RAIRO. INFORMATIQUE THÉORIQUE ET APPLICATIONS

ANDREAS WEBER

Decomposing a k-valued transducer into k unambiguous ones

RAIRO. Informatique théorique et applications, tome 30, nº 5 (1996), p. 379-413

<http://www.numdam.org/item?id=ITA_1996__30_5_379_0>

Informatique théorique et Applications/Theoretical Informatics and Applications
(vol. 30, n° 5, 1996, pp. 379-413)

DECOMPOSING A k-VALUED TRANSDUCER INTO k UNAMBIGUOUS ONES (*) (**)

by Andreas WEBER [1]

Communicated by J. BERSTEL

Abstract. – *In this article finite-valued transducers are investigated in connection with their inner structure. The transducer models considered are the normalized finite transducer (NFT) and the nondeterministic generalized sequential machine (NGSM), which is a real-time NFT. It is shown that a k-valued NGSM M can be effectively decomposed into k unambiguous NGSMs $M_1, \ldots, M_k$ such that the transduction realized by M is the union of the transductions realized by $M_1, \ldots, M_k$. Each transducer M_i has double exponential size and can be computed in deterministic double exponential time. This result can be extended to NFTs. As a consequence, the k-valued NGSMs (NFTs) and the k-ambiguous NGSMs (NFTs, respectively) realize the same class of transductions.*

Résumé. – *Dans cette article, les transducteurs d'image bornée sont examinés en liaison avec leur structure interne. Les modèles de transducteurs qui sont considérés sont les transducteurs finis normalisés (NFT) et les NGSM, qui sont des NFT à temps réel. Il est démontré qu'un NGSM d'image k-bornée M peut être effectivement décomposé en k NGSM non ambigus $M_1, \ldots, M_k$ de telle manière que la transduction réalisée par M soit égale à l'union des transductions réalisées par $M_1, \ldots, M_k$. Chaque transducteur M_i a une taille doublement exponentielle et peut être calculé en temps déterministe doublement exponentiel. On peut étendre ce résultat aux NFT. En conséquence, les NGSM (resp. NFT) d'image k-bornée et les NGSM (resp. NFT) k-ambigus réalisent la même classe de transductions.*

(*) Received May 1993.

(**) A preliminary version of this article appeared in the *Proceedings of the 1st Latin American Symposium on Theoretical Informatics, São Paulo, Brasil, 1992.*

([1]) Fachbereich Informatik, Johann Wolfgang Goethe-Universität, D-60054 Frankfurt am Main, Germany; E-mail: weber@informatik.uni-frankfurt.de. A part of the research for this article was done while the author was supported by a Postdoctoral Fellowship of the Japan Society for the Promotion of Science.

INTRODUCTION

The transducer is the classical model of a finite-state machine with output device. Informally, a transducer M may be regarded as a finite, directed, labeled graph. The vertices and edges of that graph represent the states and transitions of M, respectively. The label of an edge is the pair of words consumed and produced by the corresponding transition from the one-way input tape and on the one-way output tape of M, respectively. The machine M is called a normalized finite transducer, abbreviated NFT, if these input words always have length 0 or 1 – or a nondeterministic generalized sequential machine, abbreviated NGSM, if only length 1 appears, *i.e.*, M is a real-time transducer. The computations in an NFT M are represented by paths in the above graph. Every such path consumes an input word and produces an output word along its edges. A computation is successful if it corresponds to a path initiating and terminating at designated initial and final states, respectively. Such paths are called accepting. The transduction (or relation) realized by M is the set of pairs (x, z) of input/output words being consumed/produced by any accepting path. For each such pair (x, z), z is called a value for x in M. Two transducers are equivalent if the transductions realized by them coincide, *i.e.*, every input word has the same set of values in both machines.

The valuedness of an NFT M is the maximal number of different values for an input word or is infinite, depending on whether or not a maximum exists. For any positive integer k, the transducer M is called finite valued (k-valued, single valued) if its valuedness is finite (at most k, at most 1, respectively). It is said to be k-ambiguous (unambiguous) if any input word is consumed by at most k (at most 1, respectively) different accepting paths – and finitely ambiguous if it is k-ambiguous for some k. Evidently, every k-ambiguous transducer is k-valued and every finitely ambiguous transducer is finite valued. The converse is in general false.

It is decidable in deterministic polynomial time whether or not a given NFT is finite valued (Weber [W90]) and, for any fixed positive integer k, whether or not it is k-valued (Gurari and Ibarra [GI83]). Since ambiguity is a special case of valuedness (just replace the output word of any transition by the transition itself), the two above results remain valid if "valued" is replaced by "ambiguous". For further background on transducers the reader may consult the textbooks (Berstel [B79]) and (Gurari [G89]).

The work presented in this article is motivated by the two following structural theorems for finite-valued transducers.

(1) A finite-valued NGSM (NFT) M can be effectively decomposed into finitely many single-valued NGSMs (NFTs, respectively) $M_1, \ldots, M_N$ such that the transduction realized by M is the union of the transductions realized by $M_1, \ldots, M_N$ (Weber [W93]).

(2) A single-valued NGSM (NFT) M can be effectively transformed into an equivalent unambiguous NGSM (NFT, respectively) M' (Eilenberg [E74] and Schützenberger [Sch76], *see* Berstel [B79, Chapt. IV]).

In result (1), the integer N is *always* of exponential order. This is in the optimal range if the valuedness of M is exponential. Transducers with the latter property exist (Weber [W90]). Each machine M_i in (1) has double exponential size and can be computed in deterministic double exponential time. The machine M' in result (2) has exponential size, which is optimal in certain cases of M, and it can be computed in deterministic exponential time (Weber and Klemm [WK95], *see* Section 2).

The main result of this article (*see* Section 3) is the following theorem combining results (1) and (2).

(3) For any positive integer k, a k-valued NGSM (NFT) M can be effectively decomposed into k unambiguous NGSMs (NFTs, respectively) $M_1, \ldots, M_k$ such that the transduction realized by M is the union of the transductions realized by $M_1, \ldots, M_k$.

We want to point out that result (3) improves down to optimality the number of single-valued transducers in (1) and extends (2) from single-valued to k-valued transducers. Every machine M_i in (3) has double exponential size and can be computed in deterministic double exponential time where k appears in the second exponent each. Therefore, if the valuedness of M is of polynomial order, then this theorem yields a decomposition of M into an optimal number of unambiguous transducers, and each of them has about the same size as each of the – exponentially many – single-valued transducers provided by (1). For any fixed positive integer k result (3) states that a k-valued NGSM (NFT) M can be effectively transformed into an equivalent k-ambiguous NGSM (NFT, respectively) M' of double exponential size. In certain cases of M the size of M' is necessarily exponential (Leung [Le93]). In particular, the k-valued NGSMs (NFTs) and the k-ambiguous NGSMs (NFTs, respectively) realize the same class of transductions (*see* Section 3). Note that in general we cannot expect in result (3) that the transductions realized by $M_1, \ldots, M_k$ are pairwise disjoint (Lisovik [Li91]).

Because of reduction our main task will be to prove theorem (3) for NGSMs. Intuitively, we thus have to prove that a "difficult", *i.e.*, k-valued

NGSM M is equivalent to some effectively constructible "disjoint union" of "easy", *i.e.*, unambiguous NGSMs $M_1, \ldots, M_k$. We want to point out that one major problem for the machines $M_1, \ldots, M_k$ is that the model of a "disjoint union" does not allow any communication among them. Given an input word x, each M_i has to decide autonomously which of the values for x in M it should produce as its own value. In order to do so, the transducer M_i computes a "neighborhood" graph associated with x. The "minimal" vertices of the connected components of this graph represent all values for x in M. The machine M_i obtains its value from that minimal vertex having "rank" i in the neighborhood graph.

In order to specify in more detail the construction of the unambiguous NGSMs $M_1, \ldots, M_k$ in theorem (3) we need two main tools (*see* Section 2). The first one is a strengthening of result (1) where the single-valued NGSMs are replaced by unambiguous ones without deterioating size or complexity bounds. The second tool clarifies the notion of "neighborhood" used above.

Another method, apart from the above discussion about theorem (3), to compare k-valued and k-ambiguous transducers, for any fixed positive integer k, is to study their respective equivalence problems. The best procedure we know for deciding the equivalence of k-valued NFTs is derived from theorem (1) and requires deterministic double exponential time (Weber [W93]). In contrast to this, it is decidable in deterministic single exponential time whether or not two k-ambiguous NFTs are equivalent (Gurari and Ibarra [GI83]). Note that the first-mentioned procedure, deciding the equivalence of k-valued NFTs, does not take advantage of the fixed k. A first step to improve this procedure could be to provide in theorem (3) unambiguous transducers of single exponential size. Concerning the equivalence problem for k-ambiguous NFTs, it should be interesting to find a polynomial-time or -space algorithm. Equivalence problems for transducers are further treated in the surveys (Karhumäki [K87]) and Culik [C90]).

Result (1) remains true when the valuedness of a transducer is replaced by its length-degree (Weber [W92a]). It is an open problem whether a similar extension exists for theorem (3). We want to point out that such an extension would considerably improve the complexity of the best known algorithm for deciding the equivalence of NFTs having length-degree at most k, for any fixed positive integer k (Weber [W92a]). Result (1) also remains valid when transducers are replaced by bottom-up tree transducers (Seidl [Se94]). It is an open problem whether a similar extension exists for theorem (3). Finally, theorem (3) is used in order to show that, for any positive integer k, a certain $(k+1)$-valued distance automaton is not equivalent to any k-valued distance automaton (Weber [W94]).

1. DEFINITIONS AND NOTATIONS

1.1. General

The set of all integers is denoted by $\mathcal{Z}$. For any nonnegative integer m, the set $\{1, \ldots, m\}$ is denoted by $[m]$. For any integers i and j, the set $\{t \in \mathcal{Z} : i \leq t \leq j\}$ is denoted by $[i, j]$. For every set U the set of all subsets of U having cardinality 2 is denoted by $\binom{U}{2}$.

Let Δ be a nonempty, finite set. For every $z \in \Delta^*$ and $j \in [|z|]$, the jth letter of the word z is denoted by $z(j)$. Let $z_1, z_2 \in \Delta^*$, and let $j \in [\min\{|z_1|, |z_2|\}]$. We say that the words z_1 and z_2 *differ at position* j if $z_1(j)$ and $z_2(j)$ are distinct. We write $z_1 \sqsubseteq z_2$ if z_1 is a *prefix* of z_2, *i.e.*, $|z_1| \leq |z_2|$ and, for every $j \in [|z_1|]$, the letters $z_1(j)$ and $z_2(j)$ coincide.

The *free group* generated by Δ, denoted by $\mathrm{FG}(\Delta)$, is defined as the quotient of the free monoid $(\Delta \cup \Delta^{-1})^*$, where $\Delta^{-1} = \{b^{-1} : b \in \Delta\}$, by the congruence generated by the relations $bb^{-1} = b^{-1}b = \varepsilon$ for every $b \in \Delta$. A word $z \in (\Delta \cup \Delta^{-1})^*$ is *reduced* if it contains no factor of the form bb^{-1} or $b^{-1}b$ where $b \in \Delta$. It can be seen that every element of $\mathrm{FG}(\Delta)$ has a unique reduced representative in $(\Delta \cup \Delta^{-1})^*$ (*see* Lyndon and Schupp [LS77, Sect. I.1]). We can therefore identify in an obvious way $\mathrm{FG}(\Delta)$ with the set of reduced words in $(\Delta \cup \Delta^{-1})^*$. Let $z = b_1^{\gamma_1} \ldots b_m^{\gamma_m} \in (\Delta \cup \Delta^{-1})^*$ where $b_1, \ldots, b_m \in \Delta$ and $\gamma_1, \ldots, \gamma_m \in \{1, -1\}$. Then, the inverse of z, denoted by z^{-1}, is $b_m^{-\gamma_m} \ldots b_1^{-\gamma_1}$. The sets Δ^* and $(\Delta^{-1})^*$ are submonoids of $\mathrm{FG}(\Delta)$. For any nonnegative integer l the set $\{z \in \Delta^* : |z| \leq l\}$ is denoted by $\Delta^{\leq l}$ and the set $\{z \in \Delta^* \cup (\Delta^{-1})^* : |z| \leq l\}$ is denoted by $\Delta^{\leq \pm l}$. Let $z = z_1 \ldots z_m \in \Delta^*$ and $z' = z'_1 \ldots z'_m \in \Delta^*$ where $z_1, z'_1, \ldots, z_m, z'_m \in \Delta^*$. Then, z is a prefix of z' or z' is a prefix of z if and only if, for every $l \in [m]$, $z_l^{-1} \ldots z_1^{-1} z'_1 \ldots z'_l$ is in $\Delta^* \cup (\Delta^{-1})^*$.

Let G be a finite, undirected graph. For any vertex p of G we denote by $[p]_G$ the connected component of G to which p belongs, *i.e.*, the set of all vertices q of G being connected with p.

1.2. Transducers

Our model of a transducer is the *normalized finite transducer*, abbreviated NFT. Formally, an NFT is a 6-tuple $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ where

Q, Σ, and Δ denote nonempty, finite sets of states, input symbols, and output symbols, respectively, Q_I, $Q_F \subseteq Q$ denote sets of initial and final (or accepting) states, respectively, and δ is a finite subset of $Q \times (\Sigma \cup \{\varepsilon\}) \times \Delta^* \times Q$. Here, Σ is the input alphabet, Δ is the output alphabet, and δ is the transition relation. Each element of δ denotes a *transition*. In general, of course, the transducer M will be *nondeterministic*. We say that M is a *real-time transducer* or, by historic reasons, a *nondeterministic generalized sequential machine*, abbreviated NGSM, if δ is a finite subset of $Q \times \Sigma \times \Delta^* \times Q$. In this article we mainly deal with NGSMs. If δ is a subset of $Q \times (\Sigma \cup \{\varepsilon\}) \times \{\varepsilon\} \times Q$, then M is a *nondeterministic finite automaton with ε-moves*, abbreviated ε-NFA. If δ is a subset of $Q \times \Sigma \times \{\varepsilon\} \times Q$, then M is a *nondeterministic finite automaton*, abbreviated NFA. The latter definition is, of course, isomorphic to the usual one.

The mode of operation of M is described by paths. A *path* π (of length m) is a word

$$(q_1, x_1, z_1) \ldots (q_m, x_m, z_m)\, q_{m+1} \in (Q \times (\Sigma \cup \{\varepsilon\}) \times \Delta^*)^m \cdot Q$$

such that $(q_1, x_1, z_1, q_2), \ldots, (q_m, x_m, z_m, q_{m+1})$ are transitions. The path π leads from q_1 to q_{m+1}, consumes $x = x_1 \ldots x_m \in \Sigma^*$, produces $z = z_1 \ldots z_m \in \Delta^*$, and realizes $(x, z) \in \Sigma^* \times \Delta^*$. It is *accepting* if q_1 is an initial and q_{m+1} is a final state. It is a *cycle* if q_1 and q_{m+1} coincide. Whenever convenient we identify a transition (p, a, z, q) with the path $(p, a, z)\, q$ of length 1 and vice versa. We define $\hat{\delta}$ as the set of all $(p, x, z, q) \in Q \times \Sigma^* \times \Delta^* \times Q$ such that (x, z) is realized by some path leading from p to q. If M is real time, then δ equals $\hat{\delta} \cap Q \times \Sigma \times \Delta^* \times Q$. In this case we rename $\hat{\delta}$ by δ. If M is an ε-NFA, then $\hat{\delta}$ is a subset of $Q \times \Sigma^* \times \{\varepsilon\} \times Q$. Let $\pi_1 = \pi_1'\, q_1$ and $\pi_2 = \pi_2'\, q_2$ be paths in M leading from p_1 to q_1 and from p_2 to q_2, respectively. If q_1 and p_2 coincide, then we define the path $\pi_1 \circ \pi_2$ as $\pi_1'\, \pi_2'\, q_2$. Note that the operation "$\circ$" on paths is associative.

The *transduction* (or *relation*) *realized* by M, denoted by $T(M)$, is the set of pairs (in $\Sigma^* \times \Delta^*$) realized by the accepting paths in M. The *language recognized* by M, denoted by $L(M)$, is the domain of $T(M)$, *i.e.*, the set of words (in Σ^*) consumed by the accepting paths in M. Two NFTs are *equivalent* if the transductions realized by them coincide.

If $(x, z) \in \Sigma^* \times \Delta^*$ belongs to $T(M)$, then z is a *value* for x in M. The *valuedness* of $x \in \Sigma^*$ in M, abbreviated $\mathrm{val}_M(x)$, is the number of all different values for x. The *valuedness* of M, abbreviated $\mathrm{val}(M)$, is the supremum of the set $\{\mathrm{val}_M(x) : x \in \Sigma^*\}$. Note that, for a given $x \in \Sigma^*$, $\mathrm{val}_M(x)$ may be infinite (Weber [W90, Sect. 5]) whereas it is clearly finite if M is an NGSM. The *degree of ambiguity* of M, abbreviated $\mathrm{da}(M)$, is the minimal nonnegative integer k such that any $x \in \Sigma^*$ is consumed by at most k accepting paths or is infinite, depending on whether or not such a k exists. Evidently, $\mathrm{val}(M) \leq \mathrm{da}(M)$. Let k be a positive integer. The transducer M is *finite valued* (*k-valued, single valued*) if its valuedness is finite (at most k, at most 1, respectively). It is finitely ambiguous (*k-ambiguous, unambiguous*) if its degree of ambiguity is finite (at most k, at most 1, respectively). Whenever convenient we abbreviate "unambiguous NGSM" by UGSM and "unambiguous NFA" by UFA.

A state of M is *useful* if it appears on some accepting path. If all states of M are useful, then this machine is *trim*.

Let $M_0 = (Q_0, \Sigma, \Delta, \delta_0, Q_{I,0}, Q_{F,0})$ be another NFT. We define some local structural parameters of M and M_0. The first one, $\mathrm{diff}(\delta, \delta_0)$ denotes the minimal nonnegative integer k_1 such that, for all pairs $((p, a, z, q), (p', a, z', q'))$ of transitions in M and M_0 consuming the same $a \in \Sigma \cup \{\varepsilon\}$, $||z'| - |z||$ is at most k_1. We set $\mathrm{diff}(\delta) = \mathrm{diff}(\delta, \delta)$. The set of ε and of all words (in Δ^*) produced by the transitions of M is denoted by $\mathrm{im}(\delta)$. We set $\mathrm{iml}(\delta) = \max\{|z| : z \in \mathrm{im}(\delta)\}$.

The *size* of δ, denoted by $\|\delta\|$, is defined as 1 plus the sum of $1 + |z|$ over all transitions (p, a, z, q) of M. The *size* of M, denoted by $\|M\|$, is defined as $\#Q + \#\Sigma + \#\Delta + \|\delta\|$. Note that $\#\mathrm{im}(\delta) \leq \min\{\|\delta\|, \#(\Delta^{\leq \mathrm{iml}(\delta)})\}$, $\mathrm{diff}(\delta, \delta_0) \leq \max\{\mathrm{iml}(\delta), \mathrm{iml}(\delta_0)\}$, and $\mathrm{diff}(\delta) \leq \mathrm{iml}(\delta) \leq \|\delta\| - 1$. If M is an ε-NFA, then $\|\delta\| = 1 + \#\delta$.

Let $x = x_1 \ldots x_m \in \Sigma^*$ where $x_1, \ldots, x_m \in \Sigma \cup \{\varepsilon\}$. For any two paths $\pi = (q_1, x_1, z_1) \ldots (q_m, x_m, z_m)\, q_{m+1}$ in M and $\pi' = (q'_1, x_1, z'_1) \ldots (q'_m, x_m, z'_m)\, q'_{m+1}$ in M_0 both consuming x "in the same fashion" we define

$$\mathrm{diff}(\pi, \pi') = \max\{||z'_1 \ldots z'_l| - |z_1 \ldots z_l|| : 0 \leq l \leq m\}.$$

Note that $\mathrm{diff}(\pi, \pi')$ is at most $m \cdot \mathrm{diff}(\delta, \delta_0)$.

Let $\psi : Q_0 \to Q$ be some mapping. For any path $\pi = (s_1, x_1, z_1) \ldots (s_m, x_m, z_m) s_{m+1}$ in M_0 we define the word

$$\psi(\pi) = (\psi(s_1), x_1, z_1) \ldots (\psi(s_m), x_m, z_m)\, \psi(s_{m+1}).$$

Note that in general the word $\psi(\pi)$ is not a path in M. If $\psi(\pi)$ is a path in M, then it realizes the same pair of words as π. If moreover π'

is another path in M_0 such that $\psi(\pi')$ is a path in M, then the equality $\text{diff}(\psi(\pi), \psi(\pi')) = \text{diff}(\pi, \pi')$ holds.

2. MAIN TOOLS

In this section we prove the two following theorems.

THEOREM 2.1: *Let $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be a finite-valued NGSM. Then, there are $O(2^{\text{poly}\|M\|})$ many UGSMs $M_1, \ldots, M_N$ and UFAs $M'_1, \ldots, M'_N$ such that $T(M)$ equals $T(M_1) \cup \ldots \cup T(M_N)$ and, for every $i \in [N]$, M'_i recognizes $\Sigma^* \backslash L(M_i)$. Each of these new machines has size $O(2^{2^{\text{lin}\|M\|}})$ and can be computed in* DTIME $(2^{2^{\text{lin}\|M\|}})$. *The state sets of M_i and M'_i are independent of $i \in [N]$. Let Q_0 be the state set of $M_1, \ldots, M_N$. There is a mapping $\psi : Q_0 \to Q$ which maps any (accepting) path in M_i $(i \in [N])$ to an (accepting) path in M.*

THEOREM 2.2: *Let $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be an NGSM with n states, and let k be a positive integer. Assume that there are accepting paths $\pi_1, \ldots, \pi_{k+1}$ in M consuming the same word (in Σ^*) and producing the words $z_1, \ldots, z_{k+1} \in \Delta^*$, respectively, such that for any two distinct i_1, $i_2 \in [k+1]$ either* $\text{diff}(\pi_{i_1}, \pi_{i_2})$ *is greater than* $(n^{k+1} - 1) \cdot \text{diff}(\delta)$ *or z_{i_1} and z_{i_2} are distinct. Then, the valuedness of M is greater than k.*

Theorems 2.1 and 2.2 turn out to be the main tools in order to prove the main result of this article (Theorem 3.1) stating that a k-valued NGSM can be effectively decomposed into k UGSMs.

For every single-valued NGSM M with n states there is an equivalent UGSM M' having at most $n \cdot 2^{n-1}$ (at most 2^n) states and size at most $\|M\| \cdot 2^{n-1}$ (at most $\|M\|^5 \cdot 2^n$, respectively); the UGSM M' can be computed in DTIME $(2^{\text{lin}\|M\|})$ (Weber and Klemm [WK95, Prop. 2.1 and Thm. 2.3]). Using either of these results, it is not difficult to derive from (Weber [W93, Thms. 2.1-2.3]) a weaker version of Theorem 2.1 where the UGSMs $M_1, \ldots, M_N$ and the UFAs $M'_1, \ldots, M'_N$ each have size $O(2^{2^{2^{\text{lin}\|M\|}}})$ and can be computed in DTIME $(2^{2^{2^{\text{lin}\|M\|}}})$. Theorem 2.1 strengthens [W93, Thms. 2.1-2.3] by providing UGSMs $M_1, \ldots, M_N$ rather than single-valued NGSMs and UFAs $M'_1, \ldots, M'_N$ rather than NFAs without deterioating size or complexity bounds. In order to prove Theorem 2.1, we strengthen, modify, and combine the proofs of [WK95, Prop. 2.1] and of [W93, Thms. 2.1-2.3]. Theorem 2.2 has no special history and is proved by means of pumping methods.

In the remainder of this section we prove Theorems 2.1 and 2.2, successively. For the proof of Theorem 2.1 we need the following lemma which strengthens [WK95, Prop. 2.1].

LEMMA 2.3: *Let $M = (Q_1 \times Q_2, \Sigma, \Delta, \delta, Q_I, Q_F)$ be a single-valued NGSM such that the sequence of the Q_1-components of the states of any accepting path in M is uniquely determined by the word consumed by this path. Set $n_i = \# Q_i$ $(i = 1, 2)$. Then, there is an equivalent UGSM M' having at most $n_1 n_2 2^{n_2 - 1}$ states and size at most $\|M\| \cdot 2^{n_2-1}$. The UGSM M' can be computed in* DTIME $(\text{poly}(\|M\| \cdot 2^{n_2}))$.

Proof: Let $M = (Q_1 \times Q_2, \Sigma, \Delta, \delta, Q_I, Q_F)$, n_1, and n_2 be as in the lemma. Let us fix some total order on Q_2. We construct the NGSM $M' = (Q', \Sigma, \Delta, \delta', Q'_I, Q'_F)$ by setting

$$\begin{aligned} Q' &= \{(p, q, B) \in Q_1 \times Q_2 \times 2^{Q_2} : q \in B\}, \\ Q'_I &= \{(p, q, B) \in Q' : B = \{q' \in Q_2 : (p, q') \in Q_I\}\}, \\ Q'_F &= \{(p, q, B) \in Q' : q = \min\{q' \in B : (p, q') \in Q_F\}\}, \end{aligned}$$

and

$$\begin{aligned} \delta' = \{&((p, q, B), a, z, (p', q', B')) \in Q' \times \Sigma \times \Delta^* \times Q' : \\ &((p, q), a, z, (p', q')) \in \delta, \; q = \min\{s \in B : \text{for some} \\ &z' \in \Delta^*, ((p, s), a, z', (p', q')) \in \delta\}, \text{and } B' = \{s' \in Q_2 : \\ &\text{for some } s \in B \text{ and } z' \in \Delta^*, ((p, s), a, z', (p', s')) \in \delta\}\}. \end{aligned}$$

Obviously, $\# Q' \leq n_1 n_2 2^{n_2-1} = \# Q \cdot 2^{n_2-1}$, $\|\delta'\| \leq \|\delta\| \cdot 2^{n_2-1}$, and $\|M'\| \leq \|M\| \cdot 2^{n_2-1}$. The machine M' can be computed in DTIME (poly $(\|M\| \cdot 2^{n_2})$). Any accepting path in M' realizing some $(x, z) \in \Sigma^* \times \Delta^*$, when restricting its states to their $Q_1 \times Q_2$-components, yields an accepting path in M also realizing (x, z). Thus, $T(M')$ is included in $T(M)$. On the other hand, it is easy to show that $L(M)$ is included in $L(M')$. Since M is single valued, this altogether implies that M and M' are equivalent. It remains to be shown that M' is unambiguous.

Let $x \in L(M')$, and let π be an accepting path in M' consuming x. Since M' is single valued, the path π is uniquely determined by the sequence of its states and by x. Restricting the states of π to their $Q_1 \times Q_2$-components, the assumption of the lemma yields that the Q_1-components of the states of π are uniquely determined by x. By going through π from left to right, it is easy to see that the 2^{Q_2}-components of the states of π are uniquely determined by the Q_1-components and by x. By going through π from right to left, one observes that the Q_2-components of the states of π are uniquely determined by the Q_1- and 2^{Q_2}-components and by x. Thus, the path π is the only accepting path in M' which consumes the word x. $\square$

We want to look at two special cases of Lemma 2.3. The first one is that $n_1 = 1$. Then, the uniqueness assumption trivially holds true and can be therefore omitted; the lemma and its proof coincide with [WK95, Prop. 2.1]. The second special case is as follows. There is a given subset Q_2' of 2^{Q_2} such that, for every $p \in Q_1$, the set $B = \{q' \in Q_2 \,:\, (p,\, q') \in Q_I\}$ belongs to Q_2' and, for every $a \in \Sigma$, p, $p' \in Q_1$, and $B \in Q_2'$, the set

$$B' = \{s' \in Q_2 \,:\, \text{for some } s \in B \text{ and } z' \in \Delta^*,\\ ((p,\, s),\, a,\, z',\, (p',\, s')) \in \delta\}$$

belongs to Q_2'. Then, we observe that every useful state of the UGSM M' is in $Q_1 \times Q_2 \times Q_2'$. Using this fact, it is straightforward to replace M' by an equivalent UGSM M'' with state set $Q'' = \{(p,\, q,\, B) \in Q_1 \times Q_2 \times Q_2' \,:\, q \in B\}$ and size at most $\|M\| \cdot \#Q_2'$. The machine M'' can be computed in DTIME (poly $(\|M\| \cdot \#Q_2')$).

Proof of Theorem 2.1: Let $M = (Q,\, \Sigma,\, \Delta,\, \delta,\, Q_I,\, Q_F)$ be a finite-valued NGSM with n states. We may assume that M is trim. The set of accepting paths in M is denoted by Π. Our proof of Theorem 2.1 consists of six steps. The first four steps follow almost exactly the main lines of the proof of [W93, Thms. 2.1 and 2.2]. The last two steps are applications of Lemma 2.3.

(1) We define a set S of potential path specifications. The set S has cardinality $O\,(2^{\text{poly}\|M\|})$ and can be computed in DTIME $(2^{\text{poly}\|M\|})$.

(2) We define a mapping $\varphi \,:\, \Pi \to 2^S \backslash \{\emptyset\}$ such that every $\sigma \in \varphi\,(\pi)$ acts as a specification of the path $\pi \in \Pi$ and the following holds. If π, $\pi' \in \Pi$ realize $(x,\, z)$, $(x,\, z') \in \Sigma^* \times \Delta^*$, respectively, and if $\varphi\,(\pi) \cap \varphi\,(\pi')$ is nonempty, then z and z' coincide.

(3) For every $\sigma \in S$ we construct an NGSM $\bar{M}_\sigma$ realizing the set of all $(x,\, z) \in \Sigma^* \times \Delta^*$ being realized by some $\pi \in \Pi$ with $\sigma \in \varphi\,(\pi)$. These new machines each have size $O\,(2^{2^{\text{lin}\|M\|}})$ and can be computed in DTIME $(2^{2^{\text{lin}\|M\|}})$. Their state sets coincide being of the form $Q^{(1)} \times Q^{(2)}$ for some sets $Q^{(1)}$ and $Q^{(2)}$ of cardinality $O\,(2^{2^{\text{lin}\|M\|}})$ and $O\,(\text{poly}\|M\|)$, respectively. The sequence of the $Q^{(1)}$-components of the states of any accepting path in $\bar{M}_\sigma$ $(\sigma \in S)$ is uniquely determined by the word consumed

by this path. We define a mapping $\psi_2 : Q^{(1)} \times Q^{(2)} \to Q$ which maps any transition of $\bar{M}_\sigma$ $(\sigma \in S)$ to a transition of M and any initial (final) state of $\bar{M}_\sigma$ $(\sigma \in S)$ to an initial (final, respectively) state of M.

(4) For every $\sigma \in S$ we construct an NFA $\bar{M}'_\sigma$ recognizing the set of all $x \in \Sigma^*$ such that there is no $\pi \in \Pi$ consuming x with $\sigma \in \varphi(\pi)$. These new automata each have size $O(2^{2^{\text{lin}\|M\|}})$ and can be computed in DTIME $(2^{2^{\text{lin}\|M\|}})$. Their state sets coincide being of the form $Q^{(1)} \times Q^{(3)}$ where $Q^{(3)} = 2^{Q^{(2)}} \times [3]$. The sequence of the $Q^{(1)}$-components of the states of any accepting path in $\bar{M}'_\sigma$ $(\sigma \in S)$ is uniquely determined by the word consumed by this path. Moreover, each new automaton meets the second special case of Lemma 2.3 with a given subset of $2^{Q^{(3)}}$ of cardinality at most $(1 + 2^{\#Q^{(2)}})^3$.

From steps (1)-(4) it follows that $T(M)$ equals $\bigcup_{\sigma \in S} T(\bar{M}_\sigma)$, each NGSM $\bar{M}_\sigma$ $(\sigma \in S)$ is single valued, and each NFA $\bar{M}'_\sigma$ $(\sigma \in S)$ recognizes $\Sigma^* \backslash L(\bar{M}_\sigma)$.

(5) For every $\sigma \in S$ we transform the single-valued NGSM $\bar{M}_\sigma$ into an equivalent UGSM $\tilde{M}_\sigma$. These new machines each have size $O(2^{2^{\text{lin}\|M\|}})$ and can be computed in DTIME $(2^{2^{\text{lin}\|M\|}})$. Their state sets coincide being, say, $\tilde{Q}$. We define a mapping $\psi_1 : \tilde{Q} \to Q^{(1)} \times Q^{(2)}$ which maps any transition of $\tilde{M}_\sigma$ $(\sigma \in S)$ to a transition of $\bar{M}_\sigma$ and any initial (final) state of $\tilde{M}_\sigma$ $(\sigma \in S)$ to an initial (final, respectively) state of $\bar{M}_\sigma$.

(6) For every $\sigma \in S$ we transform the NFA $\bar{M}'_\sigma$ into an equivalent UFA $\tilde{M}'_\sigma$. These automata each have size $O(2^{2^{\text{lin}\|M\|}})$ and can be computed in DTIME $(2^{2^{\text{lin}\|M\|}})$. Their state sets coincide.

Altogether, steps (1)-(6) prove the theorem. Note that $N = \#S$. The UGSMs $\tilde{M}_\sigma$ $(\sigma \in S)$ are playing the role of $M_1, \ldots, M_N$ and the UFAs $\tilde{M}'_\sigma$ $(\sigma \in S)$ are playing the role of $M'_1, \ldots, M'_N$. The mapping ψ of the theorem is obtained by concatenating ψ_1 and ψ_2.

Execution of step (1): Let us first introduce some notations. A state $p \in Q$ is *strongly connected* with a state $q \in Q$ if there are paths in M leading from p to q and from q to p, respectively. A class with respect to the so-defined equivalence relation on Q is a *strongly connected component* of M. Let us fix an order $Q_1, Q_2, \ldots, Q_k$ of the strongly connected components of M such that if $\delta \cap Q_i \times \Sigma^* \times \Delta^* \times Q_j$ is nonempty for some $i, j \in [k]$ then $i \leq j$. For each $i \in \{1, \ldots, k\}$ set $n_i = \#Q_i$.

We define a set S of potential path specifications by setting

$$S = \bigcup_{l \geq 0} \bigcup_{1 \leq i_0 < \ldots < i_l \leq k} \bigcup_{0 < j_1 < \ldots < j_l < 2^{n+1}} (Q_I \cap Q_{i_0})$$
$$\times \prod_{\lambda=1}^{l} \Bigg[\{j_\lambda\} \times Q_{i_{\lambda-1}}$$
$$\times \Bigg(\mathrm{im}\,(\delta) \cup \Bigg\{ z \in \Delta^* \,:\, |z| \leq \Bigg(n^2 \sum_{i=i_{\lambda-1}}^{i_\lambda} n_i - 1 \Bigg) \cdot \mathrm{diff}\,(\delta) \Bigg\}$$
$$\cup \Bigg\{ (t,\, \tilde{p},\, \tilde{q}) \in \mathcal{Z} \times Q^2 \,:\, |t| \leq \Bigg(n^2 \sum_{i=i_{\lambda-1}}^{i_\lambda} n_i - 1 \Bigg) \cdot \mathrm{diff}\,(\delta),$$
$$\tilde{p} \text{ is strongly connected with } \tilde{q} \text{ (in } M) \Bigg\} \Bigg) \times Q_{i_\lambda} \Bigg] \times (Q_F \cap Q_{i_l}).$$

The set S is defined exactly as in the proof of [W93, Thm. 2.1] where it was shown that this set has cardinality $O\,(2^{\mathrm{poly}\|M\|})$. It was further shown in the proof of [W93, Thm. 2.3] that the set S can be computed in $\mathrm{DTIME}\,(2^{\mathrm{poly}\|M\|})$.

Execution of step (2): Let us first introduce some notations for a word $x \in \Sigma^*$. Let $x_1, \ldots, x_m \in \Sigma$ such that $x = x_1 \ldots x_m$. Let $\mu \in \{0, \ldots, m\}$. We define the sets

$$\mathrm{att}\,(x,\, \mu) = \{s \in Q \,:\, \text{for some } r \in Q_I \text{ and } z \in \Delta^*,\ (r,\, x_1 \ldots x_\mu,\, z,\, s) \in \delta\}$$

and

$$\mathrm{der}\,(x,\, \mu) = \{r \in Q \,:\, \text{for some } s \in Q_F \text{ and } z \in \Delta^*,\ (r,\, x_{\mu+1} \ldots x_m,\, z,\, s) \in \delta\}.$$

The sets $\mathrm{att}\,(x,\, \mu)$ and $\mathrm{der}\,(x,\, \mu)$ denote the sets of states attainable from Q_I with $x_1 \ldots x_\mu$ and derivable to Q_F with $x_{\mu+1} \ldots x_m$, respectively. We define the set $\mathrm{set}(x,\, \mu)$ as $\mathrm{att}\,(x,\, \mu) \cap \mathrm{der}\,(x,\, \mu)$.

Let us fix a total order, say, "$\leq$" on 2^Q. Let "$<$" be the corresponding nonreflexive relation on 2^Q. Given $x = x_1 \ldots x_m \in \Sigma^*$, consider the uniquely determined sets $A_1, \ldots, A_{d+1} \in 2^Q$, and words $y_1, \ldots, y_d \in \Sigma^*$ such that $x = y_1 \ldots y_d$, $d+1$ is even, and (a)-(c) hold true. (a) For all $j = 1, \ldots, d+1$, $A_j = \mathrm{set}\,(x,\, |y_1 \ldots y_{j-1}|)$. (b) For all $j = 1, \ldots, d$, if j is odd then $|(y_1 \ldots y_{j-1})\, y_j| = \max\{\mu \in \{0, \ldots, m\} \,:\, A_j = \mathrm{set}\,(x,\, \mu)\}$. (c) For all $j = 1, \ldots, d$, if j is even then y_j is in Σ. Thus, for each odd $j \in [d]$, the set A_{j+1} is the "last occurrence" of A_j in the sequence $\text{set}(x, 0), \text{set}(x, 1), \ldots, \text{set}(x, m)$. Clearly, $d \leq 2^{n+1} - 1$. Let us also consider the uniquely determined sets $A_{d+2}, \ldots, A_{2^{n+1}} \in 2^Q$ such that $\{A_1, \ldots, A_{d+1}\} \cap \{A_{d+2}, \ldots, A_{2^{n+1}}\}$ is empty and

$$A_{d+2} = A_{d+3} < A_{d+4} = A_{d+5} < \ldots < A_{2^{n+1}-1} = A_{2^{n+1}}.$$

Note that if $x \notin L(M)$ then $d = 1$, $A_1 = A_2 = \emptyset$, $\emptyset \notin \{A_3, \ldots, A_{2^{n+1}}\}$, and $A_3 = A_4 < A_5 = A_6 < \ldots < A_{2^{n+1}-1} = A_{2^{n+1}}$.

Assume that $\pi \in \Pi$ is an accepting path consuming x and producing some $z \in \Delta^*$. We are going to define the set $\varphi(\pi) \in 2^S$ of specifications of the path π.

Consider the uniquely determined paths $\pi_1, \ldots, \pi_d$ and the uniquely determined words $z_1, \ldots, z_d \in \Delta^*$ and states $p'_1, q'_1, \ldots, p'_d, q'_d \in Q$ such that $\pi = \pi_1 \circ \ldots \circ \pi_d$ and, for each $j \in [d]$, π_j realizes (y_j, z_j) and leads from p'_j to q'_j. By construction, $z = z_1 \ldots z_d$, $p'_1 \in A_1 \subseteq Q_I$, $q'_{j-1} = p'_j \in A_j$ $(j = 2, \ldots, d)$, $q'_d \in A_{d+1} \subseteq Q_F$, and $\{z_j : j \in [d], j \text{ even}\} \subseteq \text{im}(\delta)$. We define the set $J = \{j \in [d] : p'_j$ is not strongly connected with $q'_j\}$. Note that $\#J \leq k - 1$. Let $l \in \{0, \ldots, k-1\}$ and $1 \leq j_1 < \ldots < j_l \leq d$ so that $J = \{j_1, \ldots, j_l\}$. Let $1 \leq i_0 < i_1 < \ldots < i_l \leq k$ so that $p'_1 \in Q_{i_0}$, $p'_{j_\lambda} \in Q_{i_{\lambda-1}}$, $q'_{j_\lambda} \in Q_{i_\lambda}$ $(\lambda = 1, \ldots, l)$, and $q'_d \in Q_{i_l}$.

Let j be a positive integer. Let π_0 be any path in M realizing some $(y_0, z_0) \in \Sigma^* \times \Delta^*$ and leading from some state $p \in Q$ to some state $q \in Q$. Let $1 \leq i(p) \leq i(q) \leq k$ so that $p \in Q_{i(p)}$ and $q \in Q_{i(q)}$. Set $\tilde{n} = \sum_{i=i(p)}^{i(q)} n_i$. We define $\varphi_j(\pi_0) \in 2^{\Delta^* \cup \mathcal{Z} \times Q^2}$ by setting $\varphi_j(\pi_0) = \{z_0\}$ if j is even or $|z_0| \leq (n^2 \tilde{n} - 1) \cdot \text{diff}(\delta)$, and $\varphi_j(\pi_0) = \{(t, \tilde{p}, \tilde{q}) \in \mathcal{Z} \times Q^2 : |t| \leq (n^2 \tilde{n} - 1) \cdot \text{diff}(\delta)$, $\tilde{p}$ is strongly connected with $\tilde{q}$ (in M), and there is a path $\tilde{\pi}$ in M realizing $(y_0, \tilde{z})$ for some $\tilde{z} \in \Delta^*$ and leading from $\tilde{p}$ to $\tilde{q}$ such that $\text{diff}(\pi_0, \tilde{\pi}) \leq (n^3 - 1) \cdot \text{diff}(\delta)$ and $t = |z_0| - |\tilde{z}|\}$, otherwise.

We are now ready to define $\varphi(\pi) \in 2^S$ by setting

$$\varphi(\pi) = \{p'_1\} \times \prod_{\lambda=1}^{l} [\{j_\lambda\} \times \{p'_{j_\lambda}\} \times \varphi_{j_\lambda}(\pi_{j_\lambda}) \times \{q'_{j_\lambda}\}] \times \{q'_d\}.$$

The mapping $\varphi : \Pi \to 2^S$ is defined exactly as in the proof of [W93, Thm. 2.1]. It was shown in this proof that for every $\pi \in \Pi$ the set $\varphi(\pi)$ is nonempty and the following holds. If $\pi, \pi' \in \Pi$ realize (x, z), $(x, z') \in \Sigma^* \times \Delta^*$, respectively, and if $\varphi(\pi) \cap \varphi(\pi')$ is nonempty, then z and z' coincide.

Execution of step (3): Let $\sigma \in S$. We are going to construct an NFT $M_\sigma = (Q_\sigma, \Sigma, \Delta, \delta_\sigma, Q_{I,\sigma}, Q_{F,\sigma})$, which realizes the set of all $(x, z) \in \Sigma^* \times \Delta^*$ being realized by some $\pi \in \Pi$ with $\sigma \in \varphi(\pi)$ and which has the property that any path in M_σ consuming ε also produces ε. It is easy to see that M_σ is equivalent to the NGSM $\bar{M}_\sigma = (Q_\sigma, \Sigma, \Delta, \bar{\delta}_\sigma, \bar{Q}_{I,\sigma}, Q_{F,\sigma})$ where

$$\bar{Q}_{I,\sigma} = \{q \in Q_\sigma \,:\, \text{for some } p \in Q_{I,\sigma},\ (p,\, \varepsilon,\, \varepsilon,\, q) \in (\hat{\delta}_\sigma)\}$$

and

$$\bar{\delta}_\sigma = \{(p,\, a,\, z,\, q) \in Q_\sigma \times \Sigma \times \Delta^* \times Q_\sigma \,:\, \text{for some } r \in Q_\sigma,\ (p,\, a,\, z,\, r) \in \delta_\sigma \text{ and } (r,\, \varepsilon,\, \varepsilon,\, q) \in (\hat{\delta}_\sigma)\}.$$

Having constructed M_σ, we will observe that it realizes the above transduction and that $\bar{M}_\sigma$ has the other properties requested by step (3).

Let $l \geq 0$, $1 \leq i_0 < \ldots < i_l \leq k$ and $0 < j_1 < \ldots < j_l < 2^{n+1}$ such that $\sigma = (q_I, (j_1, p'_{j_1}, \sigma_{j_1}, q'_{j_1}), \ldots, (j_l, p'_{j_l}, \sigma_{j_l}, q'_{j_l}), q_F)$ where $q_I \in Q_I \cap Q_{i_0}$, $p'_{j_\lambda} \in Q_{i_{\lambda-1}}$, $q'_{j_\lambda} \in Q_{i_\lambda}$ $(\lambda = 1, \ldots, l)$, and $q_F \in Q_F \cap Q_{i_l}$. For each $j = j_\lambda \in \{j_1, \ldots, j_l\}$ set $\tilde{n}_j = \sum_{i=i_{\lambda-1}}^{i_\lambda} n_i$. Define $J = \{j_1, \ldots, j_l\}$, $J_1 = \{j \in J \,:\, \sigma_j \in \Delta^*\}$, and $J_2 = \{j \in J \,:\, \sigma_j \in \mathcal{Z} \times Q^2\}$. Note that $J = J_1 \dot{\cup} J_2$. Let $j \in J_1$, and let $z_j \in \Delta^*$ such that $z_j = \sigma_j$. Then, $z_j \in \operatorname{im}(\delta)$ or $|z_j| \leq (n^2 \tilde{n}_j - 1) \cdot \operatorname{diff}(\delta)$. Let $j \in J_2$, and let $t_j \in \mathcal{Z}$ and $\tilde{p}_j, \tilde{q}_j \in Q$ such that $(t_j, \tilde{p}_j, \tilde{q}_j) = \sigma_j$. Then, $|t_j| \leq (n^2 \tilde{n}_j - 1) \cdot \operatorname{diff}(\delta)$ and $\tilde{p}_j$ is strongly connected with $\tilde{q}_j$.

By construction of the mapping φ we know that the following holds. If, for some $j \in J_2$, j is even or if, for some $j \in J_1$, j is odd and $|z_j| > (n^2 \tilde{n}_j - 1) \cdot \operatorname{diff}(\delta)$ or j is even and $z_j \notin \operatorname{im}(\delta)$, then σ does not belong to $\varphi(\Pi)$. In this case we can select M_σ arbitrarily so that $T(M_\sigma) = \emptyset$ and $\bar{M}_\sigma$ has the other properties requested by step (3). Let us therefore assume that, for every $j \in J_2$, j is odd and that, for every $j \in J_1$, either j is odd and $|z_j| \leq (n^2 \tilde{n}_j - 1) \cdot \operatorname{diff}(\delta)$ or j is even and $z_j \in \operatorname{im}(\delta)$.

Instead of constructing the NFT M_σ in detail, we explain the desired mode of operation of an arbitrary accepting path π_σ in this machine. Assume that the path π_σ realizes $(x, z) \in \Sigma^* \times \Delta^*$. Let $x_1, \ldots, x_m \in \Sigma$ so that $x = x_1 \ldots x_m$. The reader may recall from step (2) the definition of the sets $\operatorname{att}(x, \mu)$, $\operatorname{der}(x, \mu)$, and $\operatorname{set}(x, \mu)$ $(\mu = 0, \ldots, m)$, the sets $A_1, \ldots, A_{2^{n+1}} \in 2^Q$, and the words $y_1, \ldots, y_d \in \Sigma^*$. In particular, $x = y_1 \ldots y_d$, $d + 1$ is even, and $d \leq 2^{n+1} - 1$.

The path π_σ consists of five components that correspond to five components of Q_σ, the state set of M_σ. Let $Q_\sigma = Q^{(1)} \times Q^{(2)}$ where $Q^{(1)}$ makes up the first four components of Q_σ and $Q^{(2)}$ denotes the fifth component of Q_σ. Roughly spoken, the $Q^{(1)}$-components of π_σ provide the index $j \in [d]$ of the word y_j currently consumed by π_σ. These components behave independently of σ and z. The $Q^{(2)}$-component of π_σ guesses an accepting path $\pi \in \Pi$ realizing (x, z) and uses the "current index" j provided by the $Q^{(1)}$-components in order to verify "on line" that σ belongs to $\varphi(\pi)$. Note that π_σ inherits its output word z from π.

The first component of π_σ constantly contains $(A_1, \ldots, A_{2^{n+1}}) \in (2^Q)^{2^{n+1}}$. The tuple $(A_1, \ldots, A_{2^{n+1}})$ is guessed at the beginning of π_σ. We want to point out that the sets $A_{d+2}, \ldots, A_{2^{n+1}}$ are only needed in order to make the state set Q_σ "well typed".

The next three components of π_σ drive a nondeterministic process, which verifies the correctness of the sets $A_1, \ldots, A_{2^{n+1}}$ and uses them in order to provide the index $j \in [d]$ of the word y_j currently consumed by π_σ. Assume that, for some $\mu \in \{0, \ldots, m\}$, π_σ has consumed the prefix $x_1 \ldots x_\mu$ of x. Then, the second (deterministic) and third (nondeterministic) components of π_σ contain the sets $\mathrm{att}(x, \mu)$ and $\mathrm{der}(x, \mu)$, respectively.

The fourth component of π_σ contains some $(j, \alpha) \in [d+1] \times [3]$ so that the following holds. If $\alpha = 1$, then either $j \leq d$ and π_σ can, after one transition realizing $(\varepsilon, \varepsilon)$, begin to consume the letters of y_j or π_σ guesses that $j = d+1$ and accepts. In the latter case, of course, j is even, $j > j_l$, and $A_{j+1} = A_{j+2} < A_{j+3} = A_{j+4} < \ldots < A_{2^{n+1}-1} = A_{2^{n+1}}$. If $\alpha \in \{2, 3\}$, then $j \leq d$. If $\alpha = 2$, then j is even and π_σ is ready to consume the only letter of y_j. If $\alpha = 3$ and j is even, then y_j has been completely consumed. If $\alpha = 3$ and j is odd, then either π_σ is ready to consume the next letter of y_j or y_j has been completely consumed, depending on the guess of π_σ. Whenever all letters of a word y_j have been consumed, the path π_σ increments j by 1 on a transition realizing $(\varepsilon, \varepsilon)$. At the beginning of π_σ its fourth component contains (1, 1). The distinction between the values 2 and 3 for α is needed in order to ensure that $|y_j| = 1$ for all even $j \in [d]$.

The second and third components of π_σ are used in order to check that, for each $j \in [d+1]$, $A_j = \mathrm{set}(x, |y_1 \ldots y_{j-1}|)$ and that, for each odd $j \in [d]$, A_{j+1} is the "last occurrence" of A_j in the sequence $\mathrm{set}(x, 0)$, $\mathrm{set}(x, 1), \ldots, \mathrm{set}(x, m)$. Therefore, these components contribute to the verification of the first and the fourth component of π_σ.

We observe from the mode of operation of π_σ that the sequence of the $Q^{(1)}$-components of the states of π_σ and the sequence of words consumed by its transitions are uniquely determined by x. According to the construction of $\bar{M}_\sigma$ from M_σ this implies that the sequence of the $Q^{(1)}$-components of the states of any accepting path in $\bar{M}_\sigma$ is uniquely determined by the word consumed by this path.

For the fifth component of π_σ we first of all need, for every $j \in J$, an NGSM $M_{\sigma,j}$ realizing the transduction

$$T_{\sigma,j} = \{(y_0, z_0) \in \Sigma^* \times \Delta^* : (y_0, z_0) \text{ is realized by some path } \pi_0 \text{ in } M \text{ leading from } p'_j \textit{ to } q'_j \text{ so that } \sigma_j \in \varphi_j(\pi_0)\}.$$

Informally spoken, an accepting path in $M_{\sigma,j}$ simply guesses (in its first component) a path π_0 in M leading from p'_j to q'_j, realizes the same $(y_0, z_0) \in \Sigma^* \times \Delta^*$ as π_0, and verifies (on its three other components) that σ_j belongs to $\varphi_j(\pi_0)$. The verification procedure directly arises from the definition of φ_j in step (2). The detailed construction of $M_{\sigma,j}$ is given in the proof of [W93, Thm. 2.1]. The fifth component of Q_σ, $Q^{(2)}$, is set to the state set of $M_{\sigma,j}$.

The fifth component of π_σ verifies that, for some $\pi \in \Pi$ realizing (x, z), σ belongs to $\varphi(\pi)$. Following the definition of the mapping φ this component operates as follows. For every $\lambda \in \{0, \ldots, l\}$, while π_σ consumes the words $y_{j_\lambda+1}, \ldots, y_{j_{\lambda+1}-1}$, successively, it guesses and verifies (on the first subcomponent of $Q^{(2)}$) a path in M consuming $y_{j_\lambda+1} \cdots y_{j_{\lambda+1}-1}$, producing some words $z_{j_\lambda+1}, \ldots, z_{j_{\lambda+1}-1} \in \Delta^*$, successively, and leading from q'_{j_λ} to $p'_{j_{\lambda+1}}$, where we set $j_0 = 0$, $j_{l+1} = d+1$, $q'_0 = q_I$, and $p'_{d+1} = q_F$. For every $j = j_\lambda \in J\,(\lambda \in [l])$, while π_σ consumes the word y_j, this component guesses and verifies an accepting path in $M_{\sigma,j}$ consuming y_j and producing some word $z_j \in \Delta^*$. The index $j \in [d]$ of the word y_j currently consumed by π_σ is read from its fourth component. The path π_σ inherits its output word $z = z_1 \ldots z_d \in \Delta^*$ from the combination of the above paths.

From the description of the mode of operation of the path π_σ given above we conclude (informally) that the NFT M_σ realizes the correct transduction. Moreover, every transition of M_σ consuming ε also produces ε. The detailed construction of M_σ is given in the proof of [W93, Thm. 2.1]. Our only modification is in the definition of the set of final states where we add the condition that $A_{j+1} = A_{j+2} < A_{j+3} = A_{j+4} < \ldots < A_{2^{n+1}-1} = A_{2^{n+1}}$. Therefore, we can conclude (formally) from [W93, Thms. 2.1 and 2.3] that M_σ realizes the correct transduction and that the size of $\bar{M}_\sigma$ and the time complexity for its computation are bounded as desired. Note that the purpose of our modification is to distinguish certain accepting paths in M_σ from other possible accepting paths.

The state set $Q_\sigma = Q^{(1)} \times Q^{(2)}$ of M_σ and $\bar{M}_\sigma$ is independent of σ. The detailed construction of M_σ yields that

$$Q^{(1)} = (2^Q)^{2^{n+1}} \times 2^Q \times 2^Q \times ([2^{n+1}] \times [3])$$

and

$$Q^{(2)} = Q \times Q \times [-(n^3 - 1) \cdot \mathrm{diff}\,(\delta),\ (n^3 - 1) \cdot \mathrm{diff}\,(\delta)] \times [0,\ \max\{\mathrm{iml}\,(\delta),\ 1 + (n^3 - 1) \cdot \mathrm{diff}\,(\delta)\}].$$

Note that $Q^{(1)}$ and $Q^{(2)}$ have cardinality $O\,(2^{2^{\mathrm{lin}\|M\|}})$ and $O\,(\mathrm{poly}\|M\|)$, respectively.

We define the mapping $\psi_2 : Q^{(1)} \times Q^{(2)} \to Q$ as the projection to the first Q-subcomponent of the $Q^{(2)}$-component. We observe from the detailed construction of M_σ that for any transition $(r,\ a,\ z,\ s)$ of M_σ either $(a,\ z) = (\varepsilon,\ \varepsilon)$ and $\psi_2\,(r) = \psi_2\,(s)$ or $a \in \Sigma$ and $(\psi_2\,(r),\ a,\ z,\ \psi_2\,(s))$ is a transition of M. According to the construction of $\bar{M}_\sigma$ from M_σ, this implies that ψ_2 maps any transition of $\bar{M}_\sigma$ to a transition of M. Moreover, we observe that ψ_2 maps any initial (final) state of $\bar{M}_\sigma$ to an initial (final, respectively) state of M.

Execution of step (4): Let $\sigma \in S$. Following the main lines of the construction of M_σ in step (3), we are going to determine an ε-NFA $M'_\sigma = (Q'_\sigma,\ \Sigma,\ \Delta,\ \delta'_\sigma,\ Q'_{I,\sigma},\ Q'_{F,\sigma})$ which recognizes the set of all $x \in \Sigma^*$ such that there is no $\pi \in \Pi$ consuming x with $\sigma \in \varphi\,(\pi)$. It is easy to see that M'_σ is equivalent to the NFA $\bar{M}'_\sigma = (Q'_\sigma,\ \Sigma,\ \Delta,\ \bar{\delta}'_\sigma,\ \bar{Q}'_{I,\sigma},\ Q'_{F,\sigma})$ where

$$\bar{Q}'_{I,\sigma} = \{q \in Q'_\sigma\ :\ \text{for some}\ p \in Q'_{I,\sigma},\ (p,\ \varepsilon,\ \varepsilon,\ q) \in (\hat{\delta}'_\sigma)\}$$

and

$$\bar{\delta}'_\sigma = \{(p,\ a,\ \varepsilon,\ q) \in Q'_\sigma \times \Sigma \times \{\varepsilon\} \times Q'_\sigma\ :\ \text{for some}\ r \in Q'_\sigma,\ (p,\ a,\ \varepsilon,\ r) \in \delta'_\sigma\ \text{and}\ (r,\ \varepsilon,\ \varepsilon,\ q) \in (\hat{\delta}'_\sigma)\}.$$

Having constructed M'_σ, we will observe that it recognizes the above language and that $\bar{M}'_\sigma$ has the other properties requested by step (4).

Let $\sigma = (q_I,\ (j_1,\ p'_{j_1},\ \sigma_{j_1},\ q'_{j_1}),\ \ldots,\ (j_l,\ p'_{j_l},\ \sigma_{j_l},\ q'_{j_l}),\ q_F) \in S$ and J, J_1, J_2, $\tilde{n}_j\ (j \in J)$, $z_j\ (j \in J_1)$ and $(t_j,\ \tilde{p}_j,\ \tilde{q}_j)\ (j \in J_2)$ be given as in step (3). As in step (3) we also assume here that, for every $j \in J_2$, j is odd and that, for every $j \in J_1$, either j is odd and $|z_j| \leq (n^2 \tilde{n}_j - 1) \cdot \operatorname{diff}(\delta)$ or j is even and $z_j \in \operatorname{im}(\delta)$.

We recall from step (3) the NGSM $M_{\sigma, j}$ $(j \in J)$ with state set $Q^{(2)}$ realizing the transduction $T_{\sigma, j}$. Let $j \in J$. Using the well-known subset construction we obtain from $M_{\sigma, j}$ an NFA $M'_{\sigma, j}$ which is in fact deterministic and which recognizes the language $\Sigma^* \backslash L(M_{\sigma, j})$. The state set of $M'_{\sigma, j}$ is $2^{Q^{(2)}}$. The detailed construction of $M'_{\sigma, j}$ is given in the proof of [W93, Thm. 2.2]. According to the definition of $T_{\sigma, j}$, the NFA $M'_{\sigma, j}$ recognizes the set of all $y_0 \in \Sigma^*$ such that there is no path π_0 in M consuming y_0 and leading from p'_j to q'_j with $\sigma_j \in \varphi_j(\pi_0)$.

Instead of constructing the ε-NFA M'_σ in detail we explain the desired mode of operation of an arbitrary accepting path π'_σ in this machine. Assume that the path π'_σ consumes $x \in \Sigma^*$. Let $x_1, \ldots, x_m \in \Sigma$ so that $x = x_1 \ldots x_m$. The reader may recall from step (2) the definition of the words $y_1, \ldots, y_d \in \Sigma^*$. In particular, $x = y_1 \ldots y_d$, $d + 1$ is even, and $d \leq 2^{n+1} - 1$. We further ask once again to recall the main lines of the construction of the NGSM M_σ, which has state set $Q_\sigma = Q^{(1)} \times Q^{(2)}$, in step (3).

The path π'_σ consists of five components that correspond to five components of Q'_σ, the state set of M'_σ. The first four components of Q'_σ coincide with the ones of Q_σ, the state set of M_σ. Therefore, $Q'_\sigma = Q^{(1)} \times Q^{(3)}$ where $Q^{(3)}$ denotes the fifth component of Q'_σ. Concerning the $Q^{(1)}$-components, the path π'_σ operates exactly as an accepting path in M_σ consuming x. In particular, the value $(j, \alpha) \in [d+1] \times [3]$ of the fourth component of π'_σ if $j \leq d$, determines the index j of the word y_j currently consumed by π'_σ. We observe as for M_σ that the sequence of the $Q^{(1)}$-components of the states of π'_σ and the sequence of words consumed by its transitions are uniquely determined by x. According to the construction of $\bar{M}'_\sigma$ from M'_σ this implies that the sequence of the $Q^{(1)}$-components of the states of any accepting path in $\bar{M}'_\sigma$ is uniquely determined by the word consumed by this path. At the beginning (end) of π'_σ its fourth component contains $(1, 1)$ $((d+1, 1)$, respectively). One reason for π'_σ to accept the word x is that at its end $j = d + 1 \leq j_l$. Let us assume here that $j_l < d + 1$.

The fifth component of π'_σ verifies that there is no $\pi \in \Pi$ consuming x such that σ belongs to $\varphi(\pi)$. Following the definition of the mapping φ this component verifies that (a) or (b), depending on its guess, holds true. (a) For some (guessed) $\lambda \in \{0, \ldots, l\}$ there is no path in M consuming $y_{j_\lambda + 1} \cdots y_{j_{\lambda+1} - 1}$ and leading from q'_{j_λ} to $p'_{j_{\lambda+1}}$, where we set $j_0 = 0$,

$j_{l+1} = d+1$, $q'_0 = q_I$, and $p'_{d+1} = q_F$. (b) For some (guessed) $j = j_\lambda \in J$ ($\lambda \in [l]$) there is an accepting path in $M'_{\sigma,j}$ consuming y_j. Of course, condition (a) is verified, as described below, while π'_σ consumes the words $y_{j_\lambda+1}, \ldots, y_{j_{\lambda+1}-1}$, successively, and condition (b) is verified by simulating $M'_{\sigma,j}$ while π'_σ consumes the word y_j. The index $j \in [d]$ of the word y_j currently consumed by π'_σ is read from its fourth component.

Technically the fifth component of Q'_σ is $Q^{(3)} = 2^{Q^{(2)}} \times [3]$. Recall that $2^{Q^{(2)}}$ is the state set of $M'_{\sigma,j}$ $(j \in J)$. Assume that the path π'_σ contains a state with $Q^{(3)}$-component $(B_3, \beta) \in 2^{Q^{(2)}} \times [3]$. The value of β can never decrease along π'_σ. If $\beta = 2$, then π'_σ is about to verify (a) or (b). If $\beta = 1\,(\beta = 3)$, then this verification still has to be done (is already completed, respectively). If $\beta \in \{1, 3\}$, then $B_3 = \emptyset$. Assume that $\beta = 2$, π'_σ is about to verify (a) for some $\lambda \in \{0, \ldots, l\}$, and π'_σ has consumed the prefix y of $y_{j_\lambda+1} \ldots y_{j_{\lambda+1}-1}$. Then $B_3 = \{(q, q, 0, 0)$: for some $z' \in \Delta^*$, $(q'_{j_\lambda}, y, z', q) \in \delta\}$. Assume that $\beta = 2$, π'_σ is about to verify (b) for some $j \in J$, and π'_σ has consumed the prefix y of y_j. Then B_3 is the uniquely determined state of $M'_{\sigma,j}$ reached from the initial state when consuming y.

From the description of the mode of operation of the path π'_σ given above we conclude (informally) that the ε-NFA M'_σ recognizes the correct language. The detailed construction of M'_σ is given in the proof of [W93, Thm. 2.2]. Our only modification is in the definition of the set of final states where we add the condition that $A_{j+1} = A_{j+2} < A_{j+3} = A_{j+4} < \ldots < A_{2^{n+1}-1} = A_{2^{n+1}}$. Therefore, we can conclude (formally) from [W93, Thms. 2.2 and 2.3] that M'_σ recognizes the correct language and that the size of $\bar{M}'_\sigma$ and the time complexity for its computation are bounded as desired.

The state set $Q'_\sigma = Q^{(1)} \times Q^{(3)} = Q^{(1)} \times 2^{Q^{(2)}} \times [3]$ of M'_σ and $\bar{M}'_\sigma$ is independent of σ.

In order to complete step (4), we need some routine observations about the detailed construction of the ε-NFA M'_σ and the NGSM $\bar{M}'_\sigma$. Consider the set $Q' = \{B \in 2^{Q^{(3)}}$: for every $\beta \in [3]$, $\#(B \cap (2^{Q^{(2)}} \times \{\beta\})) \leq 1\}$. Note that $\# Q' \leq (1 + 2^{\# Q^{(2)}})^3$. For every $p \in Q^{(1)}$, the set $B = \{q' \in Q^{(3)}$: $(p, q') \in \bar{Q}'_{I,\sigma}\}$ belongs to Q' and, for every $a \in \Sigma$, $p, p' \in Q^{(1)}$, and $B \in Q'$, the set

$$B' = \{s' \in Q^{(3)} : \text{for some } s \in B,\ ((p, s), a, \varepsilon, (p', s')) \in \bar{\delta}'_\sigma\}$$

belongs to Q'. We have therefore observed that the NGSM $\bar{M}'_\sigma$ meets the second special case of Lemma 2.3 with the given subset Q' of $2^{Q^{(3)}}$.

Execution of step (5): Let $\sigma \in S$. Applying Lemma 2.3 to the single-valued NGSM $\bar{M}_\sigma$ constructed in step (3), we obtain an equivalent UGSM $\tilde{M}_\sigma$ having size at most $\|\bar{M}_\sigma\| \cdot 2^{\#Q^{(2)}-1}$. The UGSM $\tilde{M}_\sigma$ can be computed in DTIME $(\text{poly}(\|\bar{M}_\sigma\| \cdot 2^{\#Q^{(2)}}))$. Note that $\|\bar{M}_\sigma\|$ and $2^{\#Q^{(2)}-1}$ are of order $O(2^{2^{\text{lin}\|M\|}})$.

Let us consider the new machine $\tilde{M}_\sigma = (\tilde{Q}, \Sigma, \Delta, \tilde{\delta}_\sigma, \tilde{Q}_{I,\sigma}, \tilde{Q}_{F,\sigma})$. According to the proof of Lemma 2.3, the state set $\tilde{Q}$ of $\tilde{M}_\sigma$ is a subset of $Q^{(1)} \times Q^{(2)} \times 2^{Q^{(2)}}$, which is independent of σ. We define the mapping $\psi_1 : \tilde{Q} \to Q^{(1)} \times Q^{(2)}$ as the projection to the $Q^{(1)} \times Q^{(2)}$-component. Let (r, a, z, s) be a transition of $\tilde{M}_\sigma$. Then, by construction of this machine, $(\psi_1(r), a, z, \psi_1(s))$ is a transition of $\bar{M}_\sigma$. Moreover, we observe that ψ_1 maps any initial (final) state of $\tilde{M}_\sigma$ to an initial (final, respectively) state of $\bar{M}_\sigma$.

Execution of step (6): Let $\sigma \in S$. Applying the second special case of Lemma 2.3 to the NFA $\bar{M}'_\sigma$ constructed in step (4), we obtain an equivalent UFA $\tilde{M}'_\sigma$ having size at most $\|\bar{M}'_\sigma\| \cdot (1 + 2^{\#Q^{(2)}})^3$. The UFA $\tilde{M}'_\sigma$ can be computed in DTIME $(\text{poly}(\|\bar{M}'_\sigma\| \cdot (1 + 2^{\#Q^{(2)}})^3))$. Note that $\|\bar{M}'_\sigma\|$ and $(1 + 2^{\#Q^{(2)}})^3$ are of order $O(2^{2^{\text{lin}\|M\|}})$. Since the state set of $\bar{M}'_\sigma$ is independent of σ, the state set of $\tilde{M}'_\sigma$ is independent of σ as well.

This completes the proof of Theorem 2.1. □

We now turn to the proof of Theorem 2.2. For this purpose we need the following word lemma.

LEMMA 2.4: *Let Δ be a nonempty, finite set. Let $z_1, \ldots, z_6 \in \Delta^*$ such that the words $z_1 z_2 z_3$ and $z_4 z_5 z_6$ are distinct. Then, there is a nonnegative integer λ_0 such that for every integer $\lambda > \lambda_0$ the words $z_1 z_2^\lambda z_3$ and $z_4 z_5^\lambda z_6$ are distinct.*

Proof: If $|z_2| \neq |z_5|$, then there is at most one nonnegative integer λ such that $|z_1 z_2^\lambda z_3| = |z_4 z_5^\lambda z_6|$. Thus, we might choose λ_0 to be either this λ, if it exists, or 0, otherwise. If $|z_2| = |z_5|$ and $|z_1 z_3| \neq |z_4 z_6|$ or if $z_2 = z_5 = \varepsilon$, then we can set $\lambda_0 = 0$. Let us therefore assume that $|z_2| = |z_5| \neq 0$ and $|z_1 z_3| = |z_4 z_6|$. Because of symmetry, we may further assume that $|z_1| \leq |z_4|$. Let $j \in [|z_1 z_2 z_3|]$ be a position at which the words $z_1 z_2 z_3$ and $z_4 z_5 z_6$ differ.

We select λ_0 to be the maximal integer λ such that $|z_1| + \lambda \cdot |z_2| < |z_4 z_5|$. By our assumptions, λ_0 is nonnegative. Assume that, for some integer $\lambda > \lambda_0$, the words $z_1 z_2^\lambda z_3$ and $z_4 z_5^\lambda z_6$ coincide. Hence, z_1 is a prefix of z_4 and z_6 is a suffix of z_3 implying that $|z_1| < j \leq |z_4 z_5|$. Let μ be the maximal integer such that $j + \mu \cdot |z_5| \leq |z_4 z_5|$. Since $j \leq |z_4 z_5|$, μ is nonnegative. Since μ is maximal, we have that $|z_4| < j + \mu \cdot |z_5|$. Since $|z_1| + \mu \cdot |z_2| < j + \mu \cdot |z_5| \leq |z_4 z_5|$ and λ_0 was maximal, we know that $\mu \leq \lambda_0 \leq \lambda - 1$. Since $\lambda > \lambda_0$, we further have that $|z_1| + \lambda \cdot |z_2| \geq |z_4 z_5|$. Hence,

$$|z_4| < j + \mu \cdot |z_5| \leq j + (\lambda - 1) \cdot |z_5| \leq |z_4 z_5^{\lambda}|$$

and

$$|z_1| < j \leq j + \mu \cdot |z_2| = j + \mu \cdot |z_5| \leq |z_4 z_5| \leq |z_1 z_2^{\lambda}|.$$

In summary, we can derive the following contradiction.

$$\begin{aligned}(z_1 z_2 z_3)(j) &= (z_1 z_2^{\lambda} z_3)(j + (\lambda - 1) \cdot |z_2|)\\ &= (z_4 z_5^{\lambda} z_6)(j + (\lambda - 1) \cdot |z_5|)\\ &= (z_4 z_5^{\lambda} z_6)(j + \mu \cdot |z_5|)\\ &= (z_1 z_2^{\lambda} z_3)(j + \mu \cdot |z_2|)\\ &= (z_1 z_2^{\lambda} z_3)(j)\\ &= (z_4 z_5^{\lambda} z_6)(j)\\ &= (z_4 z_5 z_6)(j).\end{aligned}$$

Consequently, for every integer $\lambda > \lambda_0$ the words $z_1 z_2^{\lambda} z_3$ and $z_4 z_5^{\lambda} z_6$ are distinct. □

Proof of Theorem 2.2: Let $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be an NGSM with n states, and let k be a positive integer such that the assumption of the theorem holds true. Thus, there are accepting paths $\pi_1, \ldots, \pi_{k+1}$ in M consuming the same word (in Σ^*) and producing the words $z_1, \ldots, z_{k+1} \in \Delta^*$, respectively, such that for any two distinct $i_1, i_2 \in [k+1]$ either diff (π_{i_1}, π_{i_2}) is greater than $(n^{k+1} - 1) \cdot$ diff (δ) or z_{i_1} and z_{i_2} are distinct. For any paths $\pi_1, \ldots, \pi_{k+1}$ in M consuming the same word and producing the words $z_1, \ldots, z_{k+1} \in \Delta^*$, respectively, we are going to study property (*).

(*) For any two distinct $i_1, i_2 \in [k+1]$ either (i) or (ii) holds.

(i) There are factorizations $\pi_i = \pi_{i,1} \circ \pi_{i,2} \circ \pi_{i,3}$ $(i = 1, \ldots, k+1)$ such that, for every $j \in \{1, 2, 3\}$, the paths $\pi_{1,j}, \ldots, \pi_{k+1,j}$ consume the same word, the paths $\pi_{1,2}, \ldots, \pi_{k+1,2}$ are cycles, and the lengths of the words produced by $\pi_{i_1,2}$ and $\pi_{i_2,2}$ are distinct.

(ii) The words z_{i_1} and z_{i_2} are distinct.

We wish to prove Claims 1, 2, and 3.

Claim 1: There are accepting paths $\pi_1, \ldots, \pi_{k+1}$ in M consuming the same word and having property (*).

Claim 2: Let $\pi_1, \ldots, \pi_{k+1}$ be accepting paths in M consuming the same word, say, $v \in \Sigma^*$, producing the words $z_1, \ldots, z_{k+1} \in \Delta^*$, respectively, and having property (*) such that the cardinality of $\{z_1, \ldots, z_{k+1}\}$ is maximal. Then, this cardinality is $k+1$. In particular, the valuedness of v in M is at least $k+1$.

Claim 3: Let $\pi_1, \ldots, \pi_{k+1}$ be paths in M consuming the same word and producing the words $z_1, \ldots, z_{k+1} \in \Delta^*$, respectively. Let $i_1, i_2 \in [k+1]$ be distinct such that assertion (i) does not hold for $(\pi_1, \ldots, \pi_{k+1})$ and (i_1, i_2). Then, $||z_{i_1}| - |z_{i_2}||$ is at most $(n^{k+1} - 1) \cdot \text{diff}(\delta)$.

The theorem directly follows from Claims 1 and 2. Using Claim 3, it is easy to check that the paths $\pi_1, \ldots, \pi_{k+1}$ given by the assumption of the theorem have property (*). By this we have established Claim 1.

Our next goal is to prove Claim 3. This will be done by induction on the length of the word $v \in \Sigma^*$ consumed by the given paths. Let $\pi_1, \ldots, \pi_{k+1}$ be paths in M consuming the same word $v \in \Sigma^*$ and producing the words $z_1, \ldots, z_{k+1} \in \Delta^*$, respectively. Let $i_1, i_2 \in [k+1]$ be distinct such that assertion (i) does not hold for $(\pi_1, \ldots, \pi_{k+1})$ and (i_1, i_2). The base of induction is the case that $|v| \leq n^{k+1} - 1$. In this case, we have that $||z_{i_1}| - |z_{i_2}|| \leq |v| \cdot \text{diff}(\delta) \leq (n^{k+1} - 1) \cdot \text{diff}(\delta)$.

For the induction step let us assume that $|v| \geq n^{k+1}$. Then, there are factorizations $\pi_i = \pi_{i,1} \circ \pi_{i,2} \circ \pi_{i,3}$ $(i = 1, \ldots, k+1)$ such that, for every $j \in \{1, 2, 3\}$, the paths $\pi_{1,j}, \ldots, \pi_{k+1,j}$ consume the same word, say, $v_j \in \Sigma^*$, the word v_2 is nonempty, and the paths $\pi_{1,2}, \ldots, \pi_{k+1,2}$ are cycles. Let us select any such factorizations where the length of the word v_1 is minimal. Let $z_{i,j} \in \Delta^*$ be the word produced by the path $\pi_{i,j}$ $(i = 1, \ldots, k+1,\ j = 1, 2, 3)$. Since assertion (i) does not hold for $(\pi_1, \ldots, \pi_{k+1})$ and (i_1, i_2), the lengths of $z_{i_1,2}$ and $z_{i_2,2}$ coincide. Consider the paths $\pi'_1 = \pi_{1,1} \circ \pi_{1,3}, \ldots, \pi'_{k+1} = \pi_{k+1,1} \circ \pi_{k+1,3}$ in M consuming the same word $v' = v_1 v_3 \in \Sigma^*$ and producing the words $z'_1 = z_{1,1} z_{1,3} \in \Delta^*, \ldots, z'_{k+1} = z_{k+1,1} z_{k+1,3} \in \Delta^*$, respectively. It is straightforward to check that if assertion (i) held for $(\pi'_1, \ldots, \pi'_{k+1})$ and (i_1, i_2) then it would also hold for $(\pi_1, \ldots, \pi_{k+1})$ and (i_1, i_2). Therefore, we can apply the induction hypothesis to $(\pi'_1, \ldots, \pi'_{k+1})$ and (i_1, i_2) which yields that $||z'_{i_1}| - |z'_{i_2}||$ is at most $(n^{k+1} - 1) \cdot \text{diff}(\delta)$. Consequently,

$$\begin{aligned} ||z_{i_1}| - |z_{i_2}|| &= ||z_{i_1,1}\, z_{i_1,2}\, z_{i_1,3}| - |z_{i_2,1}\, z_{i_2,2}\, z_{i_2,3}|| \\ &= ||z_{i_1,1}\, z_{i_1,3}| - |z_{i_2,1}\, z_{i_2,3}|| \\ &= ||z'_{i_1}| - |z'_{i_2}|| \\ &\leq (n^{k+1} - 1) \cdot \text{diff}(\delta). \end{aligned}$$

It remains to prove Claim 2. For this purpose let us consider accepting paths $\pi_1, \ldots, \pi_{k+1}$ in M consuming the same word, say, $v \in \Sigma^*$, producing the words $z_1, \ldots, z_{k+1} \in \Delta^*$, respectively, and having property (*) such that $k' = \#\{z_1, \ldots, z_{k+1}\}$ is maximal. Assume that at least two words in $\{z_1, \ldots, z_{k+1}\}$ coincide, say, $z_1 = z_2$. We are going to construct accepting paths $\tilde{\pi}_1, \ldots, \tilde{\pi}_{k+1}$ consuming the same word, say, $u \in \Sigma^*$, producing the words $\tilde{z}_1, \ldots, \tilde{z}_{k+1} \in \Delta^*$, respectively, and having property (*) such that $\tilde{z}_1 \neq \tilde{z}_2$ and, for any two distinct $i_1, i_2 \in [k+1]$, $z_{i_1} \neq z_{i_2}$ implies that $\tilde{z}_{i_1} \neq \tilde{z}_{i_2}$. Consequently, $\#\{\tilde{z}_1, \ldots, \tilde{z}_{k+1}\} > \#\{z_1, \ldots, z_{k+1}\} = k'$ which contradicts the maximality of k'. Thus, k' equals $k+1$ as desired.

We construct the paths $\tilde{\pi}_1, \ldots, \tilde{\pi}_{k+1}$ as follows. Property (*) applied to $(\pi_1, \ldots, \pi_{k+1})$ and $(i_1, i_2) = (1, 2)$ yields factorizations $\pi_i = \tilde{\pi}_{i,1} \circ \tilde{\pi}_{i,2} \circ \tilde{\pi}_{i,3}$ $(i = 1, \ldots, k+1)$ such that, for every $j \in \{1, 2, 3\}$, the paths $\tilde{\pi}_{1,j}, \ldots, \tilde{\pi}_{k+1,j}$ consume the same word, say, $u_j \in \Sigma^*$, the paths $\tilde{\pi}_{1,2}, \ldots, \tilde{\pi}_{k+1,2}$ are cycles, and the lengths of the words produced by $\tilde{\pi}_{1,2}$ and $\tilde{\pi}_{2,2}$ are distinct. Let $\tilde{z}_{i,j} \in \Delta^*$ be the word produced by the path $\tilde{\pi}_{i,j}$ $(i = 1, \ldots, k+1,\ j = 1, 2, 3)$. By construction, $z_i = \tilde{z}_{i,1}\, \tilde{z}_{i,2}\, \tilde{z}_{i,3}$ $(i = 1, \ldots, k+1)$. Let $i_1, i_2 \in [k+1]$ be distinct such that $z_{i_1} \neq z_{i_2}$. According to Lemma 2.4 there is a nonnegative integer λ_{i_1,i_2} such that for every integer $\lambda > \lambda_{i_1,i_2}$ the words $\tilde{z}_{i_1,1}\, \tilde{z}^{\lambda}_{i_1,2}\, \tilde{z}_{i_1,3}$ and $\tilde{z}_{i_2,1}\, \tilde{z}^{\lambda}_{i_2,2}\, \tilde{z}_{i_2,3}$ are distinct. Since $|\tilde{z}_{1,2}| \neq |\tilde{z}_{2,2}|$, there is at most one nonnegative integer λ such that $|\tilde{z}_{1,1}\, \tilde{z}^{\lambda}_{1,2}\, \tilde{z}_{1,3}| = |\tilde{z}_{2,1}\, \tilde{z}^{\lambda}_{2,2}\, \tilde{z}_{2,3}|$. Select $\lambda_{1,2}$ to be either this λ, if it exists, or 0, otherwise. Finally, define $\lambda_0 = \max(\{\lambda_{1,2}\} \cup \{\lambda_{i_1,i_2} : i_1, i_2 \in [k+1], z_{i_1} \neq z_{i_2}\})$. Now, let us fix some $\lambda > \lambda_0$ and define $\tilde{\pi}_i = \tilde{\pi}_{i,1} \circ \tilde{\pi}^{\lambda-1}_{i,2} \circ \tilde{\pi}_{i,2} \circ \tilde{\pi}_{i,3}$ $(i = 1, \ldots, k+1)$, $u = u_1\, u_2^{\lambda}\, u_3$, and $\tilde{z}_i = \tilde{z}_{i,1}\, \tilde{z}^{\lambda}_{i,2}\, \tilde{z}_{i,3}$ $(i = 1, \ldots, k+1)$. Note that the path $\tilde{\pi}_i$ arises from π_i by "inserting" $\tilde{\pi}^{\lambda-1}_{i,2}$ after $\tilde{\pi}_{i,1}$ $(i = 1, \ldots, k+1)$. Since $\lambda > \lambda_0$, we know that $|\tilde{z}_1| \neq |\tilde{z}_2|$ and that, for any two distinct i_1, $i_2 \in [k+1]$, $z_{i_1} \neq z_{i_2}$ implies that $\tilde{z}_{i_1} \neq \tilde{z}_{i_2}$.

It remains to be shown that the paths $\tilde{\pi}_1, \ldots, \tilde{\pi}_{k+1}$ have property (*). By the above, it is sufficient to prove that, for any two different i_1,

$i_2 \in [k+1]$, the validity of assertion (i) is inherited from $(\pi_1, \ldots, \pi_{k+1})$ to $(\tilde{\pi}_1, \ldots, \tilde{\pi}_{k+1})$. Let $i_1, i_2 \in [k+1]$ be different. If the lengths of $\tilde{z}_{i_1, 2}$ and $\tilde{z}_{i_2, 2}$ are distinct, then the factorizations $\tilde{\pi}_i = \tilde{\pi}_{i, 1} \circ \tilde{\pi}^{\lambda}_{i, 2} \circ \tilde{\pi}_{i, 3}$ $(i = 1, \ldots, k+1)$ guarantee assertion (i) for $(\tilde{\pi}_1, \ldots, \tilde{\pi}_{k+1})$ and (i_1, i_2).

Otherwise, if $\tilde{z}_{i_1, 2}$ and $\tilde{z}_{i_2, 2}$ have the same length, let us consider factorizations $\pi_i = \pi_{i, 1} \circ \pi_{i, 2} \circ \pi_{i, 3}$ $(i = 1, \ldots, k+1)$ such that, for every $j \in \{1, 2, 3\}$, the paths $\pi_{1, j}, \ldots, \pi_{k+1, j}$ consume the same word, say, $v_j \in \Sigma^*$, the paths $\pi_{1, 2}, \ldots, \pi_{k+1, 2}$ are cycles, and the lengths of the words produced by $\pi_{i_1, 2}$ and $\pi_{i_2, 2}$ are distinct. Then, having in mind how $\tilde{\pi}_i$ arose from π_i $(i = 1, \ldots, k+1)$, it is easy to obtain from the paths $\pi_{i, j}$ $(i = 1, \ldots, k+1, j = 1, 2, 3)$ factorizations $\tilde{\pi}_i = \pi'_{i, 1} \circ \pi'_{i, 2} \circ \pi'_{i, 3}$ $(i = 1, \ldots, k+1)$ which guarantee assertion (i) for $(\tilde{\pi}_1, \ldots, \tilde{\pi}_{k+1})$ and (i_1, i_2).

3. DECOMPOSING k-VALUED TRANSDUCERS

In this section we use the outcome of Section 2 (Theorems 2.1 and 2.2) in order to prove the main result of this article.

THEOREM 3.1: *Let $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be a k-valued NGSM, where k is a positive integer. Then, there are k UGSMs $\tilde{M}_1, \ldots, \tilde{M}_k$ and UFAs $\tilde{M}'_1, \ldots, \tilde{M}'_k$ such that $T(M)$ equals $T(\tilde{M}_1) \cup \ldots \cup T(\tilde{M}_k)$ and, for every $\kappa \in [k]$, $\tilde{M}'_\kappa$ recognizes $\Sigma^* \backslash L(\tilde{M}_\kappa)$. Each of these new machines has size $O(2^{2^{\text{poly}(\|M\|+k)}})$ and can be computed in* DTIME $(2^{2^{\text{poly}(\|M\|+k)}})$.

Informally, Theorem 3.1 states that a k-valued NGSM M is equivalent to some effectively constructible "disjoint union" of k unambiguous NGSMs of double exponential size. It turns out that these UGSMs are technically quite complicated. While consuming the same input word, they need almost their entire capability in order to carry out exactly the same "basic work" upon which they decide "on line" which output word to produce. Intuitively spoken these machines are doing so because the model of a "disjoint union" of transducers does not allow any communication among them by which they could coordinate their output words. The author believes that the missing communication is one of the main reasons why the new machines are so complicated.

Note that in the case $k = 1$ of Theorem 3.1 we can select the UGSM $\tilde{M}_1$ to be of size at most $\|M\| \cdot 2^{\#Q-1}$, using Lemma 2.3 for $n_1 = 1$. Moreover it is known that $\tilde{M}_1$ has at least $2^{\#Q} - 1$ states in certain cases of M (Leung [Le93], *see* Weber and Klemm [WK95, Prop. 2.2]). For $k \geq 2$ it is open whether or not the size of the UGSMs $\tilde{M}_1, \ldots, \tilde{M}_k$ in Theorem 3.1 can be substantially improved. We only know that, in certain cases of M, the sum of the number of states of these UGSMs is at least $2^{\#Q} - 1$ (Leung [Le93]). By reduction, Theorem 3.1 can be extended to NFTs.

THEOREM 3.2: *Let* $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ *be a* k*-valued NFT, where* k *is a positive integer. Then, there are* k *unambiguous NFTs* $\tilde{M}_1, \ldots, \tilde{M}_k$ *and unambiguous* ε*-NFAs* $\tilde{M}'_1, \ldots, \tilde{M}'_k$ *such that* $T(M)$ *equals* $T(\tilde{M}_1) \cup \ldots \cup T(\tilde{M}_k)$ *and, for every* $\kappa \in [k]$, $\tilde{M}'_\kappa$ *recognizes* $\Sigma^* \backslash L(\tilde{M}_\kappa)$. *Each of these new machines has size* $O(2^{2^{\text{poly}(\|M\|+k)}})$ *and can be computed in* $\text{DTIME}(2^{2^{\text{poly}(\|M\|+k)}})$.

Since every k-ambiguous NFT is k-valued and every "disjoint union" of k unambiguous NFTs is a k-ambiguous NFT, Theorems 3.1 and 3.2 directly imply the following theorem.

THEOREM 3.3: *For every positive integer* k*, the* k*-valued NFTs (NGSMs) and the* k*-ambiguous NFTs (NGSMs, respectively) realize the same class of transductions.*

Theorem 3.3 was first established for $k = 1$ (Eilenberg [E74] and Schützenberger [Sch76], *see* Berstel [B79, Thms. IV.4.2 and IV.4.5]). For every fixed positive integer k, it is decidable in deterministic polynomial time whether or not a given NFT is k-valued (Gurari and Ibarra [GI83]). Consequently Theorem 3.3 implies that, for every fixed positive integer k, it is decidable in deterministic polynomial time whether or not a given NFT (NGSM) is equivalent to some k-ambiguous NFT (NGSM, respectively).

The remainder of this section is devoted to the proof of Theorems 3.1 and 3.2.

Proof of Theorem 3.1: Let $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be a k-valued NGSM with n states, where k is a positive integer. Applying Theorem 2.1 to M, we obtain $O(2^{\text{poly}\|M\|})$ many UGSMs $M_1, \ldots, M_N$, and UFAs $M'_1, \ldots, M'_N$ such that $T(M)$ equals $T(M_1) \cup \ldots \cup T(M_N)$ and, for every $i \in [N]$, M'_i recognizes $\Sigma^* \backslash L(M_i)$. In order to prove that Theorem 3.1 holds for M we roughly proceed as follows. Let $x \in \Sigma^*$ be an input word. Using accepting paths in the "disjoint unions" of M_i and M'_i $(i = 1, \ldots, k)$ all consuming x we define an undirected "neighborhood graph" for x with vertices in $[N]$. The minimal vertices of the connected components of this graph represent all values for x in M. By means of Theorem 2.2 it is shown that the graph has at most k connected components. For every $\kappa \in [k]$ the new UGSM $\tilde{M}_\kappa$ is then designed to obtain its value (for x) from the minimal vertex having "rank" κ in the neighborhood graph. If all vertices of this graph have "rank" less than κ, then x is planned to be recognized by the new UFA $\tilde{M}'_\kappa$.

According to Theorem 2.1, each of the machines $M_1, \ldots, M_N$ and $M'_1, \ldots, M'_N$ has size $O(2^{2^{\mathrm{lin}\|M\|}})$ and can be computed in DTIME $(2^{2^{\mathrm{lin}\|M\|}})$. Let Q_0 and Q'_0 be the state sets provided by Theorem 2.1. Let $M_i = (Q_0, \Sigma, \Delta, \delta_i, Q_{I,i}, Q_{F,i})$ and $M'_i = (Q'_0, \Sigma, \Delta, \delta'_i, Q'_{I,i}, Q'_{F,i})$ $(i \in [N])$. We may assume that Q_0 and Q'_0 are disjoint, *i.e.*, $Q_0 \cap Q'_0 = \emptyset$. For every $i \in [N]$ we define the UGSM $M_i \dot{\cup} M'_i = (Q_0 \cup Q'_0, \Sigma, \Delta, \delta_i \cup \delta'_i, Q_{I,i} \cup Q'_{I,i}, Q_{F,i} \cup Q'_{F,i})$, where $M_i \dot{\cup} M'_i$ denotes the *disjoint union* of M_i and M'_i. Let $\psi : Q_0 \to Q$ be the mapping provided by Theorem 2.1 mapping any (accepting) path in M_i $(i \in [N])$ to an (accepting) path in M.

Let us fix the notation $\mathcal{M} = (M, M_1, \ldots, M_N)$. Let $x \in \Sigma^*$ be an input word. We define the *neighborhood graph* for x with respect to $\mathcal{M}$ and k, denoted by $\mathrm{NG}_{\mathcal{M},k}(x)$, to be the undirected graph (V, E) where

$$V = \{i \in [N] : x \in L(M_i)\}$$

and $E = \{\{i_1, i_2\} \in \binom{V}{2}$: there are accepting paths π_{i_1} in M_{i_1} and π_{i_2} in M_{i_2} both consuming x and producing $z_{i_1}, z_{i_2} \in \Delta^*$, respectively, such that $\mathrm{diff}(\pi_{i_1}, \pi_{i_2})$ is at most $n^{k+1} \cdot \mathrm{diff}(\delta)$ and z_{i_1} and z_{i_2} coincide$\}$.

FACT 3.4: *For every word $x \in \Sigma^*$ the graph $\mathrm{NG}_{\mathcal{M},k}(x)$ has at most k connected components.*

Proof: Given $x \in \Sigma^*$, let us consider the graph $G = \mathrm{NG}_{\mathcal{M},k}(x) = (V, E)$. Assume that G has $k + 1$ or more connected components. Then, there are pairwise disjoint vertices $i_1, \ldots, i_{k+1} \in V$ such that no edge in E connects any two of them. By definition of G, there are accepting paths $\pi_{i_1}, \ldots, \pi_{i_{k+1}}$ in $M_{i_1}, \ldots, M_{i_{k+1}}$, respectively, all consuming x and producing the words $z_{i_1}, \ldots, z_{i_{k+1}} \in \Delta^*$, respectively, such that, for any two distinct $\lambda_1, \lambda_2 \in \{i_1, \ldots, i_{k+1}\}$, either $\mathrm{diff}(\pi_{\lambda_1}, \pi_{\lambda_2})$ is greater than $n^{k+1} \cdot \mathrm{diff}(\delta)$ or z_{λ_1} and z_{λ_2} are distinct. Thus, $\psi(\pi_{i_1}), \ldots, \psi(\pi_{i_{k+1}})$ are accepting paths in M all consuming x and producing $z_{i_1}, \ldots, z_{i_{k+1}} \in \Delta^*$, respectively, such that, for any two distinct $\lambda_1, \lambda_2 \in \{i_1, \ldots, i_{k+1}\}$, either $\mathrm{diff}(\psi(\pi_{\lambda_1}), \psi(\pi_{\lambda_2})) = \mathrm{diff}(\pi_{\lambda_1}, \pi_{\lambda_2})$ is greater than $n^{k+1} \cdot \mathrm{diff}(\delta)$ or z_{λ_1} and z_{λ_2} are distinct. By Theorem 2.2, this implies that the valuedness of M is greater than k, a contradiction. Therefore, the graph G has at most k connected components. □

Let $x \in \Sigma^*$ be an input word. Consider the undirected graph $G = \mathrm{NG}_{\mathcal{M},k}(x) = (V, E)$. Note that $V \subseteq [N]$. Let $U_1, \ldots, U_{k'}$ be the connected components of G ordered by their minimal elements, *i.e.*,

$1 \leq \min U_1 < \min U_2 < \ldots < \min U_{k'} \leq N$. From Fact 3.4 we know that k' is at most k. For any vertex $i \in V$ its *rank* in G, abbreviated $\mathrm{rk}_G(i)$, is defined as the uniquely determined $\kappa \in \{1, \ldots, k'\}$ such that i belongs to U_κ. Analogously, such ranks can be defined in any finite, undirected graph having positive integers as vertices.

We are going to define, for each $\kappa \in [k]$, a UGSM $\tilde{M}_k$ which realizes the transduction

$$\begin{aligned} T_\kappa = \{(x, z) \in \Sigma^* \times \Delta^* \;:\; & \text{there is a vertex of the graph} \\ & \mathrm{NG}_{\mathcal{M},\mathrm{k}}(x) \text{ having rank } \kappa \text{ and } (x, z) \in T(M_{i_0}) \\ & \text{where } i_0 \text{ is the minimal such vertex}\} \end{aligned}$$

and a UFA $\tilde{M}'_\kappa$ which recognizes the language

$$\begin{aligned} L_\kappa = \{x \in \Sigma^* \;:\; & \text{all vertices of the graph} \\ & \mathrm{NG}_{\mathcal{M},\mathrm{k}}(x) \text{ have rank less than } \kappa\}. \end{aligned}$$

We further require that each of our new machines has size $O(2^{2^{\mathrm{poly}(\|M\|+k)}})$ and can be computed in $\mathrm{DTIME}(2^{2^{\mathrm{poly}(\|M\|+k)}})$. In order to see that these machines are suitable for the proof of Theorem 3.1 let us first check that $T(M)$ equals $T(\tilde{M}_1) \cup \ldots \cup T(\tilde{M}_k)$ and, for every $\kappa \in [k]$, $\tilde{M}'_\kappa$ recognizes $\Sigma^* \backslash L(\tilde{M}_\kappa)$.

By definition of the set T_κ, every $(x, z) \in T(\tilde{M}_\kappa)$ belongs to $T(M_{i_0})$ for some $i_0 \in [N]$ depending on x and κ. Hence, every $T(\tilde{M}_\kappa)$ is included in $T(M)$. On the other hand, let $(x, z) \in T(M)$, and let $i \in [N]$ such that $(x, z) \in T(M_i)$. Let κ be the rank of i in the graph $G = \mathrm{NG}_{\mathcal{M},k}(x)$, and let i_0 be the minimal vertex of G having rank κ. Since i is connected with i_0 in G, the definition of G and the fact that all $M_1, \ldots, M_N$ are single valued yields that (x, z) also belongs to $T(M_{i_0})$, *i.e.*, (x, z) belongs to $T(\tilde{M}_\kappa)$ by definition of T_κ. Consequently, $T(M)$ equals $T(\tilde{M}_1) \cup \ldots \cup T(\tilde{M}_k)$. Let $\kappa \in [k]$. According to the definition of the set L_κ, every word $x \in L(\tilde{M}'_\kappa)$ has a neighborhood graph containing no vertex of rank κ. Thus, there is no word $z \in \Delta^*$ such that (x, z) is in T_κ, *i.e.*, x does not belong to $L(\tilde{M}_\kappa)$. On the other hand, for every word $x \in \Sigma^* \backslash L(\tilde{M}'_\kappa)$ the graph $G = \mathrm{NG}_{\mathcal{M},k}(x)$ contains a vertex having rank at least κ and, therefore, also a vertex having rank exactly κ. Let i_0 be the minimal vertex of G having rank κ. Then, the word x belongs to $L(M_{i_0})$ and, by definition of T_κ, also to $L(\tilde{M}_\kappa)$. Consequently, $L(\tilde{M}'_\kappa)$ equals $\Sigma^* \backslash L(\tilde{M}_\kappa)$ as desired. It remains to construct the machines $\tilde{M}_1, \ldots, \tilde{M}_k$ and $\tilde{M}'_1, \ldots, \tilde{M}'_k$ as required above.

Let us fix some $\kappa \in [k]$. In order to define the UGSM $\tilde{M}_\kappa$ and the UFA $\tilde{M}'_\kappa$ we proceed as follows. First of all, we reformulate the definition of the sets T_κ and L_κ in a way independent of the neighborhood graph. Having the new definitions in mind, we then explain the desired mode of operation of accepting paths in $\tilde{M}_\kappa$ and $\tilde{M}'_\kappa$. After this, we define the machines $\tilde{M}_\kappa$ and $\tilde{M}'_\kappa$ in detail and check that they have the properties stated above.

Up to now, we only used the fact that the NGSMs $M_1, \ldots, M_N$ are single valued. Recall, however, that these machines are unambiguous. Let us consider accepting paths $\pi_1, \ldots, \pi_N$ in the UGSMs $M_1 \dot{\cup} M'_1, \ldots, M_N \dot{\cup} M'_N$, respectively, consuming the same word $x \in \Sigma^*$ and producing the words $z_1, \ldots, z_N \in \Delta^*$, respectively. Note that such accepting paths exist for every given input word $x \in \Sigma^*$. Since the transducer $M_i \dot{\cup} M'_i$ is a disjoint union, the path π_i is contained either completely in M_i or completely in M'_i depending on whether its first state belongs to Q_0 or to Q'_0. Consider the graph $G = \mathrm{NG}_{\mathcal{M}, k}(x) = (V, E)$. Then, $V = \{i \in [N] : \pi_i \text{ is in } M_i\}$ and $E = \{\{i_1, i_2\} \in \binom{V}{2} : \mathrm{diff}(\pi_{i_1}, \pi_{i_2}) \leq n^{k+1} \cdot \mathrm{diff}(\delta) \text{ and } z_{i_1} = z_{i_2}\}$.

This implies that the sets T_κ and L_κ can be reformulated as follows.

- $T_\kappa = \{(x, z) \in \Sigma^* \times \Delta^*$: there is an $i_0 \in [N]$ and there are accepting paths $\pi_1, \ldots, \pi_N$ in $M_1 \dot{\cup} M'_1, \ldots, M_N \dot{\cup} M'_N$, respectively, all consuming x and producing the words $z_1, \ldots, z_N \in \Delta^*$, respectively, such that $z = z_{i_0}$ and i_0 is the minimal vertex of the graph $G = (V, E)$ having rank κ where $V = \{i \in [N] : \pi_i \text{ is in } M_i\}$ and $E = \{\{i_1, i_2\} \in \binom{V}{2} : \mathrm{diff}(\pi_{i_1}, \pi_{i_2}) \leq n^{k+1} \cdot \mathrm{diff}(\delta) \text{ and } z_{i_1} = z_{i_2}\}$.
- $L_\kappa = \{x \in \Sigma^*$: there are accepting paths $\pi_1, \ldots, \pi_N$ in $M_1 \dot{\cup} M'_1, \ldots, M_N \dot{\cup} M'_N$, respectively, all consuming x and producing the words $z_1, \ldots, z_N \in \Delta^*$, respectively, such that all vertices of the graph $G = (V, E)$ have rank less than κ where $V = \{i \in [N] : \pi_i \text{ is in } M_i\}$ and $E = \{\{i_1, i_2\} \in \binom{V}{2} : \mathrm{diff}(\pi_{i_1}, \pi_{i_2}) \leq n^{k+1} \cdot \mathrm{diff}(\delta) \text{ and } z_{i_1} = z_{i_2}\}\}$.

Let us next explain the desired mode of operation of arbitrary accepting paths $\tilde{\pi}$ in $\tilde{M}_\kappa$ and $\tilde{\pi}'$ in $\tilde{M}'_\kappa$. Assume that $\tilde{\pi}$ realizes the pair $(x, z) \in \Sigma^* \times \Delta^*$ and that $\tilde{\pi}'$ consumes the word $x \in \Sigma^*$.

The path $\tilde{\pi}$ consists of three components that correspond to three components of the state set $\tilde{M}_\kappa$. The first component of $\tilde{\pi}$ constantly contains an integer $i_0 \in [N]$ which is guessed at the beginning of this path. The second component of $\tilde{\pi}$ guesses accepting paths $\pi_1, \ldots, \pi_N$ in $M_1 \dot{\cup} M'_1, \ldots, M_N \dot{\cup} M'_N$, respectively, all consuming x and producing the words $z_1, \ldots, z_N \in \Delta^*$, respectively. The path $\tilde{\pi}$ produces the same word as π_{i_0}, *i.e.*, $z = z_{i_0}$. Whether π_i ($i \in [N]$) is in M_i or in M_i' depends on the guess of $\tilde{\pi}$ at its beginning. Set $V = \{i \in [N] : \pi_i \text{ is in } M_i\}$.

The third component of $\tilde{\pi}$ provides at the end of this path the set $E = \{\{i_1, i_2\} \in \binom{V}{2} : \text{diff}(\pi_{i_1}, \pi_{i_2}) \leq n^{k+1} \cdot \text{diff}(\delta) \text{ and } z_{i_1} = z_{i_2}\}$. Thus, considering the graph $G = (V, E)$, the path $\tilde{\pi}$ can verify at its end that i_0 is the minimal vertex of G having rank κ. In order to compute the edge set E, the third component of $\tilde{\pi}$ is divided into $\binom{N}{2}$ subcomponents, indexed by all possible $\{i_1, i_2\} \in \binom{[N]}{2}$. The $\{i_1, i_2\}$-subcomponent checks whether $\{i_1, i_2\}$ belongs to the set E ($\{i_1, i_2\} \in \binom{[N]}{2}$).

Assume that the path $\tilde{\pi}$, having consumed some prefix x' of x, is in state $\tilde{p} = (i_0, (p_1, \ldots, p_N), \tilde{p}_3)$. The meaning of $i_0 \in [N]$ is explained above. Let $\pi_1', \ldots, \pi_N'$ be the paths guessed so far by the second component of $\tilde{\pi}$. Then, every π_i' ($i \in [N]$) terminates at the state $p_i \in Q_0 \cup Q_0'$, consumes x', and produces some prefix z_i' of z_i; moreover there is a path π_i'' such that π_i equals $\pi_i' \circ \pi_i''$.

The third component of the state $\tilde{p}$, *i.e.*, $\tilde{p}_3$, is of the form $((y_{\{i_1, i_2\}}, \alpha_{\{i_1, i_2\}}))_{\{i_1, i_2\} \in \binom{[N]}{2}}$ where each $(y_{\{i_1, i_2\}}, \alpha_{\{i_1, i_2\}})$ is in $\Delta^{\leq \pm n^{k+1} \cdot \text{diff}(\delta)} \times [2]$. Let $\{i_1, i_2\} \in \binom{[N]}{2}$. Note that $\{p_{i_1}, p_{i_2}\}$ is a subset of Q_0 if and only if π_{i_1} and π_{i_2} are in M_i, *i.e.*, $\{i_1, i_2\}$ is a subset of V. If $\{p_{i_1}, p_{i_2}\}$ is not contained in Q_0, then $(y_{\{i_1, i_2\}}, \alpha_{\{i_1, i_2\}}) = (\varepsilon, 2)$. Thus, in the case that $\{i_1, i_2\}$ is not contained in V the $\{i_1, i_2\}$-subcomponent of the third component of $\tilde{\pi}$ has the constant value $(\varepsilon, 2)$ which is set at the beginning of $\tilde{\pi}$. Now, let us assume that $\{p_{i_1}, p_{i_2}\}$ is a subset of Q_0. If $\text{diff}(\pi_{i_1}', \pi_{i_2}')$ is at most $n^{k+1} \cdot \text{diff}(\delta)$ and z_{i_1}' is a prefix of z_{i_2}' or z_{i_2}' is a prefix of z_{i_1}', then $(y_{\{i_1, i_2\}}, \alpha_{\{i_1, i_2\}}) = ((z_{i_1}')^{-1} z_{i_2}', 1)$. Otherwise, if $\text{diff}(\pi_{i_1}', \pi_{i_2}')$ is greater than $n^{k+1} \cdot \text{diff}(\delta)$ or if z_{i_1}' and z_{i_2}' differ at some position $j \in [\min\{|z_{i_1}'|, |z_{i_2}'|\}]$, then $(y_{\{i_1, i_2\}}, \alpha_{\{i_1, i_2\}}) = (\varepsilon, 2)$. Therefore, in the case that $\{i_1, i_2\}$ is a subset of V the $\{i_1, i_2\}$-subcomponent of the third component of $\tilde{\pi}$ begins with the value $(\varepsilon, 1)$, continues with the value $(z_{\text{diff}}, 1)$ where $z_{\text{diff}} \in \Delta^{\leq \pm n^{k+1} \cdot \text{diff}(\delta)}$ represents the "difference" of the values produced so far by the paths π_{i_1} and π_{i_2}, and switches to the constant value $(\varepsilon, 2)$ if this "difference" becomes either too large or is not defined anymore.

Finally, let us consider the terminal state $\tilde{q} = (i_0, (q_1, \ldots, q_N), \tilde{q}_3)$ of $\tilde{\pi}$ where $\tilde{q}_3$ is of the form $((y_{\{i_1, i_2\}}, \alpha_{\{i_1, i_2\}}))_{\{i_1, i_2\} \in \binom{[N]}{2}}$. For every $i \in [N]$, $q_i \in Q_0 \cup Q_0'$ is the terminal state of π_i. Let $\{i_1, i_2\} \in \binom{[N]}{2}$. If $\{i_1, i_2\}$ is not contained in V or $\text{diff}(\pi_{i_1}, \pi_{i_2})$ exceeds $n^{k+1} \cdot \text{diff}(\delta)$ or z_{i_1} and z_{i_2} differ at some position $j \in [\min\{|z_{i_1}|, |z_{i_2}|\}]$, then $(y_{\{i_1, i_2\}}, \alpha_{\{i_1, i_2\}}) = (\varepsilon, 2)$. Otherwise, $(y_{\{i_1, i_2\}}, \alpha_{\{i_1, i_2\}}) = (z_{i_1}^{-1} z_{i_2}, 1)$. Thus $\{i_1, i_2\}$ belongs to E if and only if $(y_{\{i_1, i_2\}}, \alpha_{\{i_1, i_2\}})$ equals $(\varepsilon, 1)$.

The path $\tilde{\pi}'$ behaves almost in the same way as $\tilde{\pi}$. The only difference is that the value of i_0 is constantly 1, that $\tilde{\pi}'$ produces the empty word ε, and that at the end of this path it is only verified that all vertices of the graph G have rank less than κ.

Now we are ready to construct in detail the NGSM $\tilde{M}_\kappa = (\tilde{Q}, \Sigma, \Delta, \tilde{\delta}, \tilde{Q}_I, \tilde{Q}_{F, \kappa})$ and the NFA $\tilde{M}'_\kappa = (\tilde{Q}, \Sigma, \Delta, \tilde{\delta}', \tilde{Q}_I, \tilde{Q}'_{F, \kappa})$ by setting

$$\tilde{Q} = [N] \times (Q_0 \cup Q'_0)^N \times (\Delta^{\leq \pm n^{k+1} \cdot \operatorname{diff}(\delta)} \times [2])^{\binom{N}{2}},$$

$$\begin{aligned}\tilde{\delta} = \{&((i_0, (p_1, \ldots, p_N), ((y_{\{i_1, i_2\}}, \alpha_{\{i_1, i_2\}}))_{\{i_1, i_2\} \in \binom{[N]}{2}}), a, z, \\ &(i_0, (q_1, \ldots, q_N), ((y'_{\{i_1, i_2\}}, \alpha'_{\{i_1, i_2\}}))_{\{i_1, i_2\} \in \binom{[N]}{2}})) \\ &\in \tilde{Q} \times \Sigma \times \Delta^* \times \tilde{Q} : \\ &\text{there are words } z_1, \ldots, z_N \in \Delta^* \text{ such that } z = z_{i_0}, \\ &\text{for every } i \in [N],\ (p_i, a, z_i, q_i) \in \delta_i \cup \delta'_i, \text{ and,} \\ &\text{for every } \{i_1, i_2\} \in \tbinom{[N]}{2}, \\ &\text{either } \alpha_{\{i_1, i_2\}} = \alpha'_{\{i_1, i_2\}} = 1 \text{ and } z_{i_1}^{-1} y_{\{i_1, i_2\}} z_{i_2} = y'_{\{i_1, i_2\}}, \\ &\text{or } \alpha_{\{i_1, i_2\}} = 1,\ (y'_{\{i_1, i_2\}}, \alpha'_{\{i_1, i_2\}}) = (\varepsilon, 2), \\ &\text{and } z_{i_1}^{-1} y_{\{i_1, i_2\}} z_{i_2} \notin \Delta^{\leq \pm n^{k+1} \cdot \operatorname{diff}(\delta)}, \\ &\text{or } (y_{\{i_1, i_2\}}, \alpha_{\{i_1, i_2\}}) = (y'_{\{i_1, i_2\}}, \alpha'_{\{i_1, i_2\}}) = (\varepsilon, 2)\},\end{aligned}$$

$$\begin{aligned}\tilde{Q}_I = \{&(i_0, (p_1, \ldots, p_N), ((\varepsilon, \alpha_{\{i_1, i_2\}}))_{\{i_1, i_2\} \in \binom{[N]}{2}}) \in \tilde{Q} : \\ &\text{for every } i \in [N],\ p_i \in Q_{I, i} \cup Q'_{I, i} \text{ and,} \\ &\text{for every } \{i_1, i_2\} \in \tbinom{[N]}{2}, \\ &\{p_{i_1}, p_{i_2}\} \nsubseteq Q_0 \text{ if and only if } \alpha_{\{i_1, i_2\}} = 2\},\end{aligned}$$

$$\begin{aligned}\tilde{Q}_{F, \kappa} = \{&(i_0, (q_1, \ldots, q_N), ((y_{\{i_1, i_2\}}, \alpha_{\{i_1, i_2\}}))_{\{i_1, i_2\} \in \binom{[N]}{2}}) \in \tilde{Q} : \\ &\text{for every } i \in [N],\ q_i \in Q_{F, i} \cup Q'_{F, i} \text{ and } i_0 \text{ is the minimal} \\ &\text{vertex of the graph } G = (V, E) \text{ having rank } \kappa \\ &\text{where } V = \{i \in [N] : q_i \in Q_0\} \text{ and} \\ &E = \{\{i_1, i_2\} \in \tbinom{[N]}{2} : (y_{\{i_1, i_2\}}, \alpha_{\{i_1, i_2\}}) = (\varepsilon, 1)\}\},\end{aligned}$$

$$\tilde{\delta}' = \{(r,\, a,\, \varepsilon,\, s) \in \tilde{Q} \times \Sigma \times \{\varepsilon\} \times \tilde{Q} \,:\, \text{there is a } z \in \Delta^* \text{ such that } (r,\, a,\, \varepsilon,\, s) \in \tilde{\delta}\},$$

and

$$\begin{aligned}\tilde{Q}'_{F,\kappa} = \{&(i_0,\, (q_1,\, \ldots,\, q_N),\, ((y_{\{i_1,i_2\}}, \alpha_{\{i_1,i_2\}}))_{\{i_1,i_2\} \in \binom{[N]}{2}}) \in \tilde{Q} \,: \\ &i_0 = 1, \text{ for every } i \in [N],\, q_i \in Q_{F,i} \cup Q'_{F,i}, \\ &\text{and all vertices of the graph } G = (V,\, E) \text{ have rank} \\ &\text{less than } \kappa \text{ where } V = \{i \in [N] \,:\, q_i \in Q_0\} \text{ and} \\ &E = \{\{i_1,\, i_2\} \in \tbinom{[N]}{2} \,:\, (y_{\{i_1,i_2\}}, \alpha_{\{i_1,i_2\}}) = (\varepsilon,\, 1)\}\}.\end{aligned}$$

The NGSM $\tilde{M}_\kappa$ works as desired. Thus, it is easy to establish formally that $\tilde{M}_\kappa$ realizes the transduction T_κ. In order to check that $\tilde{M}_\kappa$ is unambiguous, let us consider an arbitrary accepting path $\tilde{\pi}$ in this machine consuming the word $x \in \Sigma^*$. Since the disjoint unions $M_1 \dot{\cup} M'_1,\, \ldots,\, M_N \dot{\cup} M'_N$ are unambiguous transducers, the second and third components of the states of $\tilde{\pi}$ are uniquely determined by x. The first components of the states of $\tilde{\pi}$ are uniquely determined by κ and by the second and third components of the terminal state of this path. Finally, since the NGSMs $M_1,\, \ldots,\, M_N$ are single valued, the word x and the states of $\tilde{\pi}$ determine the sequence of words produced by the transitions of $\tilde{\pi}$. In summary, the path $\tilde{\pi}$ is uniquely determined by κ and x. Hence, $\tilde{M}_\kappa$ is a UGSM. In the same way it can be seen that $\tilde{M}'_\kappa$ is a UFA which recognizes L_κ.

Note that for each transition $(r,\, a,\, z,\, s)$ of $\tilde{M}_\kappa$ there is an integer $i_0 \in [N]$ and there are states $p,\, q \in Q_0 \cup Q'_0$ such that $(p,\, a,\, z,\, q)$ is a transition of $M_{i_0} \dot{\cup} M'_{i_0}$. Recalling the mapping $\psi \,:\, Q_0 \to Q$ and the properties of the machines $M_1,\, \ldots,\, M_N$ and $M'_1,\, \ldots,\, M'_N$ this implies that either $p,\, q \in Q'_0$ and $z = \varepsilon$ or $p,\, q \in Q_0$ and $(\psi(p),\, a,\, z,\, \psi(q))$ is a transition of M. Having this remark in mind it is straightforward to verify the upper bounds stated in the following fact.

FACT 3.5: *The following assertions on the machines* $\tilde{M}_\kappa$ and $\tilde{M}'_\kappa$ *are true.*

(i) $\# \tilde{Q}$ *is of order* $O(2^{2^{\mathrm{poly}(\|M\|+k)}})$.

(ii) $\mathrm{im}(\tilde{\delta}) \subseteq \mathrm{im}(\delta)$ *and* $\mathrm{diff}(\tilde{\delta}) \leq \mathrm{iml}(\tilde{\delta}) \leq \mathrm{iml}(\delta)$.

(iii) $\|\tilde{\delta}'\| \leq \|\tilde{\delta}\| \leq (\#\Sigma + \|\delta\|) \cdot \#\tilde{Q}^2$ *and* $\|\tilde{M}'_\kappa\| \leq \|\tilde{M}_\kappa\| \leq \|M\| \cdot (1 + \#\tilde{Q}^2)$, i.e., $\tilde{M}_\kappa$ *and* $\tilde{M}'_\kappa$ *have size of order* $O(2^{2^{\mathrm{poly}(\|M\|+k)}})$.

Given a finite, undirected graph G having positive integers as vertices, its connected components and the ranks of all of its vertices can be computed in deterministic time linear in the number of vertices and edges of this graph (*see* Cormen, Leiserson, and Rivest [CLR90, Sect. 23]). Using this result and Fact 3.5, it is easy to see that the machines $\tilde{M}_\kappa$ and $\tilde{M}'_\kappa$ can be computed in $\mathrm{DTIME}(2^{2^{\mathrm{poly}(\|M\|+k)}})$.

This completes the proof of Theorem 3.1. □

We now turn to the proof of Theorem 3.2. The proof will be by reduction to Theorem 3.1. For this purpose we first of all adopt (Weber [W93, Props. 4.5 and 4.4 (ii)]) and then follow the main lines of the proof of (Weber [W93, Thm. 4.1]).

PROPOSITION 3.6 (Weber [W93, Props. 4.5 and 4.4 (ii)]): *Let* $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ *be a finite-valued NFT with* n *states. Then, an NGSM* $M' = (Q, \Sigma', \Delta, \delta', Q_I, Q_F)$, *where* $\Sigma' = \Sigma \dot{\cup} \{a_0\}$, *effectively exists such that the following assertions are true.*

(i) $\|M\| < \|M'\| \leq \|M\| + n + 1$.

(ii) *The machines* M *and* M' *have the same valuedness.*

(iii) *For any nonnegative integer* m, *for all* $x_1, \ldots, x_m \in \Sigma$, *for all nonnegative integers* $\lambda_1, \ldots, \lambda_{m+1}$, *and for any* $z \in \Delta^*$ *we have that if* $(a_0^{\lambda_1} x_1 \ldots a_0^{\lambda_m} x_m a_0^{\lambda_{m+1}}, z) \in T(M')$, *then* $(x_1 \ldots x_m, z) \in T(M)$.

(iv) *For any nonnegative integer* m, *for all* $x_1, \ldots, x_m \in \Sigma$, *for any* $z \in \Delta^*$, *and for all integers* $\lambda_1, \ldots, \lambda_{m+1} \geq n - 1$ *we have that if* $(x_1 \ldots x_m, z) \in T(M)$, *then* $(a_0^{\lambda_1} x_1 \ldots a_0^{\lambda_m} x_m a_0^{\lambda_{m+1}}, z) \in T(M')$.

(v) *The machine* M' *can be computed in* $\mathrm{DTIME}(\mathrm{poly}\|M\|)$.

Proof of Theorem 3.2: Let $M = (Q, \Sigma, \Delta, \delta, Q_I, Q_F)$ be a k-valued NFT with n states, where k is a positive integer. Let $M^{(1)} = (Q, \Sigma^{(1)}, \Delta, \delta^{(1)}, Q_I, Q_F)$ be the NGSM associated with M in Proposition 3.6. From this proposition we obtain that $\Sigma^{(1)} = \Sigma \dot{\cup} \{a_0\}$ and that $M^{(1)}$ is a k-valued NGSM of size $\Theta(\|M\|)$ which can be computed in $\mathrm{DTIME}(\mathrm{poly}\|M\|)$. Applying Theorem 3.1 to $M^{(1)}$ we obtain k UGSMs $M_1^{(1)}, \ldots, M_k^{(1)}$ and UFAs $M_1^{(2)}, \ldots, M_k^{(2)}$ such that $T(M^{(1)})$ equals $T(M_1^{(1)}) \cup \ldots \cup T(M_k^{(1)})$ and, for every $\kappa \in [k]$, $M_\kappa^{(2)}$ recognizes $(\Sigma^{(1)})^* \backslash L(M_\kappa^{(1)})$. Each of these new machines has size $O(2^{2^{\mathrm{poly}(\|M^{(1)}\|+k)}})$ and can be computed in $\mathrm{DTIME}(2^{2^{\mathrm{poly}(\|M^{(1)}\|+k)}})$.

Let $\kappa \in [k]$. Consider the UGSM $M_\kappa^{(1)} = (Q_\kappa^{(1)}, \Sigma^{(1)}, \Delta, \delta_\kappa^{(1)}, Q_{I,\kappa}^{(1)}, Q_{F,\kappa}^{(1)})$. We associate with $M_\kappa^{(1)}$ the NFT $\tilde{M}_\kappa = (\tilde{Q}_\kappa, \Sigma, \Delta, \tilde{\delta}_\kappa, \tilde{Q}_{I,\kappa}, \tilde{Q}_{F,\kappa})$

by setting $\tilde{Q}_\kappa = Q_\kappa^{(1)} \times \{0, \ldots, n-1\}$, $\tilde{Q}_{I,\kappa} = Q_{I,\kappa}^{(1)} \times \{0\}$, $\tilde{Q}_{F,\kappa} = Q_{F,\kappa}^{(1)} \times \{n-1\}$, and

$$\begin{aligned}\tilde{\delta}_\kappa = &\{((p,\, n-1),\, a,\, z,\, (q,\, 0)) \,:\, a \in \Sigma,\, (p,\, a,\, z,\, q) \in \delta_\kappa^{(1)}\}\\ &\cup \{((p,\, j-1),\, \varepsilon,\, z,\, (q,\, j)) \,:\, j \in [n-1],\, (p,\, a_0,\, z,\, q) \in \delta_\kappa^{(1)}\}.\end{aligned}$$

Consider next the UFA $M_\kappa^{(2)} = (Q_\kappa^{(2)}, \Sigma^{(1)}, \Delta, \delta_\kappa^{(2)}, Q_{I,\kappa}^{(2)}, Q_{F,\kappa}^{(2)})$. We associate with $M_\kappa^{(2)}$ the ε-NFA $\tilde{M}'_\kappa = (\tilde{Q}'_\kappa, \Sigma, \Delta, \tilde{\delta}'_\kappa, \tilde{Q}'_{I,\kappa}, \tilde{Q}'_{F,\kappa})$ by setting $\tilde{Q}'_\kappa = Q_\kappa^{(2)} \times \{0, \ldots, n-1\}$, $\tilde{Q}'_{I,\kappa} = Q_{I,\kappa}^{(2)} \times \{0\}$, $\tilde{Q}'_{F,\kappa} = Q_{F,\kappa}^{(2)} \times \{n-1\}$, and

$$\begin{aligned}\tilde{\delta}'_\kappa = &\{((p,\, n-1),\, a,\, \varepsilon,\, (q,\, 0)) \,:\, a \in \Sigma,\, (p,\, a,\, \varepsilon,\, q) \in \delta_\kappa^{(2)}\}\\ &\cup \{((p,\, j-1),\, \varepsilon,\, \varepsilon,\, (q,\, j)) \,:\, j \in [n-1],\, (p,\, a_0,\, \varepsilon,\, q) \in \delta_\kappa^{(2)}\}.\end{aligned}$$

We observe that $\|\tilde{M}_\kappa\| \leq n \cdot \|M_\kappa^{(1)}\|$ and $\|\tilde{M}'_\kappa\| \leq n \cdot \|M_\kappa^{(2)}\|$. Thus, the machines $\tilde{M}_\kappa$ and $\tilde{M}'_\kappa$ are of size $O(2^{2^{\text{poly}(\|M\|+k)}})$. Given $M_\kappa^{(1)}$, the NFT $\tilde{M}_\kappa$ can be computed in DTIME $(\text{poly}(n + \|M_\kappa^{(1)}\|))$. Given $M_\kappa^{(2)}$, the ε-NFA $\tilde{M}'_\kappa$ can be computed in DTIME $(\text{poly}(n + \|M_\kappa^{(2)}\|))$. Therefore, the machines $\tilde{M}_\kappa$ and $\tilde{M}'_\kappa$ can be computed in DTIME $(2^{2^{\text{poly}(\|M\|+k)}})$.

Proposition 3.6 and the definition of the machines $\tilde{M}_1, \ldots, \tilde{M}_k$ and $\tilde{M}'_1, \ldots, \tilde{M}'_k$ yield for all $x_1, \ldots, x_m \in \Sigma$, $z \in \Delta^*$, and $\kappa \in [k]$ that the following assertions hold true.

- $(x_1 \ldots x_m,\, z) \in T(M)$
  if and only if $(a_0^{n-1}\, x_1 \ldots a_0^{n-1}\, x_m\, a_0^{n-1},\, z) \in T(M^{(1)})$.

- $(x_1 \ldots x_m,\, z) \in T(\tilde{M}_\kappa)$
  if and only if $(a_0^{n-1}\, x_1 \ldots a_0^{n-1}\, x_m\, a_0^{n-1},\, z) \in T(M_\kappa^{(1)})$.

- $x_1 \ldots x_m \in L(\tilde{M}'_\kappa)$
  if and only if $(a_0^{n-1}\, x_1 \ldots a_0^{n-1}\, x_m\, a_0^{n-1}) \in L(M_\kappa^{(2)})$.

From this follows that $T(M)$ equals $T(\tilde{M}_1) \cup \ldots \cup T(\tilde{M}_k)$ and, for every $\kappa \in [k]$, $\tilde{M}'_\kappa$ recognizes $\Sigma^* \backslash L(\tilde{M}_\kappa)$. Moreover, for every $\kappa \in [k]$, it is easy to see that the machine $\tilde{M}_\kappa$ ($\tilde{M}'_\kappa$) inherits from $M_\kappa^{(1)}$ ($M_\kappa^{(2)}$, respectively) the property of being unambiguous. □

ACKNOWLEDGMENT

I would like to thank Helmut Seidl for useful stimulations and valuable comments concerning the main ideas and the referee for his or her suggestions concerning the presentation.

REFERENCES

[B79] J. Berstel, Transductions and Context-Free Languages, Teubner, Stuttgart, Germany, 1979.

[CLR90] T. Cormen, C. Leiserson and R. Rivest, Introduction to Algorithms, MIT Press, Cambridge, MA, 1990.

[C90] K. Culik II, New techniques for proving the decidability of equivalence problems, *Theoretical Computer Science*, 1990, *71*, pp. 29-45.

[E74] S. Eilenberg, Automata, Languages, and Machines, Volume A, Academic Press, New York, NY, 1974.

[G89] E. Gurari, An Introduction to the Theory of Computation, Computer Science Press, Rockville, MD, 1989.

[GI83] E. Gurari and O. Ibarra, A note on finite-valued and finitely ambiguous transducers, *Mathematical Systems Theory*, 1983, *16*, pp. 61-66.

[K87] J. Karhumäki, The equivalence of mappings on languages, Proc. 4th IMYCS 1986, in: *Lecture Notes in Computer Science*, No. 281, Springer-Verlag, Berlin, Heidelberg, Germany, 1987, pp. 26-38.

[Le93] H. Leung, Separating exponentially ambiguous NFA from polynomially ambiguous NFA, Proc. 4th ISAAC 1993, in: *Lecture Notes in Computer Science*, No. 762, Springer-Verlag, Berlin, Heidelberg, Germany, 1993, pp. 221-229.

[Li91] L. P. Lisovik, personal communication, 1991.

[LS77] R. Lyndon and P. Schupp, Combinatorial Group Theory, Springer-Verlag, Berlin, Heidelberg, Germany, New York, NY, 1977.

[Sch76] M. P. Schützenberger, Sur les relations rationnelles entre monoïdes libres, *Theoretical Computer Science*, 1976, *3*, pp. 243-259.

[Se94] H. Seidl, Equivalence of finite-valued tree transducers is decidable, *Mathematical Systems Theory*, 1994, *27*, pp. 285-346.

[W90] A. Weber, On the valuedness of finite transducers, *Acta Informatica*, 1990, *27*, pp. 749-780.

[W92a] A. Weber, On the lengths of values in a finite transducer, *Acta Informatica*, 1992, *29*, pp. 663-687.

[W92b] A. WEBER, Decomposing a k-valued transducer into k unambiguous ones, Proc. 1st LATIN 1992, in: *Lecture Notes in Computer Science*, No. 583, Springer-Verlag, Berlin, Heidelberg, Germany, 1992, pp. 503-515.

[W93] A. WEBER, Decomposing finite-valued transducers and deciding their equivalence, *SIAM J. Computing*, 1993, *22*, pp. 175-202.

[W94] A. WEBER, Finite-valued distance automata, *Theoretical Computer Science*, 1994, *134*, pp. 225-251.

[WK95] A. WEBER and R. KLEMM, Economy of description for single-valued transducers, *Information and Computation*, 1995, *118*, pp. 327-340.