalence relation* $\sim_{\beta_B, \gamma_A}$ *is a congruence of finite index on* A^*.

Proof: We will abbreviate β_B by β and γ_A by γ throughout the proof. Assume $u \sim_{\beta, \gamma} v$ and $u' \sim_{\beta, \gamma} v'$. We have

$$\sigma_\gamma(u) \beta \sigma_\gamma(v) \qquad \text{and} \qquad u \gamma v$$

and similarly with u and v replaced by u' and v'. Since γ is a congruence we have $uu' \gamma vv'$. The above, the fact that $\pi_\gamma(u) = \pi_\gamma(v)$, and the fact that β is a congruence imply that

$$\sigma_\gamma(uu') = \sigma_\gamma(u) \sigma_\gamma^u(u') = \sigma_\gamma(u) \sigma_\gamma^v(u') \beta \sigma_\gamma(v) \sigma_\gamma^v(v') = \sigma_\gamma(vv').$$

Thus $uu' \sim_{\beta, \gamma} vv'$ showing that $\sim_{\beta, \gamma}$ is a congruence. This obviously is a congruence of finite index since β and γ are. □

The following lemma provides an identity criterion for $\mathbf{V} \star \mathbf{W}$.

LEMMA 2.2: *Let* A *be a finite alphabet, let* $B = F_A(\mathbf{W}) \times A$ *and let* $u, v \in A^*$. *We have*

$$\mathbf{V} \star \mathbf{W} \ \textit{satisfies} \ u = v \ \textit{if and only if} \ u \sim_{\beta_B, \gamma_A} v.$$

Consequently, an A*-generated monoid* S *belongs to* $\mathbf{V} \star \mathbf{W}$ *if and only if* S *is a morphic image of* $A^* / \sim_{\beta_B, \gamma_A}$

Proof: We will abbreviate β_B by β and γ_A by γ throughout the proof. Let $u = v$ be an identity on A. Then $u = v$ holds in $\mathbf{V} \star \mathbf{W}$ if and only if

u and v represent the same element of $F_A(\mathbf{V} \star \mathbf{W})$. By Lemma 1.1, this is equivalent to u and v having the same image under the embedding of $F_A(\mathbf{V} \star \mathbf{W})$ into $F_B(\mathbf{V}) \star F_A(\mathbf{W})$ defined by $a \mapsto ((1, a), a)$, where the left unitary action of $F_A(\mathbf{W})$ on $F_B(\mathbf{V})$ is given by $t(t', a) = (tt', a)$ for $t, t' \in F_A(\mathbf{W})$ and $a \in A$.

Let $u = a_1 \ldots a_i$ and $v = b_1 \ldots b_j$. Then, u is mapped to

$$((1, a_1) + (a_1, a_2) + \cdots + (a_1 \ldots a_{i-1}, a_i), a_1 \ldots a_i), \tag{2}$$

and v to

$$((1, b_1) + (b_1, b_2) + \cdots + (b_1 \ldots b_{j-1}, b_j), b_1 \ldots b_j), \tag{3}$$

(here, $F_B(\mathbf{V})$ is written additively). The identity $u = v$ holds in $\mathbf{V} \star \mathbf{W}$ if and only if corresponding components of the pairs (2) and (3) coincide. The condition "the first components of (2) and (3) coincide" is equivalent to $\sigma_\gamma(u)\,\beta\sigma_\gamma(v)$, and the condition "the second components of (2) and (3) coincide" is equivalent to $u\,\gamma\,v$. □

COROLLARY 2.1: *If A is a finite alphabet and if $\mathbf{V}$ and $\mathbf{W}$ are two locally finite pseudovarieties of monoids, then $\mathbf{V} \star \mathbf{W}$ is locally finite and it is decidable in polynomial time whether a finite A-generated monoid belongs to $\mathbf{V} \star \mathbf{W}$.*

Proof: Let A be a finite alphabet. A finite A-generated monoid S belongs to $\mathbf{V} \star \mathbf{W}$ if and only if S is a morphic image of $F_A(\mathbf{V} \star \mathbf{W})$ (which is isomorphic to $A^*/\sim_{\beta_B, \gamma_A}$ and hence finite). This is equivalent to saying that S satisfies all the identities of $F_A(\mathbf{V} \star \mathbf{W})$ in $|A|$ variables. But, by a theorem of Birkhoff (*see* [20]), this set of identities is finitely based and so there is a polynomial time algorithm to decide whether S belongs to $\mathbf{V} \star \mathbf{W}$. □

2.2.2. *The case* $\mathbf{V} \star\star \mathbf{W}$

Let A be a finite alphabet and let $B = F_A(\mathbf{W}) \times A \times F_A(\mathbf{W})$. If u, $v \in A^*$, we write $u \approx_{\beta_B, \gamma_A} v$ for $\tau_{\gamma_A}(u)\,\beta_B\,\tau_{\gamma_A}(v)$ and $u_{\gamma_A}\,v$.

LEMMA 23: *The equivalence relation $\approx_{\beta_B, \gamma_A}$ is a congruence of finite index on A^*.*

Proof: We will abbreviate β_B by β and γ_A by γ throughout the proof. Assume $u \approx_{\beta, \gamma} v$ and $u' \approx_{\beta, \gamma} v'$. We have

$$\tau_\gamma(u)\,\beta\tau_\gamma(v) \quad \text{and} \quad u\,\gamma\,v$$

and similarly with u and v replaced by u' and v'. Since γ is a congruence we have $uu' \gamma vv'$. The above, the facts that $\pi_\gamma(u) = \pi_\gamma(v)$ and $\pi_\gamma(u') = \pi_\gamma(v')$, and the fact that β is a congruence imply that

$$\begin{aligned}\tau_\gamma(uu') &= \tau_\gamma^{1, u'}(u)\, \tau_\gamma^{u, 1}(u') \\ &= \tau_\gamma^{1, v'}(u)\, \tau_\gamma^{v, 1}(u')\, \beta\, \tau_\gamma^{1, v'}(v)\, \tau_\gamma^{v, 1}(v') = \tau_\gamma(vv').\end{aligned}$$

Thus $uu' \approx_{\beta, \gamma} vv'$ showing that $\approx_{\beta, \gamma}$ is a congruence. This obviously is a congruence of finite index since β and γ are. □

We end this section by giving an identity criterion for $\mathbf{V} \ast\ast \mathbf{W}$.

LEMMA 24: *Let A be a finite alphabet, let $B = F_A(\mathbf{W}) \times A \times F_A(\mathbf{W})$ and let $u, v \in A^*$. We have*

$$\mathbf{V} \ast\ast \mathbf{W} \ satisfies\ u = v\ if\ and\ only\ if\ u \approx_{\beta_B, \gamma_A} v.$$

Consequently, an *A-generated monoid S belongs to $\mathbf{V} \ast\ast \mathbf{W}$ if and only if S is a morphic image of $A^* / \approx_{\beta_B, \gamma_A}$.*

Proof: We will abbreviate β_B by β and γ_A by γ throughout the proof. Let $u = v$ be an identity on A. Then $u = v$ holds in $\mathbf{V} \ast\ast \mathbf{W}$ if and only if u and v represent the same element of $F_A(\mathbf{V} \ast\ast \mathbf{W})$. By Lemma 1.2, this is equivalent to u and v having the same image under the embedding of $F_A(\mathbf{V} \ast\ast \mathbf{W})$ into $F_B(\mathbf{V}) \ast\ast F_A(\mathbf{W})$ defined by $a \mapsto ((1, a, 1), a)$, where the left unitary action of $F_A(\mathbf{W})$ on $F_B(\mathbf{V})$ is given by $t(t_1, a, t_2) = (tt_1, a, t_2)$ and the right unitary action by $(t_1, a, t_2)t = (t_1, a, t_2 t)$ for $t, t_1, t_2 \in F_A(\mathbf{W})$ and $a \in A$.

Let $u = a_1 \ldots a_i$ and $v = b_1 \ldots b_j$. Then, u is mapped to

$$((1, a_1, a_2 \ldots a_i) + (a_1, a_2, a_3 \ldots a_i) + \cdots + (a_1 \ldots a_{i-1}, a_i, 1), a_1 \ldots a_i), \tag{4}$$

and v to

$$((1, b_1, b_2 \ldots b_j) + (b_1, b_2, b_3 \ldots b_j) + \cdots + (b_1 \ldots b_{j-1}, b_j, 1), b_1 \ldots b_j), \tag{5}$$

(here, $F_B(\mathbf{V})$ is written additively). The identity $u = v$ holds in $\mathbf{V} \ast\ast \mathbf{W}$ if and only if corresponding components of the pairs (4) and (5) coincide. The condition "the first components of (4) and (5) coincide" is equivalent to $\tau_\gamma(u)\, \beta \tau_\gamma(v)$, and the condition "the second components of (4) and (5) coincide" is equivalent to $u \gamma v$.

COROLLARY 2.2: *If A is a finite alphabet and if $\mathbf{V}$ and $\mathbf{W}$ are two locally finite pseudovarieties of monoids, then $\mathbf{V} \star\star \mathbf{W}$ is locally finite and it is decidable in polynomial time whether a finite A-generated monoid belongs to $\mathbf{V} \star \star \mathbf{W}$*

Proof: The proof is similar to that of Corollary 2.1.

3. ON SEMIDIRECT PRODUCTS OF A LOCALLY FINITE PSEUDOVARIETY V BY $\mathbf{J}_m$

Fix a locally finite pseudovariety of monoids $\mathbf{V}$ and let β_A be the congruence of finite index generating $\mathbf{V}$ for the finite alphabet A. Here we assume that $\beta_A \subseteq \alpha_1$. In Section 3.1, we give a basis of identities for $\mathbf{V} \star \mathbf{J}_m$ and in Section 3.2, a basis for $\mathbf{V} \star \star \mathbf{J}_m$.

We will need the following properties of the congruence α_m or $\alpha_{\bar{m}}$ repeatedly.

LEMMA 3.1 (Simon [33]): *Let $m \geq 1$. Let A be a finite alphabet and let $u, v \in A^*$. We have $u \, \alpha_m \, uv$ (respectively $u \, \alpha_m \, vu$) if and only if there exist words $u_1, \ldots, u_m$ such that $u = u_m \ldots u_1$ (respectively $u = u_1 \ldots u_m$) and $\alpha(v) \subseteq \alpha(u_1) \subseteq \cdots \subseteq \alpha(u_m)$.*

LEMMA 3.2: *Let $m \geq 1$. Let A be a finite alphabet and let $u, v \in A^*$. If $\sigma_{\alpha_m}(u) \, \alpha_1 \, \sigma_{\alpha_m}(v)$, then $u \, \alpha_m \, v$.*

Proof: Put $u = a_1 \ldots a_i$, $v = b_1 \ldots b_j$. Since $\sigma_{\alpha_m}(u) \, \alpha_1 \, \sigma_{\alpha_m}(v)$, the letter $(\pi_{\alpha_m}(a_1 \ldots a_{i-1}), a_i)$ which is in $\sigma_{\alpha_m}(u)$ is also in $\sigma_{\alpha_m}(v)$, and the letter $(\pi_{\alpha_m}(b_1 \ldots b_{j-1}), b_j)$ which is in $\sigma_{\alpha_m}(v)$ is also in $\sigma_{\alpha_m}(u)$. So there exist $1 \leq k \leq i$ and $1 \leq \ell \leq j$ satisfying

$$(\pi_{\alpha_m}(a_1 \ldots a_{i-1}), a_i) = (\pi_{\alpha_m}(b_1 \ldots b_{\ell-1}), b_\ell),$$

$$(\pi_{\alpha_m}(b_1 \ldots b_{j-1}), b_j) = (\pi_{\alpha_m}(a_1 \ldots a_{k-1}), a_k).$$

We conclude that $\alpha_m(u) = \alpha_m(a_1 \ldots a_i) = \alpha_m(b_1 \ldots b_\ell) \subseteq \alpha_m(v)$ and $\alpha_m(v) \subseteq \alpha_m(u)$ follows similarly. □

LEMMA 3.3: *Let $k \geq 1$ and let $\bar{m}$ be a k-tuple of positive integers. Let A be a finite alphabet and let $u, v \in A^*$. If $\tau_{\alpha_{\bar{m}}}(u) \, \alpha_1 \, \tau_{\alpha_{\bar{m}}}(v)$, then $u \, \alpha_{(1, \bar{m})} \, v$ and therefore $u \, \alpha_{\bar{m}} \, v$.*

Proof: The condition $\tau_{\alpha_{\bar{m}}}(u) \, \alpha_1 \, \tau_{\alpha_{\bar{m}}}(v)$ is equivalent to $u \, \alpha_{(1, \bar{m})} \, v$. □

We will also need the following property of the congruence $\alpha_{(1,m)}$. If $u = a_1 \ldots a_n$ is a word on A and $1 \le i \le j \le n$, then $u[i, j]$, $u(i, j)$, $u(i, j]$ and $u[i, j)$ denote the segments $a_i \ldots a_j$, $a_{i+1} \ldots a_{j-1}$, $a_{i+1} \ldots a_j$, and $a_i \ldots a_{j-1}$ respectively ($u[i, i)$ denotes the empty word).

Given a finite alphabet A and a word $u \in A^+$, the (m) first positions in u are defined as follows: Let u_1 denote the smallest prefix of u such that $\alpha(u_1) = \alpha(u)$; call p_1 the last position of u_1. Then, let u_2 be the smallest prefix of $u(p_1, |u|]$ such that $\alpha(u_2) = \alpha(u(p_1, |u|])$; call p_2 the last position of u_2 if $u(p_1, |u|]$ is nonempty, otherwise let $p_2 = p_1$. Continue this way. Then having defined u_{m-1} and p_{m-1}, let u_m be the smallest prefix of $u(p_{m-1}, |u|]$ such that $\alpha(u_m) = \alpha(u(p_{m-1}, |u|])$; call p_m the last position of u_m if $u(p_{m-1}, |u|]$ is nonempty, otherwise let $p_m = p_{m-1}$. If $|\alpha(u)| = 1$ ($|\alpha(u)|$ denotes the cardinality of $\alpha(u)$), $p_1, \ldots, p_m$ are the (m) first positions in u and the procedure ends. If $|\alpha(u)| > 1$, $p_1, \ldots, p_m$ are among the (m) first positions in u. The rest are found by repeating the process to find the (m) first positions in $u[1, p_1)$ (if nonempty) and the $(m - i)$ first positions in $u(p_i, p_{i+1})$ (if nonempty) for all $1 \le i < m$. Similarly, the (m) last positions in u are defined by finding suffixes of u. Together, the (m) first and (m) last positions in u are called the (m) positions in u. These positions were defined in [9].

LEMMA 3.4 (Blanchet-Sadri [9]): *Let $m \ge 1$. Let A be a finite alphabet and let $u, v \in A^+$. Let $p_1, \ldots, p_k$ ($p_1 < \cdots < p_k$) (respectively $q_1, \ldots, q_\ell$ ($q_1 < \cdots < q_\ell$)) be the (m) positions in u (respectively v). We have $u\, \alpha_{(1,m)}\, v$ if and only if the following three conditions are satisfied:*

1. $k = \ell$.
2. $R_a^u\, p_i$ *if and only if* $R_a^u\, q_i$ *for all* $1 \le i \le k$ *and* $a \in A$.
3. $u(p_i, p_{i+1})\, \alpha_1\, v(q_i, q_{i+1})$ *for all* $1 \le i < k$.

For sections 3.1 and 3.2, fix a sequence $u_i = v_i$, $i \ge 1$ of identities on $\mathcal{X}^*$ defining $\mathbf{V}$ and call it $\mathcal{E}$.

3.1. The case $\mathbf{V} \star \mathbf{J}_m$

We now give a basis of identities for the pseudovariety $\mathbf{V} \star \mathbf{J}_m$.

Let $m \ge 1$. The basis $\mathcal{E}'_m$ consists of the following type of identities on $\mathcal{X}^*$:

$$w_m \ldots w_1 u_i = w_m \ldots w_1 v_i \tag{6}$$

where $\alpha(u_i v_i) \subseteq \alpha(w_1) \subseteq \cdots \subseteq \alpha(w_m)$, and where $i \ge 1$.

THEOREM 3.1: *Let $m \geq 1$. The pseudovariety $\mathbf{V} \star \mathbf{J}_m$ is defined by $\mathcal{E}'_m$.*

Proof: Fix $m \geq 1$. For the inclusion $\mathbf{V} \star \mathbf{J}_m \subseteq \mathbf{V}(\mathcal{E}'_m)$, we use Lemma 2.2. Let $u = v$ be any identity of type (6), that is

$$u = w_m \ldots w_1 u_i,$$
$$v = w_m \ldots w_1 v_i,$$

where $\alpha(u_i v_i) \subseteq \alpha(w_1) \subseteq \cdots \subseteq \alpha(w_m)$, and where $i \geq 1$. Then we need to show that $u \sim \beta_B, \alpha_m v$, or $\sigma_{\alpha_m}(u)\, \beta_B\, \sigma_{\alpha_m}(v)$ and $u\, \alpha_m\, v$ where $A = \alpha(uv)$ and $B = F_A(\mathbf{J}_m) \times A$. By Lemma 3.2, this amounts to verifying that $\sigma_{\alpha_m}(u)\, \beta_B\, \sigma_{\alpha_m}(v)$ (here $\beta_B \subseteq \alpha_1$ by assumption). First, we note that for every w on A satisfying $\alpha(w) \subseteq \alpha(w_1)$, we have the equality $\pi_{\alpha_m}(w_m \ldots w_1 w) = \pi_{\alpha_m}(w_m \ldots w_1)$ since $\alpha(w_1) \subseteq \cdots \subseteq \alpha(w_m)$. This comes from Lemma 3.1. It then follows that

$$\pi_{\alpha_m}(w_m \ldots w_1 w) = \pi_{\alpha_m}(w_m \ldots w_1)$$

for every prefix w of u_i since $\alpha(u_i) \subseteq \alpha(w_1)$. A similar statement can be made for every prefix w of v_i. These statements are used in the computation of $\sigma_{\alpha_m}(u)$ and $\sigma_{\alpha_m}(v)$ which follows. If $w = a_1 \ldots a_n$ on A, we will abbreviate the word

$$(\pi_{\alpha_m}(w_m \ldots w_1), a_1)(\pi_{\alpha_m}(w_m \ldots w_1), a_2) \ldots (\pi_{\alpha_m}(w_m \ldots w_1), a_n)$$

on the alphabet B by $\sigma(w)$. We have the equalities

$$\sigma_{\alpha_m}(u) = \sigma_{\alpha_m}(w_m \ldots w_1)\, \sigma(u_i),$$
$$\sigma_{\alpha_m}(v) = \sigma_{\alpha_m}(w_m \ldots w_1)\, \sigma(v_i).$$

Now, we have $\sigma(u_i)\, \beta_B\, \sigma(v_i)$ since $u_i\, \beta_A\, v_i$, and therefore $\sigma_{\alpha_m}(u)$ and $\sigma_{\alpha_m}(v)$ are β_B-equivalent. This shows that $\mathbf{V} \star \mathbf{J}_m$ satisfies $u = v$.

For the reverse inclusion, it suffices to show that if an identity $u = v$ holds in $\mathbf{V} \star \mathbf{J}_m$, then it is a consequence of $\mathcal{E}'_m$. Again by Lemma 2.2 and Lemma 3.2, our hypothesis on the identity $u = v$ means that $\sigma_{\alpha_m}(u)\, \beta_B\, \sigma_{\alpha_m}(v)$ where $A = \alpha(uv)$ and $B = F_A(\mathbf{J}_m) \times A$. Let $u' a$ be the shortest prefix of u satisfying $\pi_{\alpha_m}(u' a) = \pi_{\alpha_m}(u)$. The word u can hence be factorized as $u = u' a u''$ for some u', $u'' \in A^*$. Since $\pi_{\alpha_m}(u' a u'') = \pi_{\alpha_m}(u' a)$, there exist $w_1, \ldots, w_m \in A^+$ with $u' a = w_m \ldots w_1$ and $\alpha(u'') \subseteq \alpha(w_1) \subseteq \cdots \subseteq \alpha(w_m)$ by Lemma 3.1.

Now, let $v'b$ be the shortest prefix of v satisfying $\pi_{\alpha_m}(v'b) = \pi_{\alpha_m}(v)$ giving a factorization $v = v'bv''$ for some $v', v'' \in A^*$. We have

$$\sigma_{\alpha_m}(u) = \sigma_{\alpha_m}(u')(\pi_{\alpha_m}(u'), a)\,\sigma'(u''),$$
$$\sigma_{\alpha_m}(v) = \sigma_{\alpha_m}(v')(\pi_{\alpha_m}(v'), b)\,\sigma'(v''),$$

where for every $w = a_1 \dots a_n$ on A, the word

$$(\pi_{\alpha_m}(u), a_1)(\pi_{\alpha_m}(u), a_2) \dots (\pi_{\alpha_m}(u), a_n)$$

on the alphabet B has been abbreviated by $\sigma'(w)$ ($\pi_{\alpha_m}(u) = \pi_{\alpha_m}(v)$ by Lemma 3.2). We first note that $\alpha_m(u')$ is lacking an element of $\alpha_m(u)$ ending with a, and $\alpha_m(v')$ is lacking an element of $\alpha_m(v)$ ending with b. The sets $\alpha(\sigma_{\alpha_m}(u'))$, $\{(\pi_{\alpha_m}(u'), a)\}$, $\alpha(\sigma'(u''))$, $\alpha(\sigma_{\alpha_m}(v'))$, $\{(\pi_{\alpha_m}(v'), b)\}$ and $\alpha(\sigma'(v''))$ are pairwise disjoint except possibly for the pair $\alpha(\sigma_{\alpha_m}(u'))$, $\alpha(\sigma_{\alpha_m}(v'))$, the pair $\{(\pi_{\alpha_m}(u'), a)\}$, $\{(\pi_{\alpha_m}(v'), b)\}$ and the pair $\alpha(\sigma'(u''))$, $\alpha(\sigma'(v''))$ (this fact will be used in the rest of the proof). To see this, the letters in $\sigma_{\alpha_m}(u')$, $\sigma_{\alpha_m}(v')$ and the letters $(\pi_{\alpha_m}(u'), a)$, $(\pi_{\alpha_m}(v'), b)$ cannot appear in $\sigma'(u'')$ nor in $\sigma'(v'')$ since every letter in $\sigma'(u'')$ or $\sigma'(v'')$ has $\pi_{\alpha_m}(u)$ as first coordinate; the letter $(\pi_{\alpha_m}(u'), a)$ (respectively $(\pi_{\alpha_m}(v'), b)$) cannot appear in $\sigma_{\alpha_m}(u')$ (respectively $\sigma_{\alpha_m}(v')$) because of the choice of $u'a$ (respectively $v'b$; and the letter $(\pi_{\alpha_m}(u'), a)$ cannot appear in $\sigma_{\alpha_m}(v')$ since every letter in $\sigma_{\alpha_m}(v')$ has as first coordinate a word that is lacking an element of $\alpha_m(v)$ ending with b but contained in u' (similarly $(\pi_{\alpha_m}(v'), b)$ cannot appear in $\sigma_{\alpha_m}(u')$). Second, if $a \neq b$, then the letter $(\pi_{\alpha_m}(u'), a)$ which is in $\alpha(\sigma_{\alpha_m}(u))$ is not in $\alpha(\sigma_{\alpha_m}(v))$. We get a contradiction since $\sigma_{\alpha_m}(u)\,\alpha_1\,\sigma_{\alpha_m}(v)$. So $(\pi_{\alpha_m}(u'), a) = (\pi_{\alpha_m}(v'), b)$, yielding $a = b$. Consequently, we get $\sigma_{\alpha_m}(u')\,\beta_B\,\sigma_{\alpha_m}(v')$, and $\sigma'(u'')\,\beta_B\,\sigma'(v'')$ or $u''\,\beta_A v''$. The identity $u'' = v''$ is deducible from the defining basis of $\mathbf{V}$ since $u''\,\beta_A v''$. We hence see that $u'au'' = w_m \dots w_1 u'' = w_m \dots w_1 v'' = u' a v''$ is deducible from $\mathcal{E}'_m$. Now, since $\sigma_{\alpha_m}(u')$ and $\sigma_{\alpha_m}(v')$ are β_B-equivalent, we can repeat the process. Since u and v obviously start with the same letter ($\sigma_{\alpha_m}(u)$ and $\sigma_{\alpha_m}(v)$ have the same alphabet and their first letter is the only one to have 1 as first coordinate), the process terminates with a deduction of $u = v$ from $\mathcal{E}'_m$. □

COROLLARY 3.1: *The pseudovariety* $\mathbf{V} \star \mathbf{J}$ *is ultimately defined by* $\mathcal{E}'_m$, $m \geq 1$.

Proof: The result follows from $\mathbf{V} \star \mathbf{J} = \mathbf{V} \star \bigcup_{m \geq 1} \mathbf{J}_m = \bigcup_{m \geq 1} \mathbf{V} \star \mathbf{J}_m$ and Theorem 3.1. □

3.3.1. *A basis of identities for* $\mathbf{J}_{m_1} \star \mathbf{J}_{m_2}$

In this section, we give a basis of identities for the pseudovariety $\mathbf{J}_{m_1} \star \mathbf{J}_{m_2}$.

Let $m_1, m_2 \geq 1$. Letting $x = x_1$, the basis $(\mathcal{A}_{m_1})'_{m_2}$ consists of the following type of identities on $\mathcal{X}^+$:

$$w_{m_2} \ldots w_1 u_i \ldots u_1 x v_1 \ldots v_j = w_{m_2} \ldots w_1 u_i \ldots u_1 v_1 \ldots v_j$$

where $\alpha(u_i v_j) \subseteq \alpha(w_1) \subseteq \cdots \subseteq \alpha(w_{m_2})$, where

$$\{x\} \subseteq \alpha(u_1) \subseteq \cdots \subseteq \alpha(u_i) \quad \text{and} \quad \{x\} \subseteq \alpha(v_1) \subseteq \cdots \subseteq \alpha(v_j),$$

and where $i + j = m_1$.

COROLLARY 3.2: *Let* $m_1, m_2 \geq 1$. *The pseudovariety* $\mathbf{J}_{m_1} \star \mathbf{J}_{m_2}$ *is defined by* $(\mathcal{A}_{m_1})'_{m_2}$.

Proof: By Theorem 3.1 using the fact that $\alpha_{m_1} \subseteq \alpha_1$ and

$$\mathbf{J}_{m_1} = \mathbf{V}(\mathcal{A}_{m_1}). \quad \square$$

COROLLARY 3.3 (Blanchet-Sadri [14]): *Let* $m \geq 1$. *We have the relation* $\mathbf{J}_1 \star \mathbf{J}_m = \mathbf{J}_1^{m+1}$.

Proof: Since we are dealing with equational pseudovarieties, the equality $\mathbf{J}_1 \star \mathbf{J}_m = \mathbf{J}_1^{m+1}$ means that $\mathbf{J}_1 \star \mathbf{J}_m$ and $\mathbf{J}_1^{m+1}$ satisfy the same identities. Almeida [3] shows that $\mathbf{J}_1^{m+1}$ is defined by $\mathcal{B}_m$ and Corollary 3.2 shows that $\mathbf{J}_1 \star \mathbf{J}_m$ is defined by $(\mathcal{A}_1)'_m$. But it is easy to see that $\mathcal{B}_m$ is equivalent to $(\mathcal{A}_1)'_m$. □

The relation $\mathbf{J}_1 \star \mathbf{J} = \mathbf{R}$ is known to Brzozowski and Fich [18]. The equality $\mathbf{J}_1 \star \mathbf{J}_m = \mathbf{J}_1^{m+1}$ gives a proof that a conjecture of Pin [28] concerning tree-hierarchies of pseudovarieties of monoids is false [14] (another proof using different techniques is given in [15]). Almeida [3] implies that $\mathbf{J}_1 \star \mathbf{J}_m$ admits a finite basis of identities if and only if $m = 1$.

3.1.2. *A basis of identities for* $\mathbf{J}_{m_1} \star \ldots \star \mathbf{J}_{m_k}$

In this section, we give a basis of identities for the pseudovariety $\mathbf{J}_{m_1} \star \ldots \star \mathbf{J}_{m_k}$.

COROLLARY 3.4: *If* $k \geq 2$ *and* $m_1, \ldots, m_k$ *are positive integers, then the pseudovariety* $\mathbf{J}_{m_1} \star \ldots \star \mathbf{J}_{m_k}$ *is defined by* $(\mathcal{A}_{m_1})'_{m_2 + \cdots + m_k}$.

Proof: The proof is by induction on k. For $k = 2$, the result is Corollary 3.2. Assume the results holds for k. Now, Lemma 2.2 provides a congruence β_k generating $\mathbf{J}_{m_1} \star \ldots \star \mathbf{J}_{m_k}$. For $k = 2$, $\beta_2 = \sim_{\alpha_{m_1}, \alpha_{m_2}}$; then $\beta_{k+1} = \sim_{\beta_k, \alpha_{m_{k+1}}}$. We have $\beta_k \subseteq \alpha_{m_k} \subseteq \alpha_1$ for $k \geq 2$. Using the inductive hypothesis, Theorem 3.1 and the inclusion $\beta_k \subseteq \alpha_1$, we get that

$$\mathbf{J}_{m_1} \star \ldots \star \mathbf{J}_{m_{k+1}} = (\mathbf{J}_{m_1} \star \ldots \star \mathbf{J}_{m_k}) \star \mathbf{J}_{m_{k+1}}$$

is defined by $((\mathcal{A}_{m_1})'_{m_z+\cdots+m_k})'_{m_{k+1}}$. But the latter is equivalent to $(\mathcal{A}_{m_1})'_{m_2+\cdots+m_{k+1}}$. □

COROLLARY 3.5: *If $k \geq 2$ and $m_1, \ldots, m_k$ are positive integers, then we have the relation* $\mathbf{J}_{m_1} \star \ldots \star \mathbf{J}_{m_k} = \mathbf{J}_{m_1} \star \mathbf{J}_{m_2+\cdots+m_k}$.

Proof: Since we are dealing with equational pseudovarieties, the equality $\mathbf{J}_{m_1} \star \ldots \star \mathbf{J}_{m_k} = \mathbf{J}_{m_1} \star \mathbf{J}_{m_2+\cdots+m_k}$ means that $\mathbf{J}_{m_1} \star \ldots \star \mathbf{J}_{m_k}$ and $\mathbf{J}_{m_1} \star \mathbf{J}_{m_2+\cdots+m_k}$ satisfy the same identities. Corollary 3.2 shows that $\mathbf{J}_{m_1} \star \mathbf{J}_{m_2+\cdots+m_k}$ is defined by $(\mathcal{A}_{m_1})'_{m_2+\cdots+m_k}$ and Corollary 3.4 shows that $\mathbf{J}_{m_1} \star \ldots \star \mathbf{J}_{m_k}$ is also defined by $(\mathcal{A}_{m_1})'_{m_2+\cdots+m_k}$. □

3.1.3. *A basis of identities for* $(\mathbf{J}_1 \star \mathbf{J}_{m_1})^{\varrho} \star \mathbf{J}_{m_2}$

Given any pseudovariety of monoids $\mathbf{V}$, define $\mathbf{V}^{\varrho} = \{S^{\varrho} | S \in \mathbf{V}\}$ (here, S^{ϱ} is the monoid S reversed). The set $\mathbf{V}^{\varrho}$ is a pseudovariety of monoids. In this section, we give a basis of identities for the pseudovariety $(\mathbf{J}_1 \star \mathbf{J}_{m_1})^{\varrho} \star \mathbf{J}_{m_2}$.

Let $m_1, m_2 \geq 1$. Letting $x = x_1$ and $y = x_2$, the basis $\mathcal{C}_{m_1, m_2}$ consists of the following two types of identities on $\mathcal{X}^+$:

$$u_{m_2} \ldots u_1 x^2 v_1 \ldots v_{m_1} = u_{m_2} \ldots u_1 x v_1 \ldots v_{m_1}$$

where $\{x\} \subseteq \alpha(v_1) \subseteq \cdots \subseteq \alpha(v_{m_1}) \subseteq \alpha(u_1) \subseteq \cdots \subseteq \alpha(v_{m_2})$, and

$$u_{m_2} \ldots u_1 x y v_1 \ldots v_{m_1} = u_{m_2} \ldots u_1 y x v_1 \ldots v_{m_1}$$

where $\{x, y\} \subseteq \alpha(v_1) \subseteq \cdots \subseteq \alpha(v_{m_1}) \subseteq \alpha(u_1) \subseteq \cdots \subseteq \alpha(v_{m_2})$. The basis $\mathcal{C}_{m, m}$ turns out to be close to a basis in Section 3.2.1.

COROLLARY 3.6: *Let $m_1, m_2 \geq 1$. The pseudovariety $(\mathbf{J}_1 \star \mathbf{J}_{m_1})^{\varrho} \star \mathbf{J}_{m_2}$ is defined by $\mathcal{C}_{m_1, m_2}$.*

Proof: Let A be a finite alphabet and let $u, v \in A^*$. We have $(\mathbf{J}_1 \star \mathbf{J}_{m_1})^{\varrho}$ satisfies $u = v$ if and only if $\mathbf{J}_1 \star \mathbf{J}_{m_1}$ satisfies $u^{\varrho} = v^{\varrho}$ if and only if $\sigma_{\alpha_{m_1}}(u^{\varrho}) \, \alpha_1 \, \sigma_{\alpha_{m_1}}(v^{\varrho})$ and $u^{\varrho} \, \alpha_{m_1} \, v^{\varrho}$ (the notation w^{ϱ} refers to the reversal of w). We therefore conclude that the congruence generating $(\mathbf{J}_1 \star \mathbf{J}_{m_1})^{\varrho}$

for A is included in α_1. The latter, Theorem 3.1 and $\mathbf{J}_1 \star \mathbf{J}_{m_1} = \mathbf{V}(\mathcal{B}_{m_1})$ implies the result.

3.2. The case $\mathbf{V} \star\star \mathbf{J}_m$

We now give a basis of identities for the pseudovariety $\mathbf{V} \star\star \mathbf{J}_m$.

Let $m \geq 1$. The basis $\mathcal{E}''_m$ consists of the following type of identities on $\mathcal{X}^*$:

$$w_m \ldots w_1 u_i w'_1 \ldots w'_m = w_m \ldots w_1 v_i w'_1 \ldots w'_m \tag{7}$$

where $\alpha(u_i v_i) \subseteq \alpha(w_1) \subseteq \cdots \subseteq \alpha(w_m)$ and

$$\alpha(u_i v_i) \subseteq \alpha(w'_1) \subseteq \cdots \subseteq \alpha(w'_m),$$

and where $i \geq 1$.

THEOREM 3.2: *Let $m \geq 1$. The pseudovariety $\mathbf{V} \star\star \mathbf{J}_m$ is defined by $\mathcal{E}''_m$.*

Proof: Fix $m \geq 1$. For the inclusion $\mathbf{V} \star\star \mathbf{J}_m \subseteq \mathbf{V}(\mathcal{E}''_m)$, we use Lemma 2.4. Let $u = v$ be any identity of type (7), that is

$$u = w_m \ldots w_1 u_i w'_1 \ldots w'_m,$$
$$v = w_m \ldots w_1 v_i w'_1 \ldots w'_m,$$

where

$$\alpha(u_i v_i) \subseteq \alpha(w_1) \subseteq \cdots \subseteq \alpha(w_m),$$
$$\alpha(u_i v_i) \subseteq \alpha(w'_1) \subseteq \cdots \subseteq \alpha(w'_m),$$

and where $i \geq 1$. Then we need to show that $\tau_{\alpha_m}(u) \beta_B \tau_{\alpha_m}(v)$ and $u \alpha_m v$ where $A = \alpha(uv)$ and $B = F_A(\mathbf{J}_m) \times A \times F_A(\mathbf{J}_m)$. By Lemma 3.3, this amounts to verifying that $\tau_{\alpha_m}(u) \beta_B \tau_{\alpha_m}(v)$ (here $\beta_B \subseteq \alpha_1$ by assumption). First, we note that for every w on A satisfying

$$\alpha(w) \subseteq \alpha(w_1),$$

we have the equality $\pi_{\alpha_m}(w_m \ldots w_1 w) = \pi_{\alpha_m}(w_m \ldots w_1)$ since $\alpha(w_1) \subseteq \cdots \subseteq \alpha(w_m)$. This comes from Lemma 3.1. It then follows that $\pi_{\alpha_m}(w_m \ldots w_1 w) = \pi_{\alpha_m}(w_m \ldots w_1)$ for every prefix w of u_i since $\alpha(u_i) \subseteq \alpha(w_1)$. A similar statement can be made for every prefix w of v_i. Second, we note that for every w satisfying $\alpha(w) \subseteq \alpha(w'_1)$, we have the equality

$$\pi_{\alpha_m}(ww'_1 \dots w'_m) = \pi_{\alpha_m}(w'_1 \dots w'_m) \text{ since}$$
$$\alpha(w'_1) \subseteq \dots \subseteq \alpha(w'_m).$$

This also comes from Lemma 3.1. It then follows that

$$\pi_{\alpha_m}(ww'_1 \dots w'_m) = \pi_{\alpha_m}(w'_1 \dots w'_m)$$

for every suffix w of u_i since $\alpha(u_i) \subseteq \alpha(w'_1)$. A similar statement can be made for every suffix w of v_i. These statements are used in the computation of $\tau_{\alpha_m}(u)$ and $\tau_{\alpha_m}(v)$ which follows. If $w = a_1 \dots a_n$ on A we will abbreviate the word

$${}_m(w_m \dots w_1), a_1, \pi_{\alpha_m}(w'_1 \dots w'_m))$$
$$..(\pi_{\alpha_m}(w_m \dots w_1), a_n, \pi_{\alpha_m}(w'_1 \dots w'_m))$$

on the alphabet B by $\tau(w)$. We have the equalities

$$\tau_{\alpha_m}(u) = \tau_{\alpha_m}^{1, w'_1 \dots w'_m}(w_m \dots w_1)\, \tau(u_i)\, \tau_{\alpha_m}^{w_m \dots w_1, 1}(w'_1 \dots w'_m),$$
$$\tau_{\alpha_m}(v) = \tau_{\alpha_m}^{1, w'_1 \dots w'_m}(w_m \dots w_1)\, \tau(v_i)\, \tau_{\alpha_m}^{w_m \dots w_1, 1}(w'_1 \dots w'_m).$$

Now, we have $\tau(u_i)\, \beta_B\, \tau(v_i)$ since $u_i\, \beta_A\, v_i$ and therefore $\tau_{\alpha_m}(u)$ and $\tau_{\alpha_m}(v)$ are β_B-equivalent. This show that $\mathbf{V} \star\star \mathbf{J}_m$ satisfies $u = v$.

For the reverse inclusion, it suffices to show that if an identity $u = v$ holds in $\mathbf{V} \star\star \mathbf{J}_m$, then it is a consequence of $\mathcal{E}''_m$. Again by Lemma 2.4, our hypothesis on the identity $u = v$ means that $\tau_{\alpha_m}(u)\, \beta_B\, \tau_{\alpha_m}(v)$ and $u\, \alpha_m\, v$ with $A = \alpha(uv)$ and $B = F_A(\mathbf{J}_m) \times A \times F_A(\mathbf{J}_m)$. First of all, $\tau_{\alpha_m}(u)\, \beta_B\, \tau_{\alpha_m}(v)$ implies $u\, \alpha_{(1, m)}\, v$ by Lemma 3.3. Since u and v are either both empty or both nonempty, we treat the case where u and v are both nonempty (the other case is trivial). Let $p_1, \dots, p_k$ $(p_1 < \dots < p_k)$ (respectively $q_1, \dots, q_\ell$ $(q_1 < \dots < q_\ell)$) be the (m) positions in u (respectively v). The three conditions of Lemma 3.4 are satisfied. In fact, since $\tau_{\alpha_m}(u)\, \beta_B\, \tau_{\alpha_m}(v)$, we can say better (in the sense that the following three conditions imply the three conditions of Lemma 3.4 since $\beta_A \subseteq \alpha_1$):

- $k = \ell$.
- $R_a^u\, p_j$ if and only if $R_a^v\, q_j$ for all $1 \leq j \leq k$ and $a \in A$.
- $u(p_j, p_{j+1})\, \beta_A\, v(q_j, q_{j+1})$ for all $1 \leq j < k$ (this follows by an argument similar to that of the proof of $u''\, \beta_A\, v''$ in Theorem 3.1).

The latter implies that $u(p_j, p_{j+1}) = v(q_j, q_{j+1})$ is a consequence of $\mathcal{E}$ for all $1 \leq j < k$.

Fix j. If $u(p_j, p_{j+1})$ is nonempty, rewrite $u[1, p_j]$ as $w_m \ldots w_1$ and $u[p_{j+1}, |u|]$ as $w'_1 \ldots w'_m$ for some $w_1, \ldots, w_m, w'_1, \ldots, w'_m$ with

$$\alpha(u(p_j, p_{j+1})) \subseteq \alpha(w_1) \subseteq \cdots \subseteq \alpha(w_m),$$
$$\alpha(u(p_j, p_{j+1})) \subseteq \alpha(w'_1) \subseteq \cdots \subseteq \alpha(w'_m).$$

This can be done based on the choice of the p_j's. Since $\mathcal{E} \vdash u(p_j, p_{j+1}) = v(q_j, q_{j+1})$ we get

$$\mathcal{E}''_m \vdash w_m \ldots w_1 \, u(p_j, p_{j+1}) \, w'_1 \ldots w'_m = w_m \ldots w_1 \, v(q_j, q_{j+1}) \, w'_1 \ldots w'_m.$$

We can repeat the process for each j, and we get a deduction of $u = v$ from $\mathcal{E}''_m$. □

COROLLARY 3.7: *The pseudovariety* $\mathbf{V} \star\star \mathbf{J}$ *is ultimately defined by* $\mathcal{E}''_m$, $m \geq 1$.

Proof: The result follows from $\mathbf{V} \star\star \mathbf{J} = \bigcup_{m \geq 1} \mathbf{V} \star\star \mathbf{J}_m$ and Theorem 3.2. □

3.2.1. *A basis of identities for* $\mathbf{J}_{m_1} \star\star \mathbf{J}_{m_2}$

In this section, we give a basis of identities for the pseudovariety $\mathbf{J}_{m_1} \star\star \mathbf{J}_{m_2}$.

Let $m_1, m_2 \geq 1$. Letting $x = x_1$, the basis $(\mathcal{A}_{m_1})''_{m_2}$ consists of the following type of identities on $\mathcal{X}^+$:

$$w_{m_2} \ldots w_1 \, u_i \ldots u_1 \, x \, v_1 \ldots v_j \, w'_1 \ldots w'_{m_2}$$
$$= w_{m_2} \ldots w_1 \, u_i \ldots u_1 \, v_1 \ldots v_j \, w'_1 \ldots w'_{m_2}$$

where

$$(u_i v_j) \subseteq \alpha(w_1) \subseteq \cdots \subseteq \alpha(w_{m_2})$$

and

$$\alpha(u_i v_j) \subseteq \alpha(w'_1) \subseteq \cdots \subseteq \alpha(w'_{m_2}),$$

where

$$\{x\} \subseteq \alpha(u_1) \subseteq \cdots \subseteq \alpha(u_i)$$

and

$$\{x\} \subseteq \alpha(v_1) \subseteq \cdots \subseteq \alpha(v_j),$$

and where $i + j = m_1$.

In the case $m_1 = 1$ and $m_2 = m$, the basis $(\mathcal{A}_1)''_m$ is equivalent to the set consisting of the following two types of identities on $\mathcal{X}^+$ ($x = x_1$ and $y = x_2$):

$$u_m \ldots u_1 x^2 v_1 \ldots v_m = u_m \ldots u_1 x v_1 \ldots v_m$$

where $\{x\} \subseteq \alpha(u_1) \subseteq \cdots \subseteq \alpha(u_m)$ and $\{x\} \subseteq \alpha(v_1) \subseteq \cdots \subseteq \alpha(v_m)$, and

$$u_m \ldots u_1 xy v_1 \ldots v_m = u_m \ldots u_1 yx v_1 \ldots v_m$$

where $\{x, y\} \subseteq \alpha(u_1) \subseteq \cdots \subseteq \alpha(u_m)$ and $\{x, y\} \subseteq \alpha(v_1) \subseteq \cdots \subseteq \alpha(v_m)$

COROLLARY 3.8: *Let $m_1, m_2 \geq 1$. The pseudovariety $\mathbf{J}_{m_1} \star\star \mathbf{J}_{m_2}$ is defined by $(\mathcal{A}_{m_1})''_{m_2}$.*

Proof: By Theorem 3.2 using the facts that

$$\alpha_{m_1} \subseteq \alpha_1 \qquad \text{and} \qquad \mathbf{J}_{m_1} = \mathbf{V}(\mathcal{A}_{m_1}). \quad \square$$

COROLLARY 3.9: *Let $m \geq 1$. We have the relation $\mathbf{J}_1 \star\star \mathbf{J}_m = \mathbf{V}_{(1, m)}$. More generally, if $k \geq 1$ and $\bar{m}$ is a k-tuple of positive integers, then $\mathbf{J}_1 \star\star \mathbf{V}_{\bar{m}} = \mathbf{V}_{(1, \bar{m})}$.*

Proof: By Lemma 2.4 using the fact that $u \approx_{\alpha_1, \alpha_{\bar{m}}} v$ if and only if $u \alpha_{(1, \bar{m})} v$. $\square$

The relation $\mathbf{J}_1 \star\star \mathbf{V}_k = \mathbf{V}_{k+1, 1}$ is known to Weil (this is a particular case of Proposition 2.12 in [40]).

3.2.2. *On iterated two-sided semidirect products of* $\mathbf{J}_1$

In this section, we study some iterated two-sided semidirect products of $\mathbf{J}_1$.

Let $k \geq 2$. Letting $x = x_1$ and $y = x_2$ the basis $\mathcal{D}_k$ consists of the following two types of identities on $\mathcal{X}^+$:

$$u_{k-1} \ldots u_1 x^2 v_1 \ldots v_{k-1} = u_{k-1} \ldots u_1 x v_1 \ldots v_{k-1}$$

where $\{x\} \subseteq \alpha(u_1)$ and $\{x\} \subseteq \alpha(v_1)$, where $\alpha(u_i v_i) \subseteq \alpha(u_{i+1})$ and $\alpha(u_i v_i) \subseteq \alpha(v_{i+1})$ for $1 \leq i < k-1$, and

$$u_{k-1} \ldots u_1 xy v_1 \ldots v_{k-1} = u_{k-1} \ldots u_1 yx v_1 \ldots v_{k-1}$$

where $\{x, y\} \subseteq \alpha(u_1)$ and $\{x, y\} \subseteq \alpha(v_1)$, where $\alpha(u_i v_i) \subseteq \alpha(u_{i+1})$ and $\alpha(u_i v_i) \subseteq \alpha(v_{i+1})$ for $1 \leq i < k-1$.

COROLLARY 3.10: *Let $\mathbf{W}_i$ be the sequence of pseudovarieties of monoids defined by $\mathbf{W}_1 = \mathbf{J}_1$ and $\mathbf{W}_{i+1} = \mathbf{W}_i \star\star \mathbf{J}_1$. If $k \geq 2$, then the pseudovariety $\mathbf{W}_k$ is defined by $\mathcal{D}_k$.*

Proof: The proof is by induction on k. For $k = 2$, the result is Corollary 3.8. Assume the result holds for k. Now, Lemma 2.4 provides a congruence β_k generating $\mathbf{W}_k$. For $k = 2$, $\beta_2 = \approx_{\alpha_1, \alpha_1}$; then $\beta_{k+1} = \approx_{\beta_k, \alpha_1}$. We have $\beta_k \subseteq \alpha_1$ for $k \geq 2$. Using the inductive hypothesis, Theorem 3.2 and the inclusion $\beta_k \subseteq \alpha_1$, we get that $\mathbf{W}_{k+1} = \mathbf{W}_k \star\star \mathbf{J}_1$ is defined by $(\mathcal{D}_k)_1''$. But the latter is equivalent to $\mathcal{D}_{k+1}$.

We end this section with an iterated two-sided semidirect product of $\mathbf{J}_1$ perfectly related to the standard Ehrenfeucht-Fraïssé game.

COROLLARY 3.11: *Let* $\mathbf{W}_i'$ *be the sequence of pseudovarieties of monoids defined by* $\mathbf{W}_1' = \mathbf{J}_1$ and $\mathbf{W}_{i+1}' = \mathbf{J}_1 \star\star \mathbf{W}_i'$. *Let* $k \geq 1$, *let* A *be a finite alphabet and let* $u, v \in A^*$. *We have* $\mathbf{W}_k'$ *satisfies* $u = v$ *if and only if* $u \, \alpha_{\bar{1}_k} \, v$. *In other words,* $\mathbf{W}_k' = \mathbf{V}_{\bar{1}_k}$.

Proof: The proof is by induction on k. For $k = 1$, the result trivially holds. Assume the result holds for k. Then $\mathbf{W}_{k+1}' = \mathbf{J}_1 \star\star \mathbf{W}_k' = \mathbf{J}_1 \star\star \mathbf{V}_{\bar{1}_k}$ (by the inductive hypothesis). But the latter equals $\mathbf{V}_{(1, \bar{1}_k)}$ or $\mathbf{V}_{\bar{1}_{k+1}}$ by Corollary 3.9.

COROLLARY 3.12: *Let* $\mathbf{W}_i'$ *be the sequence of pseudovarieties of monoids defined by* $\mathbf{W}_1' = \mathbf{J}_1$ *and* $\mathbf{W}_{i+1}' = \mathbf{J}_1 \star\star \mathbf{W}_i'$. *We have the relation* $\mathbf{A} = \bigcup_{k \geq 1} \mathbf{W}_k'$.

Proof: Let $k \geq 1$ and let $\bar{m}$ be a k-tuple of positive integers. We have $\mathbf{V}_{\bar{1}_k} \subseteq \mathbf{V}_{\bar{m}} \subseteq \mathbf{V}_{\bar{1}_n}$ where $n = m_1 + \cdots + m_k$ [6]. We have then $\mathbf{A} = \bigcup_{k \geq 1} \mathbf{V}_{\bar{1}_k} = \bigcup_{k \geq 1} \mathbf{W}_k'$ by Corollary 3.11. □

REFERENCES

1. D. ALBERT, R. BALDINGER and J. RHODES, Undecidability of the identity problem for finite semigroups, *Journal of Symbolic Logic*, 1992, *57*, pp. 179-192.
2. J. ALMEIDA, Semidirect products of pseudovarieties from the universal algebraist's point of view, *Journal of Pure and Applied Algebra*, 1989, *60*, pp. 113-128.
3. J. ALMEIDA, On iterated semidirect products of finite semilattices, *Journal of Algebra*, 1991, *142*, pp. 239-254.
4. J. ALMEIDA, *Semigrupos Finitos e Álgebra Universal* (Institute of Mathematics and Statistics of the University of São Paulo, 1992; *Finite Semigroups and Universal Algebra* (World Scientific, Singapore, 1994.
5. J. ALMEIDA and P. WEIL, Free profinite semigroups over semidirect products, *Izvestiya Vysshikh Učebnykh Zavedeniĭ Matematica*, 1995, *1*, pp. 3-31.
6. F. BLANCHET-SADRI, Some logical characterizations of the dot-depth hierarchy and applications, Ph. D. Thesis, McGill University, 1989.

7. F. Blanchet-Sadri, Games, equations and the dot-depth hierarchy, *Computers and Mathematics with Applications*, 1989, *18*, pp. 809-822.
8. F. Blanchet-Sadri, On dot-depth two, *RAIRO Informatique Théorique et Applications*, 1990, *24*, pp. 521-529.
9. F. Blanchet-Sadri, Games, equations and dot-depth two monoids, *Discrete Applied Mathematics*, 1992, *39*, pp. 99-111.
10. F. Blanchet-Sadri, The dot-depth of a generating class of aperiodic monoids is computable, *International Journal of Foundations of Computer Science*, 1992, *3*, pp. 419-442.
11. Blanchet-Sadri, Equations and dot-depth one, *Semigroup Forum*, 1993, *47*, pp. 305-317.
12. F. Blanchet-Sadri, Equations and monoid varieties of dot-depth one and two, *Theoretical Computer Science*, 1994, *123*, pp. 239-258.
13. F. Blanchet-Sadri, On a complete set of generators for dot-depth two, *Discrete Applied Mathematics*, 1994, *50*, pp. 1-25.
14. F. Blanchet-Sadri, Equations on the semidirect product of a finite semilattice by a $\mathcal{J}$-trivial monoid of height k, *RAIRO Informatique Théorique et Applications*, 1995, *29*, pp. 157-170.
15. F. Blanchet-Sadri, Some logical characterizations of the dot-depth hierarchy and applications, *Journal of Computer and System Sciences*, 1995, *51*, pp. 324-337.
16. F. Blanchet-Sadri, Inclusion relations between some congruences related to the dot-depth hierarchy, *Discrete Applied Mathematics*, 1996, *68*, pp. 33-71.
17. F. Blanchet-Sadri and X. H. Zhang, Equations on the semidirect product of a finite semilattice by a finite commutative monoid, *Semigroup Forum*, 1994, *49*, pp. 67-81.
18. J. A. Brzozowski and F. E. Fich, Languages of $\mathcal{R}$-trivial monoids, *Journal of Computer and System Sciences*, 1980, *20*, pp. 32-49.
19. J. A. Brzozowski and R. Knast, The dot-depth hierarchy of star-free languages is infinite, *Journal of Computer and System Sciences*, 1978, *16*, pp. 37-55.
20. S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, New York, 1981.
21. Cohen R. S. and J. A. Brzozowski, Dot-depth of star-free events, *Journal of Computer and System Sciences*, 1971, *5*, pp. 1-15.
22. A. Ehrenfeucht, An application of games to the completeness problems for formalized theories, *Fundamenta Mathematicae*, 1961, *49*, pp. 129-141.
23. S. Eilenberg, *Automata, Languages, and Machines, Vol. A*, Academic Press, New York, 1974; *Vol. B*, Academic Press, New York, 1976.
24. S. Eilenberg and M. P. Schützenberger, On pseudovarieties, *Advances in Mathematics*, 1976, *19*, pp. 413-418.
25. C. Irastorza, Base non finie de variétés, in *STACS'85*, Lecture Notes in Computer Science, Springer-Verlag, Berlin, 1985, *182*, pp. 180-186.
26. D. Perrin and J. E Pin, First order logic and star-free sets, *Journal of Computer and System Sciences*, 1986, *32*, pp. 393-406.
27. J. E. Pin, *Variétés de Langages Formels*, Masson, Paris, 1984; *Varieties of Formal Languages*, North Oxford Academic, London, 1986 and Plenum, New York, 1986.
28. J. E. Pin, Hiérarchies de concaténation, *RAIRO Informatique Théorique et Applications*, 1984, *18*, pp. 23-46.
29. J. E. Pin, On semidirect products of two finite semilattices, *Semigroup Forum*, 1984, *28*, pp. 73-81.
30. J. Reiterman, The Birkhoff theorem for varieties of finite algebras, *Algebra Universalis*, 1982, *14*, pp. 1-10.

31. J. Rhodes and B. Tilson, The kernel of monoid morphisms, *Journal of Pure and Applied Algebra*, 1989, *62*, pp. 227-268.
32. I. Simon, Hierarchies of events of dot-depth one, Ph. D. Thesis, University of Waterloo, 1972.
33. I. Simon, Piecewise testable events in *Proc. 2nd GI Conf.*, Lecture Notes in Computer Science, 1975, *33*, Springer-Verlag, Berlin, pp. 214-222.
34. P. Stiffler, Extension of the fundamental theorem of finite semigroups, *Advances in Mathematics*, 1973, *11*, pp. 159-209.
35. H. Straubing, Finite semigroup varieties of the form $\mathbf{V} \star \mathbf{D}$, *Journal of Pure and Applied Algebra*, 1985, *36*, pp. 53-94.
36. H. Straubing and P. Weil, On a conjecture concerning dot-depth two languages, *Theoretical Computer Science*, 1992, *104*, pp. 161-183.
37. W. Thomas, Classifying regular events in symbolic logic, *Journal of Computer and System Sciences*, 1982, *25*, pp. 360-376.
38. W. Thomas, An application of the Ehrenfeucht-Fraïssé game in formal language theory, *Mémoires de la Société Mathématique de France*, 1984, *16*, pp. 11-21.
39. B. Tilson, Categories as algebra: an essential ingredient in the theory of semigroups, *Journal of Pure and Applied Algebra*, 1987, *48*, pp. 83-198.
40. P. Weil, Closure of varieties of languages under products with counter, *Journal of Computer and System Sciences*, 1992, *45*, pp. 316-339.