INFORMATIQUE THÉORIQUE ET APPLICATIONS

HELEN CAMERON
DERICK WOOD

Binary trees, fringe thickness and minimum path length

Informatique théorique et applications, tome 29, nº 3 (1995), p. 171-191
<http://www.numdam.org/item?id=ITA_1995__29_3_171_0>

Informatique théorique et Applications/Theoretical Informatics and Applications
(vol. 29, n° 3, 1995, p. 171 à 191)

BINARY TREES, FRINGE THICKNESS AND MINIMUM PATH LENGTH (*)

by Helen CAMERON (1) and Derick WOOD (2)

Communicated by C. CHOFFRUT

Abstract. – *We solve the following problem: Characterize the minimum-path-length binary trees with respect to size and fringe thickness, where the fringe thickness of a tree is the difference between the lengths of shortest and longest root-to-frontier paths. This result demonstrates that minimum path length is, in this setting, more amenable to analysis than maximum path length.*

Résumé. – *Nous résolvons le problème suivant : caractériser les arbres binaires de longueur de chemins minimum pour une taille et une épaisseur de frange données, où l'épaisseur de la frange est égale à la différence entre les longueurs d'un plus court et d'un plus long chemin de la racine à la frontière. Ce résultat démontre que, dans ce contexte, la longueur minimale d'un chemin se prête mieux à l'analyse que la longueur maximale.*

1. INTRODUCTION

We argue that one method of measuring the efficiency of a class of trees is to compute the average number of comparisons made by an insert, delete, or member operation in each tree in the class. But the average number of comparisons made by one of these operations in a given tree is the average length of a path from the root to a node. Moreover, the average path length of a tree (with respect to a uniform probability distribution on the items in the tree) is its path length (the sum of the lengths of the path from the root to each node in the tree) divided by the number of nodes in the tree. Hence, the path length is an important measure of the efficiency of a class of trees,

(*) Received january 1993; revised june 1994; accepted october 1994. This work was supported under grants from the Natural Sciences and Engineering Research Council of Canada and the Information Technology Research Centre of Ontario.

(1) Department of Computer Science, University of Manitoba, Winnipeg, Manitoba, R3T 2N2, Canada.

(2) Department of Computer Science, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.

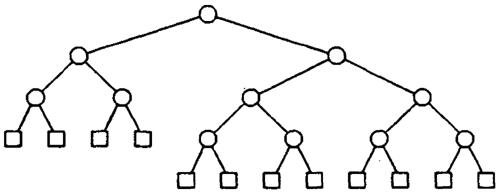

Figure 1. – A minimum path length binary tree of size 11.

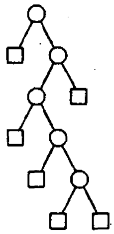

Figure 2. – A maximum path length binary tree of size 5.

in its own right. In addition, the minimum and maximum values of the path length, for each size of tree, are also important, since they determine the range of possible path lengths and, hence, the range of values of the average number of comparisons made by an operation in trees of the same size.

Knuth [Knu73] showed that a binary tree has the minimum path length among all binary trees of size N if and only if the external nodes (nodes with no children) appear on exactly two levels in the tree and those two levels are consecutive; see Figure 1. The external path length of such a tree is

$$(N+1)\,(\log_2 (N+1) + 1 + \theta - 2^{\theta}),$$

where $\theta = \lceil \log_2 (N+1) \rceil - \log_2 (N+1) \in [0,\, 1)$. A binary tree has the maximum path length among all binary trees of size N if and only if it has at most one internal node per level; that is, a binary tree has the maximum path length for its size if and only if every internal node has at most one internal child; see, for example, the tree in Figure 2. The external path length of such a tree is

$$\frac{N\,(N+3)}{2}.$$

The path lengths of most binary trees fall somewhere in the middle of this range, rather than at the extremes; therefore, there have been attemps to refine these bounds. Nievergelt and Wong [NW73] give an upper bound for

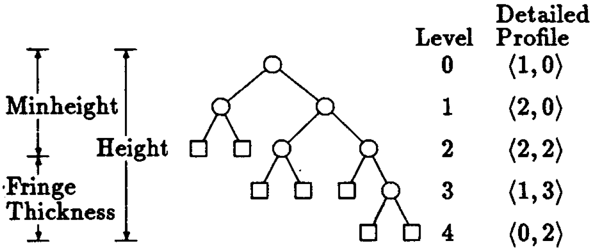

Figure 3. – An example binary tree. It has size 6, height 4, minheight 2 (that is, it has a *Bin*(2) prefix), and fringe thickness 2. Furthermore, its external path length is 21.

the path length of a binary tree T in terms of the weight (the number of external nodes) and the maximum weight balance of T's subtrees. Klein and Wood [KW89] derive the upper bound

$$(N+1)\left(\log_2(N+1)+\Delta-\log_2\Delta-\Psi(\Delta)\right)$$

for the external path length of a binary tree of size N and fringe thickness Δ, where $\Psi(\Delta) \geq 0.6622\ldots$

Driven by similar concerns, we want to characterize the minimum path length trees for each size *and* fringe thickness. (The corresponding problem for maximum path length trees is still open; Klein and Wood [KW89] and Cameron and Wood [Cam91, CW94] have obtained partial results.) Recently, De Santis and Persiano [DP94] derived an attainable lower bound for the path length of binary trees of a given size and fringe thickness, when the fringe thickness is less than half of the size. We solve the minimum-path-length problem completely using an approach based on some of DeSantis and Persiano's intermediate results and on their methodology.

2. DEFINITIONS

We now give the basic definitions and results for binary search trees. Many of the following definitions are illustrated in Figure 3. The trees that we consider are extended trees; that is, the nodes of each tree are of two types: **internal nodes** (nodes that have at least one child) and **external nodes** (nodes with no children). **A binary tree** is a tree in which every internal node has exactly two children.

The size of a binary tree T is the number of internal nodes in the tree; it is denoted by *size* (T). The **height** of a tree T is the number of edges on a longest root-to-frontier path; it is denoted by $ht\,(T)$. The **level** of a node in a tree is the distance of the node from the root of the tree, where the distance is the number of edges on the path from the root to the node. Thus, the root is a level 0, its children (if any) are at level 1, their children are at level 2, and so on.

DEFINITION 2.1: *The* **minheight** *of binary tree T, denoted by $minht\,(T)$, is the minimum level containing an external node; that is $minht\,(T)$ is the number of edges on a shortest path from the root to an external node.*

DEFINITION 2.2: *The* **fringe thickness** *of a tree T is the difference between the lengths of a longest and a shortest path from the root to an external node; that is, the fringe thickness is $ht\,(T) - minht\,(T)$*

Note that if we are given any two of the values, height, minheight, and fringe thickness, then we can compute the third value.

We can ignore the placement of nodes in the arguments that we use throughout this paper; we need to known only how far each node is from the root of the tree. The (external-node) profile of an extended binary tree is an appropriate abstraction; it was introduced by De Santis and Persiano [DP94]. The **(external-node) profile** of a binary tree is the sequence $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_h)$ of the numbers of external nodes on each level in the tree. We employ the following shorthand notation for profiles: a^b represents

$$\underbrace{a, \ldots, a,}_{b}$$

* represents a nonnegative integer, and + represents a positive integer. Not all sequences of integers are profiles of binary trees; therefore, we shall use the Kraft Equality to determine whether a sequence of integers is a profile.

PROPOSITION 2.1 **(Kraft Equality)**: *Let $l_1, \ldots, l_{N+1}$ be $N+1$ nonnegative integers, for some $N \geq 0$; then there is an extended binary tree with a total of $N+1$ external nodes on levels $l_1, \ldots, l_{N+1}$ if and only if*

$$\sum_{i=1}^{N+1} 2^{-l_i} = 1.$$

The **perfect binary tree of height h**, denoted by $Bin\,(h)$, is the only binary tree of height h whose external nodes all appear on one level. A recursive definition $Bin\,(h)$ is given in Figure 4. Level i of $Bin\,(h)$ contains 2^i nodes and each node on that level is the root of a $Bin\,(h-i)$ subtree.

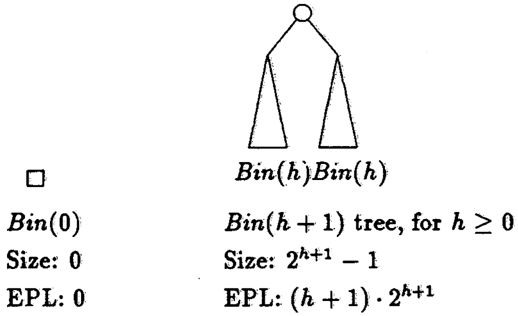

Figure 4. – A recursive definition of the $Bin(h)$ tree.

A **snake of height h**, denoted by $Snake(h)$, is any binary tree of height h that consists of a chain of h internal nodes, one on each of the levels $0, \ldots, h-1$; Figure 5 displays an example of a $Snake(h)$.

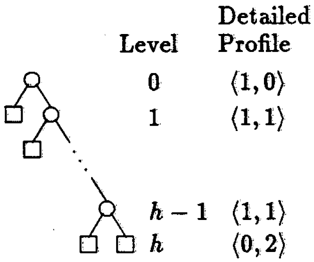

Figure 5. – A snake of height h. A $Snake(h)$ has size h and its external path length is $h \cdot (h+3)/2$.

DEFINITION 2.3: *A binary tree has a* **binary prefix of height b** *(or a $Bin(b)$ prefix), if the levels $0, \ldots, b-1$ contain only internal nodes and level b contains at least one external node.*

Since the root of every nonempty binary tree is an internal node, every nonempty binary tree has at least a $Bin(1)$ prefix. Note that the height of the binary prefix of a binary tree is the minheight of the tree.

Let T be a binary tree and let $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_h)$ be its profile. The **external path length** of T is denoted by EPL (T) and is defined to be $EPL(T) = \sum_{i=0}^{h} i \cdot \varepsilon_i$. In other words, the external path length is the sum, over all external nodes, of their distances from the root.

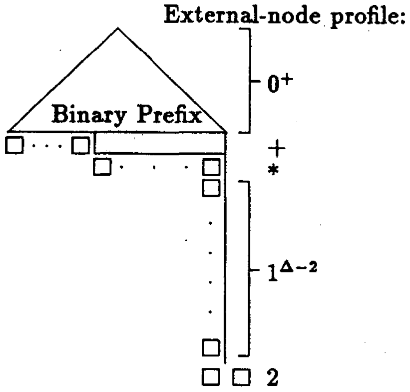

Figure 6. – **A right-weighted** $(a,\ \Delta,\ N)$**-tree, where** $a = -1$.

3. TWO TRIVIAL CASES

The two trivial cases are when the fringe thickness is 0 or 1. The perfect binary trees are the only binary trees with fringe thickness 0. There is a binary tree of size N and fringe thickness 0 if and only if $N + 1$ is a power of two. In contrast, there is a binary tree of size N and fringe thickness 1 if and only if $N + 1$ is not a power of two. All binary trees of size N and fringe thickness 1 have the same profile; each has height $\lceil \log_2 (N + 1) \rceil$ and exactly $N - 2^{\lfloor \log_2 (N+1) \rfloor} + 1$ internal nodes on the next-to-last level, level $\lfloor \log_2 (N + 1) \rfloor$. Thus, we assume that $\Delta > 1$ in the following sections.

4. $(a,\ \Delta,\ N)$-TREES

De Santis and Persiano [DP94] show that the binary trees that have minimum path length, for a given size N and fringe thickness Δ, are $(a,\ \Delta,\ N)$-trees. Figures 6, 7 and 8 provide examples of the three kinds of right-weighted $(a,\ \Delta,\ N)$-trees.

Definition 4.1: *For* $N \geq 3$, $2 \leq \Delta \leq N - 1$, *and* $-1 \leq a \leq \Delta - 2$, *an* $(a,\ \Delta,\ N)$**-tree** *is a binary tree of size* N *and fringe thickness* Δ *that has one of the following profiles:*

- $(0^+, +, *, 1^{\Delta-2}, 2)$, *if* $a = -1$.
- $(0^+, 1, 0^{\Delta-2}, *, *)$, *if* $a = \Delta - 2$.
- $(0^+, 1, 0^a, *, *, 1^{\Delta-(a+3)}, 2)$, *otherwise.*

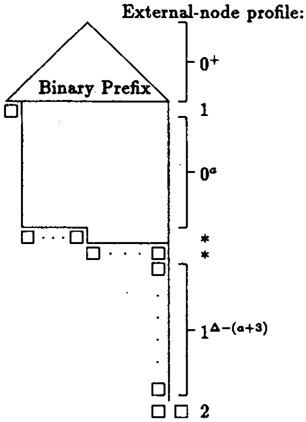

Figure 7. – A right-weighted $(a,\ \Delta,\ N)$**-tree, where** $-1 < a < \Delta - 2$.

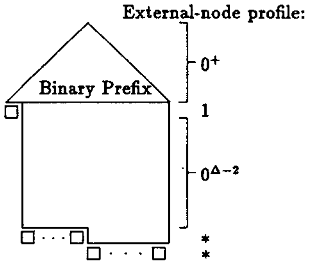

Figure 8. – A right-weighted $(a,\ \Delta,\ N)$**-tree, where** $a = \Delta - 2$.

PROPOSITION 4.1 **(De Santis and Persiano [DP94])**: *Let* $N \geq 3$ *and* $2 \leq \Delta \leq N - 1$. *If* T *has the minimum path length among all binary trees of size* N *and fringe thickness* Δ, *then* T *is an* $(a,\ \Delta,\ N)$*-tree, for some* a.

Given N, Δ, and a, what conditions must they satisfy to guarantee the existence of an $(a,\ \Delta,\ N)$-tree? Certainly, we must have $2 \leq \Delta \leq N - 1$ and $-1 \leq a \leq \Delta - 2$, but these conditions are not sufficient. As a first step in finding sufficient conditions, we show how to bound the *'s and +'s in the profiles of the $(a,\ \Delta,\ N)$-trees, when we are also given the minheight b.

LEMMA 4.1: *Let* $N \geq 3$, $2 \leq \Delta \leq N - 1$, $-1 \leq a \leq \Delta - 2$, *and* $b \geq 1$. *If there is an* $(a,\ \Delta,\ N)$*-tree with minheight* b, *then it has one of the following profiles:*

• $(0^b, *_1, *_2, 1^{\Delta-2}, 2)$, *where* $1 \leq *_1 \leq 2^b - 1$ *and* $*_2 = 2\,(2^b - *_1) - 1$, *if* $a = -1$.

• $(0^b, 1, 0^{\Delta-2}, *_1, *_2)$, *where* $0 \leq *_1 \leq 2^{\Delta-1}(2^b - 1) - 1$ *and* $*_2 = 2(2^{\Delta-1}(2^b - 1) - *_1)$, *if* $a = \Delta - 2$.

• $(0^b, 1, 0^a, *_1, *_2, 1^{\Delta-(a+3)}, 2)$, *where* $0 \leq *_1 \leq 2^{a+1}(2^b - 1) - 1$ *and* $*_2 = 2(2^{a+1}(2^b - 1) - *_1) - 1$, *otherwise.*

Proof: Let T be an (a, Δ, N)-tree with minheight b. We use the profile of T and the following fact to compute the numbers of internal nodes on each level, which enable us to derive bounds on $*_1$ and relate $*_1$ to $*_2$. Letting ι_i denote the number of internal nodes on level i, it is clear that the number of nodes on level $i + 1$ is $2\iota_i$, because internal nodes have two children and external nodes have no children. Moreover, the number of nodes on level $i + 1$ can also be expressed as $\iota_{i+1} + \varepsilon_{i+1}$.

Because T has minheight b, the numbers of internal nodes on levels 0 to $b - 1$ is $2^0, 2^1, \ldots, 2^{b-1}$. The numbers of internal nodes on the remaining levels depend on the parameter a. We consider the three possible values of a in turn.

If $-1 < a < \Delta - 2$, then $(0^b, 1, 0^a, *_1, *_2, 1^{\Delta-(a+3)}, 2)$ is the profile of T. Because there are 2^{b-1} internal nodes on level $b - 1$ and we are given the numbers of external nodes on levels b to $b + a$, we can compute the numbers of internal nodes on levels b to $b + a$. The numbers of internal nodes on levels b through $b + a$ is, therefore, $2^b - 1, 2^1(2^b - 1), \ldots, 2^a(2^b - 1)$.

Similarly, we can compute, from the frontier upwards, the numbers of internal nodes on levels $b + a + 3, b + a + 4, \ldots, b + \Delta$ to be $1, \ldots, 1, 0$.

We can now bound $*_1$, the number of external nodes on level $b + a + 1$. Since there are nodes on levels $b + a + 2$ onward, there is at least one internal node on level $b + a + 1$. Furthermore, since there are $2^a(2^b - 1)$ internal nodes on level $b + a$, there are $2^{a+1}(2^b - 1)$ nodes on level $b + a + 1$; therefore, $0 \leq *_1 \leq 2^{a+1}(2^b - 1) - 1$. In addition, because there are two nodes on level $b + a + 3$, there is exactly one internal node on level $b + a + 2$; therefore, $*_2 = 2(2^{a+1}(2^b - 1) - *_1) - 1$.

If $a = -1$, then $(0^b, *_1, *_2, 1^{\Delta-2}, 2)$ is the profile of T. Since T has minheight b, there must be at least one external node on level b; that is,

$*_1 \geq 1$. By similar aruguments, we can show that the numbers of internal nodes is:

$$\begin{aligned} 1,\, 2, \ldots,\, 2^{b-1} &\Leftarrow Bin\,(b)\,\text{prefix},\\ 2^b - *_1,\, 1 &\Leftarrow \text{Levels}\, b \text{ and } b+1\\ 1, \ldots,\, 1 &\Leftarrow \text{Levels}\, b+2, \ldots,\, b+\Delta-1,\\ 0 &\Leftarrow \text{Level}\, b+\Delta, \end{aligned}$$

where $1 \leq *_1 \leq 2^b - 1$. Furthermore, $*_2 = 2\,(2^b - *_1) - 1$.

Finally, if $a = \Delta - 2$, then $(0^b,\, 1,\, 0^{\Delta-2},\, *_1,\, *_2)$ is the profile of T. In this case, the numbers of internal nodes is

$$\begin{aligned} 1,\, 2, \ldots,\, 2^{b-1} &\Leftarrow Bin\,(b)\,\text{prefix},\\ 2^b - 1,\, 2\,(2^b - 1), \ldots, 2^{\Delta-2}\,(2^b - 1) &\Leftarrow \text{Levels}\, b, \ldots, b+\Delta-2,\\ 2^{\Delta-1}\,(2^b - 1) - *_1 &\Leftarrow \text{Level}\, b+\Delta-1,\\ 0 &\Leftarrow \text{Level}\, b+\Delta, \end{aligned}$$

where $0 \leq *_1 \leq 2^{\Delta-1}\,(2^b - 1) - 1$. Furthermore, $*_2 = 2\,(2^{\Delta-1}\,(2^b - 1) - *_1)$. □

LEMMA 4.2: *Let $\Delta \geq 2$, $-1 \leq a \leq \Delta - 2$ and $b \geq 1$. Then, there is an (a, Δ, N)-tree with minheight b if and only if*

$$(2^b - 1)\, 2^{a+1} + \Delta - a \leq N + 1 \leq (2^b - 1)\, 2^{a+2} + \Delta - (a+1).$$

Proof: We will examine the case $-1 < a < \Delta - 2$; the proofs for $a = -1$ and $a = \Delta - 2$ are similar.

($\Rightarrow$) Assume that there is an (a, Δ, N)-tree with minheight b.

By Lemma 4.1, if $-1 < a < \Delta - 2$, then an (a, Δ, N)-tree with minheight b must have profile $(0^b,\, 1,\, 0^a,\, *_1,\, *_2,\, 1^{\Delta-(a+3)},\, 2)$, where $0 \leq *_1 \leq 2^{a+1}\,(2^b - 1) - 1$ and $*_2 = 2\,(2^{a+1}\,(2^b - 1) - *_1) - 1$. Hence, we see that the size of the tree satisfies the equation

$$\begin{aligned} N + 1 &= 1 + *_1 + 2\,(2^{a+1}\,(2^b - 1) - *_1) - 1 + \Delta - (a+3) + 2\\ &= 2^{a+2}\,(2^b - 1) + \Delta - a - 1 - *_1. \end{aligned}$$

Using the inequalities for $*_1$, we conclude that,

$$2^{a+1}(2^b-1)+\Delta-a \le N+1 \le 2^{a+2}(2^b-1)+\Delta-a-1.$$

($\Rightarrow$) Assume that N satisfies

$$2^{a+1}(2^b-1)+\Delta-a \le N+1 \le 2^{a+2}(2^b-1)+\Delta-a-1.$$

To show that there is an $(a,\ \Delta,\ N)$-tree of minheight b and profile $(0^b,\ 1,\ 0^a,\ *_1,\ *_2,\ 1^{\Delta-(a+3)},\ 2)$, we let $*_1 = 2^{a+2}(2^b-1)+\Delta-a-2-N$ and $*_2 = 2(2^{a+1}(2^b-1)-*_1)-1 = 2N-2^{a+2}(2^b-1)-2\Delta+2a+3$. Now, we must show that $E=(0^b,\ 1,\ 0^a,\ *_1,\ *_2,\ 1^{\Delta-(a+3)},\ 2)$ is the profile of an $(a,\ \Delta,\ N)$-tree of minheight b.

If E is the profile of some binary tree T, then T has size $N = *_1 + *_2 + \Delta - (a+3) + 2$, fringe thickness Δ, minheight b, and T is an $(a,\ \Delta,\ N)$-tree. Now, we establish that E is the profile of a binary tree by showing that $*_1$ and $*_2$ are nonnegative, and that E satisfies the Kraft equality. The inequality $*_1 \ge 0$ follows directly from the inequality $N+1 \le 2^{a+2}(2^b-1)+\Delta-a-1$ and the inequality $*_2 \ge 0$ follows the inequality $2^{a+1}(2^b-1)+\Delta-a \le N+1$. Now,

$$\sum_{i=0}^{b+\Delta} e_i\,2^{-i} = 2^{-b}+*_1\,2^{-(b+a+1)}+*_2\,2^{-(b+a+2)}+\sum_{i=b+a+3}^{b+\Delta-1} 2^{-i}+2\cdot 2^{-(b+\Delta)},$$

where e_i is the i-th element of E. Substituting the values of $*_1$ and $*_2$ and using the identity $\sum_{i=0}^{n} 2^{-i} = (2^{n+1}-1)/2^n$

$$\begin{aligned}\sum_{i=0}^{b+\Delta} e_i\,2^{-i} &= 2^{-b}(2^{a+2}(2^b-1)+\Delta-a-2-N)\,2^{-(b+a+1)}\\ &\quad+(2N-2^{a+2}(2^b-1)-2\Delta+2a+3)\,2^{-(b+a+2)}\\ &\quad+(2^{b+\Delta}-1)\,2^{-(b+\Delta-1)}-(2^{b+a+3}-1)\,2^{-(b+a+2)}\\ &\quad+2^{-(b+\Delta-1)}\\ &= 2^{-b}+2^{a+2}(2^b-1)\,2^{-(b+a+1)}-2\cdot 2^{-(b+a+1)}\\ &\quad+3\cdot 2^{-(b+a+2)}-2^{-(b+\Delta)}+2^{-(b+a+2)}\\ &= 1.\end{aligned}$$

Thus, by the Kraft Equality, E is the profile of an $(a,\ \Delta,\ N)$-tree of minheight b. □

5. CHARACTERIZATION OF MINIMUM-PATH-LENGTH TREES

We begin the approach to our main theorem (Theorem 5.1) with the definition of a value that will be the minheight of minimum-path-length trees.

DEFINITION 5.1: *Let $b(a,\ \Delta,\ N)$ be the value*

$$\left\lfloor \log_2 \left(\frac{N + 1 + 2^{a+1} + a - \Delta}{2^{a+1}} \right) \right\rfloor.$$

We now establish necessary and sufficient conditions for the existence of $(a,\ \Delta,\ N)$-trees.

THEOREM 5.1: *Let $N \geq 3$, $2 \leq \Delta \leq N - 1$, and $-1 \leq a \leq \Delta - 2$. Then, there is an $(a,\ \Delta,\ N)$-tree if and only if $b(a,\ \Delta,\ N) \geq 1$ and*

$$\frac{N + 1 + 2^{a+1} + a - \Delta}{2^{a+1}} < 2^{b(a,\ \Delta,\ N)+1} - 1.$$

Furthermore, each $(a,\ \Delta,\ N)$-tree has minheight $b(a,\ \Delta,\ N)$.

Proof:

($\Rightarrow$) Assume that there is an $(a,\ \Delta,\ N)$-tree T. Now, T has minheight b, for some $b > 0$. By Lemma 4.2, the size N of T satisfies the inequality

$$(2^b - 1)\, 2^{a+1} + \Delta - a \leq N + 1 \leq (2^b - 1)\, 2^{a+2} + \Delta - (a + 1),$$

which yields the inequality

$$(2^b - 1)\, 2^{a+1} \leq N + 1 - \Delta + a \leq (2^b - 1)\, 2^{a+1} - 1.$$

Now, algebraic manipulation gives

$$2^b - 1 \leq \frac{N + 1 + 2^{a+1} + a - \Delta}{2^{a+1}} < 2^{b+1} - 1$$

and, since b is an integer, $\log_2 (2^{b+1} - 1) < b + 1$, and

$$b(a,\ \Delta,\ N) = \left\lfloor \log_2 \frac{N + 1 + 2^{a+1} + a - \Delta}{2^{a+1}} \right\rfloor,$$

we can conclude that $b \leq b(a, \Delta, N) < b + 1$; therefore, $b = b(a, \Delta, N)$. In other words, $b(a, \Delta, N)$ is the minheight of the (a, Δ, N)-tree T. Since the minheight of a nonempty binary tree is at least 1, $b(a, \Delta, N) \geq 1$. Furthermore, since

$$\frac{N + 1 + 2^{a+1} + a - \Delta}{2^{a+1}} < 2^{b+1} - 1$$

and $b = b(a, \Delta, N)$,

$$\frac{N + 1 + 2^{a+1} + a - \Delta}{2^{a+1}} < 2^{b(a, \Delta, N)+1} - 1.$$

($\Leftarrow$) Suppose that $b(a, \Delta, N) \geq 1$ and

$$\frac{N + 1 + 2^{a+1} + a - \Delta}{2^{a+1}} < 2^{b(a, \Delta, N)+1} - 1.$$

Since

$$b(a, \Delta, N) = \left\lfloor \log_2 \frac{N + 1 + 2^{a+1} + a - \Delta}{2^{a+1}} \right\rfloor,$$

$$2^{b(a, \Delta, N)} \leq \frac{N + 1 + 2^{a+1} + a - \Delta}{2^{a+1}} < 2^{b(a, \Delta, N)+1}.$$

But

$$\frac{N + 1 + 2^{a+1} + a - \Delta}{2^{a+1}} < 2^{b(a, \Delta, N)+1} - 1,$$

so

$$2^{b(a, \Delta, N)} \leq \frac{N + 1 + 2^{a+1} + a - \Delta}{2^{a+1}} < 2^{b(a, \Delta, N)+1} - 1.$$

We now reverse the argument in the first part of the proof to obtain the inequality

$$(2^{b(a, \Delta, N)} - 1)\, 2^{a+1} + \Delta - a \leq N + 1 \leq (2^{b(a, \Delta, N)} - 1)\, 2^{a+2} + \Delta - (a + 1).$$

Since $b(a, \Delta, N) \geq 1$, by Lemma 4.2, there is a (a, Δ, N)-tree of minheight $b(a, \Delta, N)$. $\square$

It is easy to understand Condition 1 ($b(a, \Delta, N) \geq 1$) in Theorem 5.1: since $b(a, \Delta, N)$ is the minheight of the (a, Δ, N)-tree (if it exists), it must be at least one. Condition 2,

$$\frac{N+1+2^{a+1}+a-\Delta}{2^{a+1}} < 2^{b(a, \Delta, N)+1} - 1,$$

relates to the number of "slots" for nodes on level $b+a+1$, where $b = b(a, \Delta, N)$, versus the number of internal nodes left to be placed on level $b+a+1$ after the rest of the (a, Δ, N)-tree has been constructed. Consider building an (a, Δ, N)-tree, where $-1 < a < \Delta - 2$, by starting with a bin of N internal nodes and removing nodes to build parts of the tree, leaving the nodes on level $b+a+1$ to last. We build the tree as follows:

Levels 0 to $b-1$: By Theorem 5.1, b is the minheight of the tree. Therefore, remove $2^b - 1$ internal nodes to build the binary prefix.

Levels b to $b+a$: From the profile, the numbers of external nodes on levels b to $b+a$ are $1, 0^a$. Tne numbers of internal nodes on level $b-1$ is 2^{b-1}. By applying the formula

$$2 \cdot \iota_{i-1} = \iota_i + \varepsilon_i,$$

where ι_j is the number of internal nodes on level j and ε_j is the number of external nodes on level j, to levels b to $b+a$, we conclude that each of the $2^b - 1$ internal nodes on level b is the root of a subtree with a $Bin(a+1)$ prefix. Therefore, remove $(2^b - 1)(2^{a+1} - 1)$ internal nodes to build $(2^b - 1) Bin(a+1)$ binary prefixes for the subtrees that are rooted on level b.

Levels $b+a+2$ to $b+\Delta$: The external node profile for levels $b+a+3$ to $b+\Delta$ is $(1^{\Delta-(a+3)}, 2)$. By applying the above formula for $i = b+\Delta, \ldots, b+a+2$, we see that there can only be one internal node on level $b+a+2$ and it is the root of a $Snake(\Delta - (a+2))$ subtree. Therefore, remove $\Delta - (a+2)$ internal nodes to build a $Snake(\Delta - (a+2))$ rooted on level $b+a+2$.

Figure 9 displays the construction thus far when $N = 11$, $\Delta = 5$ and $a = 1$. The $N + 2 + a - \Delta - 2^{a+1} \cdot (2^b - 1)$ internal nodes that are left in the bin must be placed on level $b+a+1$. There are, however, only $2^{a+2} \cdot (2^b - 1)$ "slots" for nodes on level $b+a+1$, since each of the $Bin(a+1)$ subtrees rooted on level b has 2^{a+2} places for children on level $b+a+1$. Therefore,

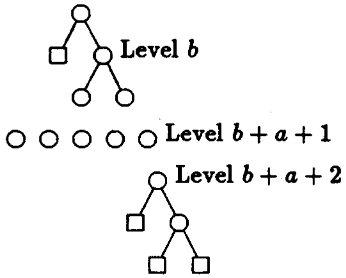

Figure 9. – Constructing an (a, Δ, N)-tree, where $N = 11$, $\Delta = 5$ and $a = 1$.

we must have $N+2+a-\Delta-2^{a+1}\cdot(2^b-1) \leq 2^{a+2}\cdot(2^b-1)$. Rearranging this inequality gives Condition 2,

$$\frac{N+1+2^{a+1}+a-\Delta}{2^{a+1}} < 2^{b\,(a,\,\Delta,\,N)+1}-1.$$

We now examine the difference of $b\,(a,\,\Delta,\,N)$ and $b\,(a+1,\,\Delta,\,N)$, where $b\,(a+1,\,\Delta,\,N) \geq 1$ and $a \geq -1$.

LEMMA 5.2: *Assume that* $\Delta \leq N-1$ *and* $a \geq -1$. *Then*

$$b\,(a,\,\Delta,\,N) - b\,(a+1,\,\Delta,\,N)$$

is either 0 *or* 1.

Proof:

$$\begin{aligned} &b\,(a,\,\Delta,\,N) - b\,(a+1,\,\Delta,\,N) \\ &\quad= \left\lfloor \log_2 \frac{N+1+2^{a+1}+a-\Delta}{2^{a+1}} \right\rfloor - \left\lfloor \log_2 \frac{N+1+2^{a+1}+a+1-\Delta}{2^{a+2}} \right\rfloor \\ &\quad= \lfloor \log_2 (N+1+2^{a+1}+a-\Delta) \rfloor - (a+1) \\ &\qquad - \lfloor \log_2 (N+1+2^{a+2}+a+1-\Delta) \rfloor + (a+2) \\ &\quad= 1 + \lfloor \log_2 (N+1+2^{a+1}+a-\Delta) \rfloor \\ &\qquad - \lfloor \log_2 (N+1+2^{a+1}+a-\Delta+2^{a+1}+1) \rfloor. \end{aligned}$$

Let $c = \lfloor \log_2 (N+1+2^{a+1}+a-\Delta) \rfloor$. Clearly, $2^c \leq N+1+2^{a+1}+a-\Delta < 2^{c+1}$. Since $\Delta \leq N-1$ and $a \geq -1$, we have $2^{a+1} < N+1+2^{a+1}+a-\Delta$; that is, $a+1 \leq c$. Thus $2^{a+1}+1 \leq 2^{a+2} \leq 2^{c+1}$ and we can conclude that

$$2^c < N+1+2^{a+1}+a-\Delta+2^{a+1}+1 < 2^{c+2};$$

in other words,

$$0 \leq \lfloor \log_2 (N + 1 + 2^{a+1} + a - \Delta + 2^{a+1} + 1) \rfloor$$
$$- \lfloor \log_2 (N + 1 + 2^{a+1} + a - \Delta) \rfloor \leq 1.$$

Finally, $b(a, \Delta, N) - b(a+1, \Delta, N) = 1 + \lfloor \log_2 (N+1+2^{a+1}+a-\Delta) \rfloor - \lfloor \log_2 (N+1+2^{a+1}+a-\Delta+2^{a+1}+1) \rfloor$, and $b(a, \Delta, N) - b(a+1, \Delta, N)$ is either 0 or 1. □

Corollary 5.3: *Let* $N \geq 3$, $2 \leq \Delta \leq N - 1$ *and* $-1 \leq a$. *If* $b(a, \Delta, N) < 1$, *then, for all* a', $a \leq a'$, *there are no* (a', Δ, N)*-trees. If* $b(a, \Delta, N) \geq 1$, *then* $b(a'', \Delta, N) \geq 1$, *for all* a'', $-1 \leq a'' < a$.

Proof: Using Lemma 5.2, we can show by induction that $b(a', \Delta, N) \leq b(a, \Delta, N)$, for all $a' \geq a$, and that $b(a'', \Delta, N) \geq b(a, \Delta, N)$, for all a'', $-1 \leq a'' < a$. Hence, by Theorem 5.1, the result follows. □

What is the largest $a \geq -1$ such that $b(a, \Delta, N) \geq 1$?

Lemma 5.4: *Let* $N \geq 3$ *and* $2 \leq \Delta \leq N - 1$. *Let* $\overline{a}$ *be the largest integer such that* $N + 1 - \Delta \geq 2^{\overline{a}+1} - \overline{a}$. *Then,* $\overline{a}$ *is the largest integer such that* $b(\overline{a}, \Delta, N) \geq 1$.

Proof: Since $N+1-\Delta \geq 2^{\overline{a}+1} - \overline{a}$, we have $N+1+2\,\overline{a}+1+\overline{a}-\Delta \geq 2^{\overline{a}+2}$. By taking the logarithms and rearranging, we obtain

$$\log_2 \frac{N + 1 + 2^{\overline{a}+1} + \overline{a} - \Delta}{2^{\overline{a}+1}} \geq 1;$$

therefore, $b(\overline{a}, \Delta, N) \geq 1$.

We now have to prove that $\overline{a}$ is the largest integer such that $b(\overline{a}, \Delta, N) \geq 1$. Consider any integer $a' > \overline{a}$; therefore, because of the assumption about $\overline{a}$, $N + 1 - \Delta < 2^{a'+1} - a'$. Now, repeating the first part of the proof with a', we find that $b(a', \Delta, N) < 1$. □

Given that there are binary trees of size N and fringe thickness Δ, for which a are there (a, Δ, N)-trees? Suppose that $\overline{a}$ is the largest value such that $b(\overline{a}, \Delta, N) \geq 1$. By Corollary 5.3, since $b(a, \Delta, N) < 1$, for all $a > \overline{a}$. Furthermore, $b(a, \Delta, N) \geq 1$, for all a, $-1 \leq a \leq \overline{a}$. To determine whether there is an (a, Δ, N)-tree, we must discover the values of a for which

$$\frac{N + 1 + 2^{a+1} + a - \Delta}{2^{a+1}} \neq 2^{b(a, \Delta, N)+1} - 1.$$

We now show that, for any two consecutive values a and $a + 1$ in the range $[-1, \overline{a}]$, there is an (a, Δ, N)- or $(a + 1, \Delta, N)$-tree.

LEMMA 5.5: *Let a be an integer such that $-1 \leq a \leq \Delta - 2$ and $b(a+1, \Delta, N) \geq 1$. Then there is an (a, Δ, N)- or an $(a+1, \Delta, N)$-tree.*

Proof: Since $b(a+1, \Delta, N) \geq 1$, by Corollary 5.3, $b(a, \Delta, N) \geq 1$.

We will prove by contradiction that there is an (a, Δ, N)- or $(a+1, \Delta, N)$-tree. Assume that there is neither an (a, Δ, N)-tree nor an $(a+1, \Delta, N)$-tree. By Theorem 5.1, since $b(a, \Delta, N) \geq 1$ and there is no (a, Δ, N)-tree,

$$\frac{N + 1 + 2^{a+1} + a - \Delta}{2^{a+1}} = 2^{b(a, \Delta, N)+1} - 1.$$

This equation gives

$$N + 1 + 2^{a+1} + a - \Delta = 2^{a+b(a, \Delta, N)+2}. \tag{1}$$

Similarly, since $b(a+1, \Delta, N) \geq 1$ and there is no $(a+1, \Delta, N)$-tree,

$$\frac{N + 1 + 2^{a+2} + a + 1 - \Delta}{2^{a+2}} = 2^{b(a+1, \Delta, N)+1} - 1,$$

which yields

$$N + 1 + 2^{a+3} + a + 1 - \Delta = 2^{a+b(a+1, \Delta, N)+3}. \tag{2}$$

Combining Equations 1 and 2, we get

$$2^{a+2} + 1 = 2^{a+2} \cdot (2^{b(a+1, \Delta, N)+1} - 2^{b(a, \Delta, N)}). \tag{3}$$

By Lemma 5.2, $0 \leq b(a, \Delta, N) - b(a+1, \Delta, N) \leq 1$. If $b(a, \Delta, N) = b(a+1, \Delta, N)$, then Equation 3 becomes

$$2^{a+2} + 1 = 2^{b(a, \Delta, N)+a+2}.$$

Now, if $2^{a+2} + 1$ is a power of two, then $a = -2$, which provides a contradiction. On the other hand, if $b(a, \Delta, N) = b(a+1, \Delta, N) + 1$, then Equation 3 implies that $2^{a+2} + 1 = 0$, which also provides a contradiction. Therefore, there is an (a, Δ, N)- or $(a+1, \Delta, N)$-tree. $\square$

Since there is an (a, Δ, N)-tree for at least every other a in the range $[1, \overline{a}]$, we can deduce which binary trees, of a given size and fringe thickness, have the minimum path length. We first need to define two new functions; see De Santis and Persiano [DP94].

DEFINITION 5.2: *Let $N \geq 3$, $2 \leq \Delta \leq N-1$ and $-1 \leq a$. Let $\alpha(\Delta)$ be the integer part of the unique solution to the equation $x + 2^{x+1} = \Delta$,*

$$X(a, \Delta, N) = N - \Delta + 2^{a+1} + a + 2,$$

and

$$\begin{aligned} F(a, \Delta, N) = {} & (N+1)(\lceil \log_2 X(a, \Delta, N) \rceil + 1) \\ & - 2^{\lceil \log_2 X(a, \Delta, N) \rceil} + 2^{a+2} \\ & + \frac{a(a+1)}{2} + \frac{\Delta(\Delta - 3)}{2} - a\Delta - 2. \end{aligned}$$

THEOREM 5.6: *Let $N \geq 3$ and $2 \leq \Delta \leq N-1$. Let $\bar{a}$ be the largest integer such that $b(\bar{a}, \Delta, N) \geq 1$. If $\bar{a} > \alpha(\Delta)$, then the binary trees of size N and fringe thickness Δ that have a minimum path length are:*

- *the $(\alpha(\Delta), \Delta, N)$-trees, if*

$$\frac{N + 1 + 2^{\alpha(\Delta)+1} + \alpha(\Delta) - \Delta}{2^{\alpha(\Delta)+1}} < 2^{b(\alpha(\Delta), \Delta, N)+1} - 1;$$

- *whichever ones of the $(\alpha(\Delta) - 1, \Delta, N)$- and the $(\alpha(\Delta) + 1, \Delta, N)$-trees have the smaller path length, otherwise.*

If $\bar{a} \leq \alpha(\Delta)$, then the binary trees of size N and fringe thickness Δ that have minimum path length are:

- *the $(\bar{a}, \Delta, N)$-trees, if*

$$\frac{N + 1 + 2^{\bar{a}+1} + \bar{a} - \Delta}{2^{\bar{a}+1}} = 2^{b(\bar{a}, \Delta, N)+1} - 1;$$

- *the $(\bar{a} - 1, \Delta, N)$-trees, otherwise.*

Proof: The proof is immediate from the preceding results and the following result of De Santis and Persioano [DP94]. □

PROPOSITION 5.1: *Let $N \geq 3$, $2 \leq \Delta \leq N-1$ and $0 \leq a \leq \Delta - 2$. If there is an $(a-1, \Delta, N)$-tree, then*

$$F(a, \Delta, N) \begin{cases} \leq F(a-1, \Delta, N) & if \quad a \leq \alpha(\Delta); \\ \geq F(a-1, \Delta, N) & if \quad a > \alpha(\Delta). \end{cases}$$

Otherwise, if there are (a, Δ, N)- and $(a-2, \Delta, N)$-trees, then

$$F(a, \Delta, N) \begin{cases} \leq F(a-2, \Delta, N) & if \quad a \leq \alpha(\Delta); \\ \geq F(a-2, \Delta, N) & if \quad a > \alpha(\Delta). \end{cases}$$

a	$\dfrac{N+1+2^{a+1}+a-\Delta}{2^{a+1}}$	$b(a, \Delta, N)$	$2^{b(a, \Delta, N)+1}-1$	(a, Δ, N)-tree exists?
–1	7	2	7	No, Condition 2 fails.
0	4.5	2	7	Yes.
1	3	1	3	No, Condition 2 fails.
2	2.125	1	3	Yes.
3	1.625	0	1	No, both conditions fail.

Figure 10. – The existence of (a, Δ, N)-trees for $N = 11$ and $\Delta = 5$.

6. A SECOND LOOK AT (a, Δ, N)–TREES

Lemma 5.5 seems to suggest that there may be some values of N and Δ for which the (a, Δ, N)-tree does not exist for every other value of a in the range $[-1, \overline{a}]$. We prove a stronger statement than that. We prove that the (a, Δ, N)-tree does not exist for at most two values of a in the range $[-1, \overline{a}]$.

It is possible that no (a, Δ, N)-tree exists for exactly two different values of a in the range $[-1, \overline{a}]$; that is, there may be exactly two values of a in the range $[-1, \overline{a}]$ for which

$$b(a, \Delta, N) \geq 1$$

holds, but

$$\frac{N+1+2^{a+1}+a-\Delta}{2^{a+1}} < 2^{b(a, \Delta, N)+1}-1$$

does not hold. For example, for $N = 11$ and $\Delta = 5$, $\overline{a} = 2$ and no (a, Δ, N)-tree exists for $a = -1$ and $a = 1$; see Figure 10.

We show that the (a, Δ, N)-tree does not exist for at most two values of a such that $-1 \leq a \leq \min(\overline{a}, \Delta - 2)$ in two steps. First, we show that if Condition 2 does not hold for some a in $[1, \overline{a}]$, then $b(a, \Delta, N) = b(a+1, \Delta, N)$. Then, we show that $b(a, \Delta, N) = b(a+1, \Delta, N)$ for at most two values of a in $[-1, \overline{a}]$.

LEMMA 6.1: *Let $N \geq 3$ and $1 < \Delta < N$. If*

$$\frac{N+1+2^{a+1}+a-\Delta}{2^{a+1}} \geq 2^{b(a, \Delta, N)+1}-1,$$

for some $a \in [-1, \overline{a}]$, then $b(a, \Delta, N) = b(a+1, \Delta, N)$.

Proof: By Lemma 5.2, either $b(a+1, \Delta, N) = b(a, \Delta, N)$ or $b(a+1, \Delta, N) = b(a, \Delta, N) - 1$. Suppose $b(a+1, \Delta, N) = b(a, \Delta, N) - 1$.

Since $b(a+1, \Delta, N) < b(a, \Delta, N)$ and

$$b(a+1, \Delta, N) = \left\lfloor \log_2 \frac{N+1+2^{a+2}+a+1-\Delta}{2^{a+2}} \right\rfloor,$$

we have

$$2^{b(a+1, \Delta, N)} \leq \frac{N+1+2^{a+2}+a+1-\Delta}{2^{a+2}} < 2^{b(a, \Delta, N)}.$$

Multiplying by 2 and subtracting $(2^{a+1}+1)/2^{a+1}$, we have

$$\frac{N+1+2^{a+1}+a-\Delta}{2^{a+1}} < 2^{b(a, \Delta, N)+1} - 1 - \frac{1}{2^{a+1}},$$

which contradicts

$$\frac{N+1+2^{a+1}+a-\Delta}{2^{a+1}} \geq 2^{b(a, \Delta, N)+1} - 1.$$

Therefore, we must have $b(a+1, \Delta, N) = b(a, \Delta, N)$. □

Note that the converse (if $b(a+1, \Delta, N) = b(a, \Delta, N)$, then $(N+1+2^{a+1}+a-\Delta)/2^{a+1} \geq 2^{b(a, \Delta, N)+1} - 1$) is not true. For example, for $N = 17$, $\Delta = 5$, and $a = 2$,

$$b(a, \Delta, N) = \left\lfloor \log_2 \frac{23}{8} \right\rfloor = 1$$

and

$$b(a+1, \Delta, N) = \left\lfloor \log_2 \frac{32}{16} \right\rfloor = 1,$$

but

$$\frac{23}{8} = \frac{N+1+2^{a+1}+a-\Delta}{2^{a+1}} < 2^{b(a, \Delta, N)+1} - 1 = 3.$$

LEMMA 6.2: *Let $N \geq 3$ and $1 < \Delta < N$. Then $b(a, \Delta, N) = b(a+1, \Delta, N)$ for at most two values of a in* $[-1, \min(\overline{a}, \Delta-2)]$.

Proof: How many values of a are there in $[-1, \min(\overline{a}, \Delta-2)]$? We show that $\overline{a} \leq \lfloor \log_2 (N+1-\Delta) \rfloor$, so that there are at most $\lfloor \log_2 (N+1-\Delta) \rfloor + 2$ values of a in $[-1, \min(\overline{a}, \Delta-2)]$.

Assume instead that $\overline{a} > \lfloor \log_2 (N + 1 - \Delta) \rfloor$; that is, assume $\overline{a} = \lfloor \log_2 (N + 1 - \Delta) \rfloor + c$, for some $c \geq 1$. Therefore,

$$(4) \qquad 2^{\overline{a}-c} \leq N + 1 - \Delta < 2^{\overline{a}-c+1}.$$

Since $\overline{a}$ is the largest integer a such that $2^{a+1} - a \leq N + 1 - \Delta$, we have

$$(5) \qquad 2^{\overline{a}+1} \leq N + 1 - \Delta + \overline{a}.$$

From Inequality 4 and Inequality 5, we can conclude that

$$2^{\overline{a}+1} - 2^{\overline{a}-c+1} \leq \overline{a},$$

or

$$(6) \qquad 2^{\overline{a}} \leq \frac{2^{c-1}}{2^c - 1}\,\overline{a}.$$

since c is some positive integer,

$$\frac{2^{c-1}}{2^c - 1} \leq 1.$$

Since $\overline{a}$ is some integer no smaller than -1,

$$2^{\overline{a}} > \overline{a}.$$

Therefore,

$$2^{\overline{a}} > \frac{2^{c-1}}{2^c - 1}\,\overline{a}.$$

which contradicts Inequality 6.

By Lemma 5.2, $b(a, \Delta, N)$ takes on each of the values between $b(\overline{a}, \Delta, N) = 1$ (by the definition of $\overline{a}$) and $b(-1, \Delta, N) = \lfloor \log_2 (N+1-\Delta) \rfloor$ as a goes from -1 to $\overline{a}$. Since $b(a, \Delta, N)$ takes on $\lfloor \log_2 (N + 1 - \Delta) \rfloor$ distinct values for at most $\lfloor \log_2 (N + 1 - \Delta) \rfloor + 2$ different values of a, we can have $b(a, \Delta, N) = b(a + 1, \Delta, N)$ at most twice. $\square$

Because there can be two values of a in the range $[-1, \overline{a}]$ for which Condition 2 does not hold, we must still prove Lemma 5.5 (that for two consecutive values of a in $[-1, \overline{a}]$, either the (a, Δ, N)-tree or the $(a+1, \Delta, N)$-tree exists) as it stands in order to prove Theorem 5.6.

7. CONCLUDING REMARKS

We have characterized the minimum-path-length binary trees, for all sizes and fringe thicknesses. Recently, De Prisco *et al.* [DP94] also characterized the minimum-path-length binary trees for fringe thickness Δ and size N, where $\Delta \geq (N+1)/2$.

The characterization of the maximum-path-length binary trees, for all sizes and fringe thicknesses, is still unsolved. Cameron [Cam91] and Cameron and Wood [CW94] give a partial solution by characterizing the maximum-path-length binary trees, for all sizes, fringe thicknesses, *and heights*. The determination of the heights that guarantee maximum path length is the crucial unsolved problem.

REFERENCES

[CAM91]. H. Cameron, *Extremal Cost Binary Trees*, PhD thesis, University of Waterloo, 1991.

[CW94]. H. Cameron and D. Wood, Maximal path length of binary trees, *Discrete Applied Mathematics*, 1994, *55(1)*, pp. 15-35.

[DP94]. R. De Prisco, G. Parlati and G. Persiano, On the path length of trees with known fringe, Unpublished manuscript, 1994.

[DP94]. A. De Santis and G. Persiano, Tight upper and lower bounds on the path length of binary trees, *SIAM Journal on Computing*, 1994, *23(1)*, pp. 12-23.

[Knu73]. D. E. Knuth, *The Art of Computer Programming, Vol. 3: Sorting and Searching*, Addison-Wesley, Reading, MA, 1973.

[KW89]. R. Klein and D. Wood, On the path length of binary trees, *Journal of the ACM*, 1989, *36(2)*, pp. 280-289.

[NW73]. J. Nievergelt and Chak-Kuen Wong, Upper bounds for the total path length of binary trees, *Journal of the ACM*, January 1973, *20(1)*, pp. 1-6.