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ON MORPHICALLY GENERATED FORMAL POWER SERIES (*)

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Abstract. – *We define morphically generated formal power series and study their properties. These series are obtained by interpreting various L systems as mechanisms to generate power series. The wellknown classes of rational and algebraic power series are obtained as a special case.*

Résumé. – *Nous définissons les séries formelles engendrées par morphismes, et étudions leurs propriétés. Ces séries sont obtenues en interprétant divers L-systèmes comme mécanismes de génération de séries formelles. Les classes bien connues des séries formelles rationnelles et algébriques s'obtiennent comme cas particuliers de cette construction.*

1. INTRODUCTION

Formal power series play an important role in many diverse areas of theoretical computer science and mathematics [1], [9], [10], [20]. The classes of power series studied most often in connection with automata, grammars and languages are the rational and algebraic series. It is wellknown that each regular (resp. context-free) language is the support of a rational (resp. an algebraic) series. However, the rational series are able to model also nonregular phenomena. So the transition from regular languages to rational power series constitutes a very essential generalization. It is also wellknown that many problems concerning parallel rewriting and L systems lead to rational series [11], [12]. Rational series are also widely used in combinatorics and nonlinear control theory.

In language theory formal power series often provide a powerful tool for obtaining deep decidability results [9], [20]. A brilliant example is

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the solution of the equivalence problem for finite deterministic multitape automata given by Harju and Karhumäki [5].

In this paper we search for a common generalization to the theories of formal power series and L systems. Our approach sheds new light on both theories and opens up new interesting avenues for further research. We are going to define and study formal power series obtained by morphic iteration. These series are generated by suitably modified L systems. We give a simple example.

Suppose A is a semiring and Σ is a finite alphabet. Denote the semiring of formal polynomials over Σ with coefficients in A by $A\langle\Sigma^*\rangle$ and assume that $h : A\langle\Sigma^*\rangle \to A\langle\Sigma^*\rangle$ is a semiring morphism. Such a morphism necessarily satisfies $h(\lambda) = \lambda$. We suppose also that $h(a \cdot \lambda) = a \cdot \lambda$ holds for every $a \in A$. Finally, assume $\omega \in A\langle\Sigma^*\rangle$. Now define the sequence $r^{(i)}\,(i \geq 0)$ by $r^{(0)} = \omega$, $r^{(i+1)} = h(r^{(i)})$. Then $\lim r^{(i)}$, if it exists, is a morphically generated series. Of course, we have to specify the convergence used in the limit process. In our work we allow also more complicated iteration. Instead of $r^{(i+1)} = h(r^{(i)})$ we might have, e.g., $r^{(i+1)} = ah_1(r^{(i)}) + h_2(r^{(i)})\,h_3(r^{(i)})$, where a is a letter and h_1, h_2, h_3 are, not necessarily distinct, morphisms of $A\langle\Sigma^*\rangle$.

Hence, to define a morphically generated series we have to specify the semiring $A\langle\langle\Sigma^*\rangle\rangle$, the convergence, the mode of iteration, the morphisms used in the iteration and the initial point. Therefore we consider 5-tuples $(A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}, P, \varphi, \omega)$ referred to as Lindenmayerian series generating systems. Here $\mathcal{D}$ specifies the convergence, P is a polynomial specifying the mode of iteration and φ gives the morphisms.

A brief outline of the contents of the paper follows. In Section 2 we define Lindenmayerian series generating systems, shortly, LS systems, and LS series. In Section 3 we study fixed point properties of LS series and the possibilities to generate LS series monotonically. Both issues are very important in the theory of LS series. In Section 4 we define ELS series which are of the form $r \odot \mathrm{char}(\Delta^*)$ where r is an LS series. We establish basic closure properties of ELS series and show that algebraic series are ELS series. In Section 5 we study decidability questions concerning LS and ELS series.

This paper is essentially self-contained. Only the rudiments concerning formal languages (*see* [19]), power series (*see* [1], [9], [20]) and L systems (*see* [12]) are assumed. However, the motivation of our work might be easier to grasp if the reader has more extensive previous knowledge about formal power series, L systems and their applications (*see* also [13], [14]).

Our work has close connections to earlier work concerning language equations (*see* [2]-[4], [7], [15]-[18], [6]).

2. DEFINITIONS AND EXAMPLES

It is assumed that the reader is familiar with the basics of the theories of semirings and formal power series (*see* [1], [9]). Notions and notations that are not defined are taken from [9].

If A is a semiring and Σ is an alphabet, not necessarily finite, the *semiring of formal power series with coefficients in A and (noncommuting) variables in Σ* is denoted by $A\langle\langle\Sigma^*\rangle\rangle$. If $r \in A\langle\langle\Sigma^*\rangle\rangle$ we denote

$$r = \sum_{w\in\Sigma^*} (r, w)\, w, \qquad r_n = \sum_{|w|\le n} (r, w)\, w \quad (n \ge 0)$$

and

$$\operatorname{supp}(r) = \{w | (r, w) \ne 0\}.$$

The set $\operatorname{supp}(r)$ is called the *support* of r. The subsemiring of $A\langle\langle\Sigma^*\rangle\rangle$ consisting of the series having a finite support is denoted by $A\langle\Sigma^*\rangle$. The elements of $A\langle\Sigma^*\rangle$ are referred to as *polynomials.*

In the sequel we use the notion of convergence introduced in [9]. We denote by $\mathcal{D}_d = (D_d, \lim_d)$ the convergence in $A\langle\langle\Sigma^*\rangle\rangle$ which is obtained when the discrete convergence in A is transferred to $A\langle\langle\Sigma^*\rangle\rangle$ as explained in [9]. Also $\mathcal{D}_d$ is called the *discrete convergence*. It is easy to see that $\mathcal{D}_d$ is multiplicative (*see* [8]).

Suppose A is a commutative semiring and $h : \Sigma^* \to A\langle\Sigma^*\rangle$ is a monoid morphism. (Here $A\langle\Sigma^*\rangle$ is regarded as a multiplicative monoid.) Then we extend h to a semiring morphism

$$h : A\langle\Sigma^*\rangle \to A\langle\Sigma^*\rangle$$

by

$$h(P) = \sum (P, w)\, h(w), \qquad P \in A\langle\Sigma^*\rangle.$$

Notice that the assumption of commutativeness is needed in the verification that indeed $h(r_1 r_2) = h(r_1)\, h(r_2)$ for $r_1, r_2 \in A\langle\Sigma^*\rangle$. In the sequel we always tacitly extend a morphism $h \in \mathrm{Hom}(\Sigma^*, A\langle\Sigma^*\rangle)$ to a semiring morphism $h : A\langle\Sigma^*\rangle \to A\langle\Sigma^*\rangle$ as explained above. Notice that $\mathrm{Hom}(\Sigma^*, A\langle\Sigma^*\rangle)$ can be identified with the set

$$\{h : A\langle\Sigma^*\rangle \to A\langle\Sigma^*\rangle \mid h \text{ is a semiring morphism and } h(a\cdot\lambda) = a\cdot\lambda \text{ for any } a \in A\}.$$

Suppose $r \in A\langle\langle\Sigma^*\rangle\rangle$ and that $\mathcal{D} = (D, \lim)$ is a convergence in $A\langle\langle\Sigma^*\rangle\rangle$. If $h : A\langle\Sigma^*\rangle \to A\langle\Sigma^*\rangle$ is a morphism, we say that $h(r)$ is defined if and only if $\lim h(r_n)$ exists. Then we denote, of course,

$$h(r) = \lim h(r_n).$$

In what follows X is a denumerably infinite alphabet of variables. Furthermore, Σ will always be a finite alphabet.

DEFINITION 2.1: Suppose A is a commutative semiring and Σ is a finite alphabet. An *interpretation* φ over (A, Σ) is a mapping from X to $\mathrm{Hom}(\Sigma^*, A\langle\Sigma^*\rangle)$.

DEFINITION 2.2: A *Lindenmayerian series generating system,* shortly, an LS system, is a 5-tuple $G = (A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}, P, \varphi, \omega)$ where A is a commutative semiring, Σ is a finite alphabet, $\mathcal{D}$ is a convergence in $A\langle\langle\Sigma^*\rangle\rangle$, P is a polynomial in $A\langle(X\cup\Sigma)^*\rangle$, φ is an interpretation over (A, Σ) and ω is a polynomial in $A\langle\Sigma^*\rangle$.

The series generated by an LS system is obtained by iteration. Before the precise definition we need a notation.

Suppose $P(x_1, \ldots, x_n) \in A\langle(X\cup\Sigma)^*\rangle$ and $s^{(1)}, \ldots, s^{(n)} \in A\langle\langle\Sigma^*\rangle\rangle$, where A is a commutative semiring. Then the series $P(s^{(1)}, \ldots, s^{(n)})$ is defined recursively as follows:

$$\begin{aligned}
a(s^{(1)}, \ldots, s^{(n)}) &= a, \qquad a \in A,\\
w(s^{(1)}, \ldots, s^{(n)}) &= w, \qquad w \in \Sigma^*,\\
x_i(s^{(1)}, \ldots, s^{(n)}) &= s^{(i)}, \qquad 1 \le i \le n,\\
(P_1 + P_2)(s^{(1)}, \ldots, s^{(n)}) &= P_1(s^{(1)}, \ldots, s^{(n)}) + P_2(s^{(1)}, \ldots, s^{(n)}),\\
(P_1 P_2)(s^{(1)}, \ldots, s^{(n)}) &= P_1(s^{(1)}, \ldots, s^{(n)}) \cdot P_2(s^{(1)}, \ldots, s^{(n)}),\\
&P_1, P_2 \in A\langle(X\cup\Sigma)^*\rangle.
\end{aligned}$$

DEFINITION 2.3: Suppose $G = (A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}, P(x_1, \ldots, x_n), \varphi, \omega)$ is an LS system. Denote $h_i = \varphi(x_i)$ for $1 \leq i \leq n$. Define the sequence $(r^{(j)})\,(j = 0, 1, \ldots)$ recursively by

$$r^{(0)} = \omega,$$
$$r^{(j+1)} = P(h_1(r^{(j)}), \ldots, h_n(r^{(j)})), \qquad j \geq 0.$$

If $\lim r^{(j)}$ exists we denote

$$S(G) = \lim r^{(j)}$$

and say that $S(G)$ is the *series generated* by G. The sequence $(r^{(j)})$ is the *approximation sequence associated to* G. A series r is called an LS *series* if there exists an LS system G such that $r = S(G)$. A series r is an LS *series with* $\omega = 0$ if there exists an LS system $G = (A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}, P, \varphi, 0)$ such that $r = S(G)$.

Example 2.4: Suppose $G_1 = (\Sigma, H, \omega)$ is a T0L system (*see* [12]). Suppose $H = \{h_1, \ldots, h_k\}$. For each i define the morphism $\bar{h}_i$ by

$$\bar{h}_i(a) = \text{char}(h_i(a)) \quad \text{for every } a \in \Sigma.$$

Define the LS system G by $G = (\mathbb{B}\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, \omega + x_1 + \cdots + x_k, \varphi, 0)$ where $\varphi(x_i) = \bar{h}_i$. Then $S(G)$ exists and equals the characteristic series of the T0L language generated by G_1.

Example 2.5: Suppose $G_1 = (\Sigma, g, \omega)$ is a D0L system (*see* [12]) such that $L(G_1)$ is infinite. Denote the Parikh vector of a word $u \in \Sigma^*$ by $\Pr(u)$. Choose a letter $c \notin \Sigma \cup X$. Denote by A the semiring of square matrices of order $\text{card}(\Sigma)$ with entries in $\mathbb{N}$ and by A_1 the subsemiring of A generated by the growth matrix M of G_1. Define the morphism $\bar{g} : (\Sigma \cup c)^* \to A_1\langle(\Sigma \cup c)^*\rangle$ by

$$\bar{g}(\sigma) = g(\sigma), \quad \sigma \in \Sigma, \qquad \bar{g}(c) = M\,c.$$

Define the LS system G by $G = (A_1\langle\langle(\Sigma \cup c)^*\rangle\rangle, \mathcal{D}_d, P(x), \varphi, 0)$ where $P(x) = c\omega + x$ and $\varphi(x) = \bar{g}$. Furthermore, denote $\pi = \Pr(\omega)$ and $\eta = (1, \ldots, 1)^T$. Then

$$S(G) = \sum_{n \geq 0} M^n\, cg^n(\omega),$$
$$\pi\, S(G) = \sum_{n \geq 0} \Pr(g^n(\omega))\, cg^n(\omega)$$

and

$$\pi S(G)\eta = \sum_{n\geq 0} |g^n(\omega)|\, cg^n(\omega).$$

This example shows that the coefficients of an LS series can record much more than just the multiplicities with which the elements are obtained in a generating process. This is an applicationwise important property of formal power series in general.

Example 2.6: Denote $\Sigma = \{a, b\}$, $P(x) = a + x^2$ and define $\varphi(x)$ by $\varphi(x)(a) = b$, $\varphi(x)(b) = a$. Define an LS system G by

$$G = (\mathbb{N}\langle\langle \Sigma^* \rangle\rangle,\ \mathcal{D}_d,\ P,\ \varphi,\ a + b).$$

It is not difficult to see that $S(G)$ equals the unique quasiregular solution of the equation

$$r = a + bb + br^2 + r^2 b + r^4.$$

In general, if G is an LS system, $S(G)$ does not necessarily exist.

Example 2.7: Denote $\Sigma = \{a, b\}$ and $P(x) = x$. Define $\varphi_1(x)$ by $\varphi_1(x)(a) = b$, $\varphi_1(x)(b) = a$ and $\varphi_2(x)$ by $\varphi_2(x)(a) = a$, $\varphi_2(x)(b) = 2b$. Furthermore, denote $\omega_1 = a + b$ and $\omega_2 = a$. Define the LS system G_{ij} by

$$G_{ij} = (\mathbb{N}\langle\langle \Sigma^* \rangle\rangle,\ \mathcal{D}_d,\ P,\ \varphi_i,\ \omega_j)$$

$(1 \leq i,\ j \leq 2)$. Then $S(G_{11}) = a + b$, $S(G_{12})$ and $S(G_{21})$ do not exist and $S(G_{22}) = a$.

In Examples 2.5-2.7 the systems G, G_{ij} are deterministic in the sense that for each $\sigma \in \Sigma$, $\varphi(x)(\sigma)$ is a monomial. Such systems can be considered as generalizations of DT0L systems (*see* also [6]).

We conclude this section with an example showing that the existence of the series generated by an LS system depends on the choice of the convergence and the axiom.

Example 2.8: A sequence $\alpha \in \mathbb{R}^{\mathbb{N}}$ is called an *Euler sequence* if and only if the sequence $\left(\sum_{j=0}^{n} \binom{n}{j} \alpha(j)/2^n\right)$ is a Cauchy sequence (*see* [9]). Define the limit function on the set D_E of Euler sequences by

$$\lim\nolimits_E \alpha = \lim_{n\to\infty} \sum_{j=0}^{n} \binom{n}{j} \alpha(j)/2^n.$$

Denote $\mathcal{D}_E = (D_E, \lim_E)$. Transfer this convergence to $\mathbb{R}\langle\langle\Sigma^*\rangle\rangle$ and denote also the resulting convergence by $\mathcal{D}_E$. Define the LS system G by $G = (\mathbb{R}\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_E, P(x), \varphi, 0)$ where $\Sigma = \{a\}$ and $P(x) = a + x$ and $\varphi(x)(a) = -a$. Then the approximation sequence $(r^{(j)})$ is given by

$$r^{(2n)} = 0, \quad r^{(2n+1)} = a, \qquad n \geq 0.$$

Therefore $\lim r^{(j)} = 2^{-1} a$ and $S(G)$ exists. If the Euler convergence were replaced by $\mathcal{D}_d$, $S(G)$ would not exist. On the other hand, if also the axiom 0 were replaced by $2^{-1} a$, the series $S(G)$ would exist.

3. FIXED POINT PROPERTIES OF LS SERIES

In this section we show that in many cases the series generated by an LS system can be characterized as the minimal solution of a polynomial equation involving morphisms. We also study the possibilities to generate LS series monotonically. Both questions are of fundamental importance in the theory of LS series.

DEFINITION 3.1: Suppose $G = (A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}, P(x_1, \ldots, x_n), \varphi, \omega)$ is an LS system. The series $r \in A\langle\langle\Sigma^*\rangle\rangle$ is a *fixed point* of G if $\varphi(x_i)(r)$ is defined for $1 \leq i \leq n$ and

$$r = P(\varphi(x_1)(r), \ldots, \varphi(x_n)(r)).$$

There are LS systems having infinitely many fixed points and there are LS systems having no fixed points at all. Indeed, consider the LS systems $G_i = (\mathbb{N}\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P_i(x), \varphi, 0)$, $i = 1, 2$, where $P_1(x) = x$, $P_2(x) = a + 2x$ and $\varphi(x)$ is the identity morphism ($a \in \Sigma$). Clearly any $r \in \mathbb{N}\langle\langle\Sigma^*\rangle\rangle$ is a fixed point of G_1 whereas G_2 has no fixed points. If $G_3 = (\mathbb{Z}\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P_2(x), \varphi, 0)$ then G_3 has the unique fixed point $r = -a$.

By definition, the LS system $G = (A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}, P(x_1, \ldots, x_n), \varphi, \omega)$ is *nonerasing* if $(\varphi(x_i)(\sigma), \lambda) = 0$ for every i $(1 \leq i \leq n)$ and $\sigma \in \Sigma$.

THEOREM 3.2: *Suppose $G = (A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P(x_1, \ldots, x_n), \varphi, \omega)$ is a nonerasing LS system. If $S(G)$ exists, it is a fixed point of G.*

Proof: Let $(r^{(i)})$ be the approximation sequence associated to G. Denote $h_j = \varphi(x_j)$, $1 \leq j \leq n$. By the assumption, $S(G) = \lim_{i \to \infty} r^{(i)}$ exists.

Because $(h_j(\sigma), \lambda) = 0$ for every $\sigma \in \Sigma$, $\lim_{i\to\infty} h_j(r^{(i)})$ and $h_j(\lim_{i\to\infty} r^{(i)})$ exist and are equal. Because

$$r^{(i+1)} = P(h_1(r^{(i)}), \ldots, h_n(r^{(i)}))$$

and $\mathcal{D}_d$ is multiplicative, we have

$$\lim r^{(i+1)} = P(h_1(\lim r^{(i)}), \ldots, h_n(\lim r^{(i)})).$$

This implies the claim. $\square$

The assumption that G is nonerasing is necessary in Theorem 3.2.

Example 3.3: Denote $\Sigma = \{a, b, c\}$ and $P(x, y) = x + y$. Define $h = \varphi(x)$ by $h(a) = a^2$, $h(b) = \lambda$, $h(c) = a^2$ and $g = \varphi(y)$ by $g(a) = 0$, $g(b) = b$, $g(c) = bc$. Now define the LS system G by $G = (\mathbb{B}\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P(x, y), \varphi, \omega)$ where $\omega = a^2 + bc$. It is easy to see that the $i + 1$'th term of the approximation sequence $(r^{(i)})$ associated to G is given by

$$r^{(i)} = a^{2^{i+1}} + a^{2^i} + \cdots + a^4 + a^2 + b^{i+1} c \qquad (i \geq 1).$$

Therefore $r = S(G) = \sum_{s\geq 1} a^{2^s}$. Because

$$P(h(r), g(r)) = h(r) = \sum_{s\geq 1} a^{2^{s+1}},$$

r is not a fixed point of P.

We show next that for partially ordered semirings the assumption that G is nonerasing can be replaced by other assumptions.

Suppose A is a partially ordered semiring under $\leq$. The relation $\leq$ is extended to $A\langle\langle\Sigma^*\rangle\rangle$ by $r_1 \leq r_2$ if and only if $(r_1, w) \leq (r_2, w)$ holds for all $w \in \Sigma^*$. Under this relation $A\langle\langle\Sigma^*\rangle\rangle$ is a partially ordered semiring. Suppose $G = (A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}, P(x_1, \ldots, x_n), \varphi, \omega)$ is an LS system and $\omega_1 \in A\langle\Sigma^*\rangle$. A fixed point $r \geq \omega_1$ of G is called the *minimal fixed point of G over ω_1* if $r \leq r'$ whenever r' is a fixed point of G such that $r' \geq \omega_1$. By definition, an LS system $G = (A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}, P(x_1, \ldots, x_n), \varphi, \omega)$ with the approximation sequence $(r^{(i)})$ is *reduced* if for each j $(1 \leq j \leq n)$ there exist a nonnegative integer i and $w \in \operatorname{supp}(r^{(i)})$ such that $\varphi(x_j)(w) \neq 0$. We say that a partially ordered semiring A *preserves strict inequality* if $a < b$ implies $a + c < b + c$ (resp. $ac < bc$ and $ca < cb$) for any $c \in A$ (resp. for any $c \in A$, $c \neq 0$), for every $a, b \in A$.

THEOREM 3.4: *Suppose A is a partially ordered semiring and*

$$G = (A\langle\langle\Sigma^*\rangle\rangle,\ \mathcal{D}_d,\ P(x_1,\ \ldots,\ x_n),\ \varphi,\ \omega)$$

is a reduced LS system. Denote the approximation sequence of G by $(r^{(i)})$. If $\omega \leq r^{(1)}$, the sequence $(r^{(i)})$ is monotonic, i.e., *satisfies*

$$r^{(0)} \leq r^{(1)} \leq r^{(2)} \leq \cdots$$

Suppose, furthermore, that A preserves strict inequality. Then, if $S(G)$ exists, it is the minimal fixed point of G over ω.

Proof: Denote $h_j = \varphi(x_j)$, $1 \leq j \leq n$.

If $a,\ b \in A$ and $a \leq b$, then $ap \leq bp$ for any $p \in A\langle\Sigma^*\rangle$. Therefore, if $p_1,\ p_2 \in A\langle\Sigma^*\rangle$ and $h \in \mathrm{Hom}(\Sigma^*,\ A\langle\Sigma^*\rangle)$, then $p_1 \leq p_2$ implies $h(p_1) \leq h(p_2)$. Hence, if $r^{(i)} \leq r^{(i+1)}$, then

$$\begin{aligned} r^{(i+1)} &= P(h_1(r^{(i)}),\ \ldots,\ h_n(r^{(i)})) \\ &\leq P(h_1(r^{(i+1)}),\ \ldots,\ h_n(r^{(i+1)})) = r^{(i+2)} \qquad (i \geq 0). \end{aligned}$$

This proves the first claim.

Suppose then that $r = S(G) = \lim_{i\to\infty} r^{(i)}$ exists. Clearly $r^{(i)} \leq r$ for any i. Fix j $(1 \leq j \leq n)$. Suppose now that $h_j(r)$ does not exist. Then there exists $w \in \Sigma^*$ such that the set $\{v \in \mathrm{supp}(r) \mid w \in \mathrm{supp}(h_j(v))\} = \{v_k \mid k \in \mathbb{N}\}$ is infinite. Furthermore, if we denote

$$s(\alpha) = \sum_{k=0}^{\alpha} (r,\ v_k)(h_j(v_k),\ w),$$

there exists a growing sequence (α_t) of nonnegative integers such that

$$s(\alpha_t) < s(\alpha_{t+1})$$

for any $t \geq 0$. In fact, because A preserves strict inequality, we can choose $\alpha_t = t$.

Suppose now that $a_0 x_{t_1} \ldots x_j \ldots x_{t_m} a_m$ is a term of $\mathrm{supp}(P)$, where the a's belong to Σ^*. Choose words $u_1,\ \ldots,\ u_{j-1},\ u_{j+1},\ \ldots,\ u_m \in \mathrm{supp}(r)$ such that $h_{t_\beta}(u_\beta) \neq 0$ for $1 \leq \beta \leq m$, $\beta \neq j$. Denote by $\bar{w}$ one of the shortest words in $\mathrm{supp}(a_0 h_{t_1}(u_1) \ldots h_{t_{j-1}}(u_{j-1}) a_{j-1} w a_j \ldots h_{t_m}(u_m) a_m)$. Choose a positive integer i_0 such that

$$(r^{(i)},\ u) = (r,\ u)$$

whenever $i \geq i_0$ and $|u| \leq \max\{|u_1|,\ \ldots,\ |u_{j-1}|,\ |u_{j+1}|,\ \ldots,\ |u_m|,\ |\bar{w}|\}$.

Fix $i \geq i_0$. Then there exists a positive integer i_1 such that

$$(h_j(r^{(i)}), w) < (h_j(r^{(i_1)}), w).$$

But then

$$(r^{(i_1+1)}, \bar{w}) > (r^{(i+1)}, \bar{w}) = (r, \bar{w}),$$

which is impossible. Therefore $h_j(r)$ exists and

$$\lim_{i \to \infty} h_j(r^{(i)}) = h_j(r)$$

for $1 \leq j \leq n$. It now follows as in the proof of Theorem 3.2 that r is a fixed point of G. Clearly $\omega \leq r$.

Suppose then that r' is a fixed point of G such that $\omega \leq r'$. It follows inductively that $r^{(i)} \leq r'$ for any i. Therefore $r \leq r'$. $\square$

Note that the condition $\omega \leq r^{(1)}$ trivially holds if $\omega = 0$.

Theorem 3.4 holds true for many partially ordered semirings which do not preserve strict inequality. It holds, e.g., for each semiring having no infinite ascending chains. (If A is a partially ordered semiring, $\{a_i | i \in \mathbb{N}\} \subseteq A$ is an *ascending chain* if $a_0 < a_1 < a_2 < \ldots$). Specially, Theorem 3.4 holds true for $\mathbb{B}$. If, in Theorem 3.4, the LS system G is nonerasing, Theorem 3.4 holds for any partially ordered semiring A.

We state the converse of Theorem 3.4 for $\mathbb{N}$, although, again, the same argument applies to many other cases.

COROLLARY 3.5: *Suppose* $G = (\mathbb{N}\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P(x_1, \ldots, x_n), \varphi, \omega)$ *is a reduced LS system such that*

$$\omega \leq P(\varphi(x_1)(\omega), \ldots, \varphi(x_n)(\omega)).$$

Then G *has a minimal fixed point over* ω *if and only if* $S(G)$ *exists. If both do exist, they are equal.*

Proof: Suppose $s \in \mathbb{N}\langle\langle\Sigma^*\rangle\rangle$ is the minimal fixed point of G over ω. If $(r^{(i)})$ is the approximation sequence associated to G, then $r^{(i)} \leq s$ for each i. Hence $S(G)$ exists. By Theorem 3.4, $S(G)$ is the minimal fixed point of G over ω. Therefore $s = S(G)$. $\square$

Corollary 3.5 shows that for the class of (reduced) LS systems $G = (\mathbb{N}\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P(x_1, \ldots, x_n), \varphi, \omega)$ satisfying

$$\omega \leq P(\varphi(x_1)(\omega), \ldots, \varphi(x_n)(\omega))$$

the limit approach and the fixed point approach coincide. The same observation holds true in many other cases. It will be seen below, however, than in general the limit approach is preferable.

It is an interesting question whether every LS series can be generated monotonically. More specifically, if $r = S(G)$ where G is an LS system, does there exist an LS system G' such that the approximation sequence associated to G' is monotonic and $S(G) = S(G')$. The answer turns out to be negative in general.

Example 3.6: Denote $\Sigma = \{a, b, d, e\}$, $P(x) = x$ and define $h = \varphi(x)$ by $h(a) = a$, $h(b) = b$, $h(d) = ab + adb$ and $h(e) = ba + bea$. Furthermore, denote $G = (\mathbb{N}\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P, \varphi, d + e)$. Then the approximation sequence $(r^{(n)})$ associated to G is given by

$$r^{(n)} = \sum_{i=1}^{n} (a^i b^i + b^i a^i) + a^n db^n + b^n ea^n$$

$(n \geq 0)$. Therefore

$$S(G) = \sum_{i=1}^{\infty} (a^i b^i + b^i a^i).$$

We show that $r = S(G)$ cannot be generated monotonically. Suppose on the contrary that $S(G) = S(G')$ where

$$G' = (\mathbb{N}\langle\langle\Sigma_1^*\rangle\rangle, \mathcal{D}_d, P_1(x_1, \ldots, x_m), \varphi_1, \omega)$$

is a reduced LS system such that the approximation sequence $(s^{(n)})$ associated to G' is monotonic. First, because $s^{(n)} \leq S(G') = S(G)$ for all n, we suppose without loss of generality that $\Sigma_1 = \{a, b\}$. By Theorem 3.4 we have

$$r = P_1(\varphi_1(x_1)(r), \ldots, \varphi_1(x_m)(r)).$$

Clearly P_1 is linear in X. Therefore,

$$\lambda \notin \operatorname{supp}(\varphi_1(x_j)(a)) \cup \operatorname{supp}(\varphi_1(x_j)(b)) \qquad (1 \leq j \leq m).$$

Hence each word in $\operatorname{supp}(\varphi_1(x_j)(a))$ (resp. $\operatorname{supp}(\varphi_1(x_j)(b))$) is a positive power of a letter. Also, $\varphi_1(x_j)(a)$ (resp. $\varphi_1(x_j)(b)$) is a monomial and $P_1(x_1, \ldots, x_m) = P_0 + x_1 + \ldots + x_m$ where $P_0 \leq r$ is a polynomial. Furthermore, for each j there exists a positive integer $e(j)$ such that

$\varphi_1(x_j)(a) = \sigma_1^{e(j)}$ and $\varphi_1(x_j)(b) = \sigma_2^{e(j)}$ where $\{\sigma_1, \sigma_2\} = \{a, b\}$. But then necessarily $m = 1$ and $r = P_0 + \varphi_1(x_1)(r)$. Hence $e(1) = 1$ and $P_0 = 0$. Because this is impossible, $S(G)$ cannot be generated monotonically.

Notice that $S(G)$ is indeed a fixed point of G. However, so is every series in $\mathbb{N}\langle\langle\{a, b\}^*\rangle\rangle$. Therefore, this example shows that the limit approach is often preferable to the fixed point approach.

We conclude this section by a topological condition on the set of fixed points of an LS system G guaranteeing that $S(G)$ can be generated monotonically.

THEOREM 3.7: *Suppose A is a partially ordered semiring such that for no $a \in A$ there is an infinite chain $\{a_j\}$ such that $a_j < a$ for all j. Furthermore, suppose $G = (A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P(x_1, \ldots, x_m), \varphi, \omega)$ is a nonerasing LS system such that $S(G)$ exists. Then $S(G)$ can be generated monotonically if there does not exist a sequence $s(n)$ of fixed points of G such that $\lim s(n) = S(G)$ and $s(n) < S(G)$ for each n. In particular, if G has only finitely many fixed points smaller than $S(G)$, then $S(G)$ can be generated monotonically.*

Proof: Suppose that there does not exist a sequence $s(n)$ of fixed points of G such that $\lim s(n) = S(G)$ and $s(n) < S(G)$ for each n. Denote

$$r = S(G), \qquad r_k = \sum_{|w| \le k} (r, w)\, w$$

and

$$G_k = (A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P(x_1, \ldots, x_m), \varphi, r_k)\,(k \ge 0).$$

By Theorem 3.2, $S(G)$ is a fixed point of G. Therefore, because G is nonerasing, $r_k \le P(\varphi(x_1)(r_k), \ldots, \varphi(x_m)(r_k))$ for each k. By Theorem 3.4, the approximation sequence $s^{(i)}$ associated to G_k is monotonic. Clearly $s^{(i)} \le S(G)$ for each i. Therefore $S(G_k)$ exists, $S(G_k) \le S(G)$ and $S(G_k)$ is a fixed point of G for any k. Because $S(G_k) \ge r_k$ we have $\lim S(G_k) = S(G)$. By the assumption there exists an integer t such that $S(G_t) = S(G)$. This implies the claim. □

4. ELS SERIES

In this section we define and study ELS systems and series. An ELS series is of the form $r \odot \operatorname{char}(\Delta^*)$ where $r \in A\langle\langle\Sigma^*\rangle\rangle$ is an LS series and $\Delta \subseteq \Sigma$. This generalization is well motivated for many reasons. Intuitively, it means that we pay less attention to the way the series is generated and more attention to what the series can tell us. Therefore, the generalization is very desirable applicationwise. The introduction of the Hadamard product corresponds to the use of nonterminals in language theory. Notice, however, that in a sense, it is possible to use nonterminals already in connection with LS series (*see* Example 3.6). Intuitively, only "vanishing" nonterminals are available in LS systems, whereas ELS systems have also "nonvanishing" nonterminals.

DEFINITION 4.1: An ELS *system* is a construct

$$G = (A\langle\langle\Sigma^*\rangle\rangle,\ \mathcal{D},\ P,\ \varphi,\ \omega,\ \Delta)$$

consisting of the LS system $U(G) = (A\langle\langle\Sigma^*\rangle\rangle,\ \mathcal{D},\ P,\ \varphi,\ \omega)$ called the *underlying system* of G and a subset Δ of Σ. If $S(U(G))$ exists, G generates the series

$$S(G) = S(U(G)) \odot \operatorname{char}(\Delta^*).$$

A series r is called an ELS *series* if there exists an ELS system G such that $r = S(G)$. A series r is called an ELS *series with* $\omega = 0$ if there exists an ELS system $G = (A\langle\langle\Sigma^*\rangle\rangle,\ \mathcal{D},\ P,\ \varphi,\ 0,\ \Delta)$ such that $r = S(G)$.

Example 4.2: Suppose $L \subseteq \Sigma^*$ is an ET0L language (*see* [12]). By an obvious modification of Example 2.4 one can show that $\operatorname{char}(L) \in \mathbb{B}\langle\langle\Sigma^*\rangle\rangle$ is an ELS series with $\omega = 0$.

Example 4.3: Denote $\Sigma = \{a,\ b,\ \bar{a},\ \bar{b},\ \bar{c},\ \bar{d}\}$ and $P(x,\ y) = \overline{ab} + \overline{cd} + \bar{a}x\bar{b} + \bar{c}\,y\,\bar{d} + z$. Define $\varphi(x)$ and $\varphi(y)$ by

$$\varphi(x)(\sigma) = \begin{cases} \sigma & \text{if} \quad \sigma = \bar{a} \text{ or } \sigma = \bar{b} \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi(y)(\sigma) = \begin{cases} \sigma & \text{if} \quad \sigma = \bar{c} \text{ or } \sigma = \bar{d} \\ 0 & \text{otherwise.} \end{cases}$$

Define $\varphi(z)$ by $\bar{a} \to a$, $\bar{b} \to b$, $\bar{c} \to b$, $\bar{d} \to a$, $a \to 0$, $b \to 0$. Define the ELS system G by $G = (\mathbb{N}\langle\langle\Sigma^*\rangle\rangle,\ \mathcal{D}_d,\ P,\ \varphi,\ 0,\ \{a,\ b\})$. Then the approximation sequence $(r^{(n)})$ associated to $U(G)$ is given by

$$r^{(n)} = \sum_{i=1}^{n} (\bar{a}^i\,\bar{b}^i + \bar{c}^i\,\bar{d}^i) + \sum_{i=1}^{n-1} (a^i\,b^i + b^i\,a^i).$$

Therefore $S(G) = \sum_{i \geq 1} (a^i b^i + b^i a^i)$. Hence $r = \sum_{i \geq 1} (a^i b^i + b^i a^i)$ is an ELS series with $\omega = 0$. It was shown in Example 3.6 that r is not an LS series with $\omega = 0$.

In what follows we pay special attention to LS and ELS series with $\omega = 0$. Many series interesting from L systems point of view belong to this class (*see* Examples 2.4, 2.5 and 4.2). It has already been seen that the series of this class have many nice properties not possessed by LS and ELS series in general (*see* Theorem 3.4).

In the definition of an LS system and the approximation sequence associated to an LS system only one polynomial is used. This is no restriction in the framework of ELS series.

DEFINITION 4.4: A *vector of* LS *systems* of dimension $k \geq 1$ is a k-tuple $\overline{G} = ((A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}, P_i(x_{11}, \ldots, x_{1k}, \ldots, x_{n1}, \ldots, x_{nk}), \varphi_i, \omega_i))_{1 \leq i \leq k}$ of LS systems. The *approximation sequence* $((r_{j,1}, \ldots, r_{j,k}))_{j \geq 0}$ *associated to* G is defined recursively by

$$\begin{aligned} r_{0,s} &= \omega_s, \\ r_{j+1,s} &= P_s(\varphi_s(x_{11})(r_{j,1}), \ldots, \varphi_s(x_{1k})(r_{j,k}), \ldots, \\ &\qquad \varphi_s(x_{n1})(r_{j,1}), \ldots, \varphi_s(x_{nk})(r_{j,k})), \qquad 1 \leq s \leq k. \end{aligned}$$

If $\lim_{j \to \infty} r_{j,s}$ exists for every $1 \leq s \leq k$, we denote

$$S(\overline{G}) = (\lim r_{j,1}, \ldots, \lim r_{j,k})$$

and say that $S(\overline{G})$ is the *(vector of) series generated by* $\overline{G}$.

In connection with ELS systems and series we want to emphasize that when we consider a polynomial $P(x_1, \ldots, x_n)$ we do not assume that each x_i actually has an occurrence in P.

For each $i \in \mathbb{N}$, suppose $\Sigma^{(i)}$ is an isomorphic copy of Σ and that $\text{copy}_i : \Sigma \to \Sigma^{(i)}$ is a bijective mapping. Furthermore, suppose $\Sigma^{(i)} \cap \Sigma^{(j)} = \varnothing$ for $i \neq j$. Also, suppose that for each $i \in \mathbb{N}$, $X^{(i)}$ is an isomorphic copy of X and that $X^{(i)} \cap X^{(j)} = \varnothing$ if $i \neq j$. Furthermore, assume $(X \cup \bigcup_{i \geq 0} X^{(i)}) \cap (\Sigma \cup \bigcup_{i \geq 0} \Sigma^{(i)}) = \varnothing$. Extend the mapping copy_i from $\Sigma \cup X$ to $\Sigma^{(i)} \cup X^{(i)}$ such that the restriction $\text{copy}_i : X \to X^{(i)}$ is bijective. The mapping copy_i is extended in the natural way from $A\langle\langle(\Sigma \cup X)^*\rangle\rangle$ to

$A\langle\langle(\Sigma^{(i)} \cup X^{(i)})^*\rangle\rangle$. Hence, if $r \in A\langle\langle\Sigma^*\rangle\rangle$, $\mathrm{copy}_i(r) \in A\langle\langle(\Sigma^{(i)})^*\rangle\rangle$ is the isomorphic copy of r over $\Sigma^{(i)}$.

THEOREM 4.5: *Suppose* $\overline{G} = ((A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P_i(x_{11}, \ldots, x_{1k}, \ldots, x_{n1}, \ldots, x_{nk}), \varphi_i, \omega_i))_{1\leq i\leq k}$ *is a vector of LS systems such that* $S(\overline{G}) = (r^{(1)}, \ldots, r^{(k)})$ *exists and* $r^{(1)}, \ldots, r^{(k)}$ *are quasiregular. Then* $r^{(s)}$ *is an ELS series for any* s. *If, furthermore, A is partially ordered and* $\omega_1 = \cdots = \omega_k = 0$, *then* $r^{(s)}$ *is an ELS series with* $\omega = 0$, *for any* s.

Proof: Denote the approximation sequence associated to $\overline{G}$ by $((r_{t,1}, \ldots, r_{t,k}))$. We suppose without loss of generality that $(r_{t,s}, \lambda) = 0$ for any $t \geq 0$, $1 \leq s \leq k$. If necessary, we change $\omega_1, \ldots, \omega_k$. It suffices to show that there is an LS system G such that

$$S(G) = \sum_{s=1}^{k} \mathrm{copy}_s(r^{(s)}).$$

Define the system $G = (A\langle\langle(\Sigma^{(1)} \cup \cdots \cup \Sigma^{(k)})^*\rangle\rangle, \mathcal{D}_d, P, \varphi, \omega)$ as follows. First,

$$P = \sum_{s=1}^{k} \mathrm{copy}_s(P_s)$$

and

$$\omega = \sum_{s=1}^{k} \mathrm{copy}_s(\omega_s).$$

Furthermore,

$$\varphi(\mathrm{copy}_s(x_{ij}))(\sigma) = \begin{cases} \mathrm{copy}_s(\varphi_s(x_{ij})(\sigma')) & \text{if } \sigma = \mathrm{copy}_j(\sigma') \in \Sigma^{(j)} \\ 0 & \text{if } \sigma \notin \Sigma^{(j)} \end{cases}$$

$1 \leq s \leq k$, $1 \leq i \leq n$, $1 \leq j \leq k$.

Denote now the approximation sequence associated to G by $(q^{(t)})$. It follows inductively that

$$q^{(t)} = \sum_{s=1}^{k} \mathrm{copy}_s(r_{t,s}).$$

This implies the first claim.

If A is partially ordered and $\omega_1 = \cdots = \omega_k = 0$, the approximation sequence associated to $\overline{G}$ is monotonic. Hence the assumption concerning

$r^{(1)}, \ldots, r^{(k)}$ implies that $(r_{t,s}, \lambda) = 0$ for any $t \geq 0$, $1 \leq s \leq k$. Therefore at the beginning of the proof we do not have to change the axioms and so $\omega = 0$. $\square$

In what follows we consider also vectors of LS systems which have different Σ's in their first components. This is merely a notational simplification.

In the rest of this section we always use the convergence $\mathcal{D}_d$.

LEMMA 4.6: *If $r, s \in A\langle\langle\Delta^*\rangle\rangle$ are quasiregular ELS series, then so are $r + s$ and rs. If, furthermore, A is partially ordered and r and s are ELS series with $\omega = 0$, so are $r + s$ and rs.*

Proof: We prove the claim for rs. The other case is similar. Suppose

$$r = S(G_1) \odot \operatorname{char}(\Delta_1^*),$$
$$s = S(G_2) \odot \operatorname{char}(\Delta_2^*)$$

where

$$G_1 = (A\langle\langle\Sigma_1^*\rangle\rangle, \mathcal{D}_d, P_1(x_{11}, \ldots, x_{n1}), \varphi_1, \omega_1)$$

and

$$G_2 = (A\langle\langle\Sigma_2^*\rangle\rangle, \mathcal{D}_d, P_2(x_{12}, \ldots, x_{n2}), \varphi_2, \omega_2)$$

are LS systems. Define the 3-dimensional vector of LS systems $\overline{G}$ by $\overline{G} = (G_1, G_2, G_3)$ where

$$G_3 = (A\langle\langle(\Sigma_1 \cup \Sigma_2)^*\rangle\rangle, \mathcal{D}_d, P_3(x_{31}, x_{32}), \varphi_3, 0).$$

Here $P_3(x_{31}, x_{32}) = x_{31}x_{32}$ and $\varphi_3(x_{3i})(\sigma) = \sigma$ if $\sigma \in \Delta_i$ and $\varphi_3(x_{3i})(\sigma) = 0$ otherwise $(i = 1, 2)$. Then the approximation sequence associated to $\overline{G}$ is $(r^{(j)}, s^{(j)}, (r^{(j-1)} \odot \operatorname{char}(\Delta_1^*))(s^{(j-1)} \odot \operatorname{char}(\Delta_2^*)))$ where $(r^{(j)})$ and $(s^{(j)})$ are the approximation sequences associated to G_1 and G_2, respectively. By Theorem 4.5, the series rs an is ELS series. $\square$

The easy proof of the following claim is omitted. Notice, however, that Lemma 4.7 is not a particular case of Lemma 4.6.

LEMMA 4.7: *Suppose $r \in A\langle\langle\Delta^*\rangle\rangle$ is a quasiregular ELS series and $a \in A$. Then ar is an ELS series. If, furthermore, A is partially ordered and r is an ELS series with $\omega = 0$, so is ar.*

THEOREM 4.8: *Suppose $R(x_1, \ldots, x_n) \in A\langle(X \cup \Sigma)^*\rangle$ is a quasiregular polynomial. If $r_1, \ldots, r_n \in A\langle\langle\Sigma^*\rangle\rangle$ are quasiregular ELS series, so is $R(r_1, \ldots, r_n)$. If, furthermore, A is partially ordered and each r_i is an ELS series with $\omega = 0$, so is $R(r_1, \ldots, r_n)$.*

Suppose $r \in A\langle\langle\Sigma^*\rangle\rangle$ is an ELS series. Consider the series $\bar{r}$ which is obtained from r by replacing each occurrence of σ by an ELS series r_σ for each $\sigma \in \Sigma$. It is an open question what conditions on the series r and r_σ $(\sigma \in \Sigma)$ guarantee that $\bar{r}$ is an ELS series. Theorem 4.8 is the special case where r is a quasiregular polynomial. Below we consider the case where each r_σ is a polynomial.

THEOREM 4.9: *Suppose* $G = (A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P, \varphi, \omega, \Delta)$ *is an ELS system such that* $S(G)$ *exists and is quasiregular. Furthermore, suppose* $h : \Delta^* \to A\langle\Delta_1^*\rangle$ *is a morphism such that* $(h(\sigma), \lambda) = 0$ *for every* $\sigma \in \Delta$. *Then* $h(S(G))$ *is an ELS series. If, furthermore, A is partially ordered and* $S(G)$ *is an ELS series with* $\omega = 0$, *so is* $h(S(G))$.

Proof: Denote by $(r^{(j)})$ the approximation sequence associated to $U(G)$. Extend h to a morphism from Σ^* to $A\langle\Delta_1^*\rangle$ by $h(\sigma) = 0$ if $\sigma \in \Sigma - \Delta$. Then $\lim h(r^{(j)})$ exists and equals $h(S(G))$.

Denote $P = P(x_{11}, \ldots, x_{n1})$ and $P_2(x_{21}) = x_{21}$. Define a two-dimensional vector of LS systems by $\overline{G} = (U(G), G_2)$ where $G_2 = (A\langle\langle(\Sigma \cup \Delta_1)^*\rangle\rangle, \mathcal{D}_d, P_2, \varphi_2, 0)$. Here $\varphi_2(x_{21}) = h_2$ where $h_2(\sigma) = h(\sigma)$ if $\sigma \in \Sigma$ and $h_2(\sigma) = 0$ otherwise. Now the approximation sequence associated to $\overline{G}$ is $(r^{(j)}, h(r^{(j-1)}))$. By Theorem 4.5, $h(S(G)) = \lim h(r^{(j-1)})$ is an ELS series. $\square$

The following theorem is a direct consequence of Theorem 4.5 and the definition of an algebraic series (*see* [9]).

THEOREM 4.10: *If* $r \in A^{\mathrm{alg}}\langle\langle\Sigma^*\rangle\rangle$ *is quasiregular, then* r *is an ELS series with* $\omega = 0$.

To conclude this section we show that erasing is a necessary facility in ELS sytems. This should be contrasted with the fact that $\mathcal{L}(\mathrm{ET0L}) = \mathcal{L}(\mathrm{EPT0L})$ (*see* [12]). By definition, an ELS system G is nonerasing if the LS system $U(G)$ is nonerasing.

LEMMA 4.11: *Suppose* $G = (\mathbb{N}\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P(x_1, \ldots, x_m), \varphi, 0)$ *is a nonerasing LS system such that* $r = S(G)$ *exists and* $(r, \lambda) = 0$. *Then there exists a real number* $\delta > 0$ *such that*

$$(r, w) \leq 2^{\delta|w|^2}$$

for every $w \in \Sigma^*$.

Proof: We sketch the basic ideas of the proof; the details are left to the reader.

We suppose without loss of generality that

$$P(x_1, \ldots, x_m) = n_1 x_1 + \cdots + n_k x_k + P_1(x_1, \ldots, x_m)$$

where the support of no term of P_1 belongs to X and $n_1, \ldots, n_k$ are positive integers. Denote $h_i = \varphi(x_i)$ for $i = 1, \ldots, m$ and $H = \{h_1, \ldots, h_k\}$. Because $r = S(G)$ exists, for each word $w \in \operatorname{supp}(r)$ there exists a positive integer $M(w)$ such that if $i > M(w)$ and $g_1, \ldots, g_i \in H$ then

$$\operatorname{supp}(g_1 \ldots g_i(w)) \cap \Sigma^{|w|} = \emptyset.$$

This implies that there do not exist $w \in \operatorname{supp}(r)$ and $g_1, \ldots, g_j \in H$ such that $w \in \operatorname{supp}(g_1 \ldots g_j(w))$. Therefore $M(w) \leq |\Sigma|^{|\Sigma|}$ for all $w \in \operatorname{supp}(r)$. Now the claim can be shown inductively. □

THEOREM 4.12: *There exists an LS series r with $\omega = 0$ such that if G_1 is a nonerasing ELS system with $\omega = 0$ then $S(G_1) \neq r$.*

Proof: Denote $G = (\mathbb{N}\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P, \varphi, 0)$ where $\Sigma = \{a, \bar{a}, b, c\}$, $P(x) = bc(a + \bar{a}) + x$ and $\varphi(x) = h$ is defined by $h(a) = (a + \bar{a})^2 + \lambda$, $h(\bar{a}) = \lambda$, $h(b) = b$, $h(c) = bc$. Denote the approximation sequence associated to G by $(r^{(n)})$. It is seen inductively that there exists a sequence $(s^{(n)}) \in (\mathbb{N}\langle\langle\Sigma^*\rangle\rangle)^{\mathbb{N}}$ such that

$$r^{(n+1)} = r^{(n)} + b^{n+1} c s^{(n)}$$

for all $n \geq 0$. Therefore $r = S(G)$ exists. It is easily seen that

$$b^{n+1} c (a + \bar{a})^{2^n} \leq r$$

for all $n \geq 0$. Therefore

$$(r, b^{n+2} c) \geq 2^{2^n}.$$

Hence the claim follows by Lemma 4.11. □

5. DECIDABILITY QUESTIONS

In this section we briefly discuss decidability questions concerning LS and ELS series.

By the definition due to Eilenberg, a semiring A (with $0 \neq 1$) is *positive*, if for all $a, b \in A$ the following conditions are satisfied.

(1) If $a + b = 0$ then $a = b = 0$.

(2) If $ab = 0$ then $a = 0$ or $b = 0$.

If A is a positive semiring, then the mapping $\psi : A \to \mathbb{B}$ defined by

$$\psi(a) = \begin{cases} 1 & \text{if} \quad a \neq 0 \\ 0 & \text{if} \quad a = 0 \end{cases}$$

is a semiring morphism. This implies the following lemma. In the statement of the lemma ψ stands for the unique extension

$$\psi : A\langle (X \cup \Sigma)^* \rangle \to \mathbb{B}\langle (X \cup \Sigma)^* \rangle$$

satisfying $\psi(w) = w$ for all $w \in (X \cup \Sigma)^*$.

LEMMA 5.1: *Suppose A is a positive semiring and*

$$G = (A\langle\langle \Sigma^* \rangle\rangle, \mathcal{D}_d, P, \varphi, 0)$$

is an LS system such that $S(G)$ exists. Denote

$$G' = (\mathbb{B}\langle\langle \Sigma^* \rangle\rangle, \mathcal{D}_d, P', \varphi', 0)$$

where $P' = \psi(P)$ and $\varphi'(x)(\sigma) = \psi(\varphi(x)(\sigma))$ for any $x \in X$, $\sigma \in \Sigma$. Then $S(G')$ exists and $S(G') = \operatorname{char}(\operatorname{supp}(S(G)))$.

THEOREM 5.2: *Suppose A is a positive semiring and*

$$G = (A\langle\langle \Sigma^* \rangle\rangle, \mathcal{D}_d, P, \varphi, 0)$$

is an LS system such that $S(G)$ exists. Then $\operatorname{supp}(S(G))$ *is a recursive set.*

Proof: By Lemma 5.1 we assume without restriction that $A = \mathbb{B}$. By the assumption $r = S(G)$ is the least fixed point of G. Define $F = \{(u, \Delta) \mid u \in \Sigma^* \text{ and } \Delta \subseteq \Sigma\}$. If $f = (u, \Delta) \in F$, the length of f equals $|u|$. We say that $w \in \Sigma^*$ has form $(u, \Delta) \in F$ if $w = \sigma_1 \ldots \sigma_m$, $u = \sigma_{i_1} \ldots \sigma_{i_k}$ $(0 \leq k \leq m, 1 \leq i_j \leq m)$ and $\{\sigma_j \mid j \notin \{i_1, \ldots, i_k\}\} \subseteq \Delta$. Here the σ's are letters of Σ. We say that $f \in F$ holds if $\operatorname{supp}(r)$ has a word of the form f. If $f_1, \ldots, f_s$ hold we say that $f_1 \wedge \cdots \wedge f_s$ holds.

To prove the theorem it suffices to show that it is decidable, given $f = (u, \Delta) \in F$, whether or not f holds. We proceed inductively on $|u|$

and suppose that we already know the answer if $u = \lambda$. We proceed with the assumption that $P(0, \ldots, 0)$ does not have a term of the form f. Because r is the least fixed point of G, there exist $f_{11}, \ldots, f_{1k_1}, \ldots, f_{t1}, \ldots, f_{tk_t}$ of length at most $|f|$ such that f holds if and only if at least one of $f_{11} \wedge \ldots \wedge f_{1k_1}, \ldots, f_{t1} \wedge \ldots \wedge f_{tk_t}$ holds. By induction, we can decide all f_α's with length less than $|f|$. For each f_α we also check whether the fact that f_α holds follows from the fact that $P(0, \ldots, 0)$ has a term of the form f_α. If we do not find an expression which holds we are left with a set $C_1 \subseteq \{f' \in F \mid |f'| = |f|\}$ such that f holds if and only if C_1 has an element which holds. Next we repeat the process with each $f' \in C_1$. Unless we find an element of C_1 which holds we are left with a set $C_2 \subseteq \{f' \in F \mid |f'| = |f|\}$ and we have to decide whether C_2 has an element which holds. Now we continue in the same way. If the same set C is obtained twice, f does not hold. $\square$

COROLLARY 5.3: *Suppose A is a positive semiring and*

$$G = (A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P, \varphi, 0, \Delta)$$

is an ELS system such that $S(G)$ exists. Then $\operatorname{supp}(S(G))$ *is recursive.*

THEOREM 5.4: *Suppose A is a positive semiring and*

$$G = (A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P, \varphi, 0)$$

is an LS system such that $S(G)$ exists and $\varphi(x)(\sigma) \neq 0$ for all x and $\sigma \in \Sigma$. Then it is decidable whether or not $\operatorname{supp}(S(G))$ *is infinite.*

Proof: By Lemma 5.1 we suppose without restriction that $A = \mathbb{B}$. The claim was proved in [6], provided that $\varphi(x)(\sigma) \in \Sigma^*$ for any x and $\sigma \in \Sigma$. No essentially new ideas are needed to prove Theorem 5.4. $\square$

We show next that every property which is undecidable for context-free languages is undecidable for LS series, too.

LEMMA 5.5: *Suppose $L \subseteq \Sigma^*$ is a context-free language. Then* $\operatorname{char}(L) \in \mathbb{B}\langle\langle\Sigma^*\rangle\rangle$ *is an LS series.*

Proof: Suppose $L = L(G)$ where $G = (Y, \Sigma, R, \omega)$ is a context-free grammar in the Greibach normal form. Without restriction we suppose $Y \cap X = \emptyset$. Define the LS system $G_1 = (\mathbb{B}\langle\langle Y^*\rangle\rangle, \mathcal{D}_d, P(x), \varphi, \omega)$ by $P(x) = x$ and $\varphi(x) = h$ where $h(\sigma) = \sigma$ for $\sigma \in \Sigma$ and

$h(y) = \gamma_1 + \cdots + \gamma_n$ for $y \in Y - \Sigma$ where $y \to \gamma_1, \ldots, y \to \gamma_n$ are the productions for y in G. It is not difficult to see that $S(G_1) = \mathrm{char}(L)$. □

The following theorem lists some of the most basic undecidability results concerning LS series.

THEOREM 5.6: *Suppose* $r = S(G_1)$ *and* $s = S(G_2)$ *are given LS series where* $G_i = (\mathbb{B}\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P_i, \varphi_i, \omega_i)$ *are LS systems* $(i = 1, 2)$. *It is undecidable whether or not*

(i) $r = s$,

(ii) $r \leq s$,

(iii) $r \odot s = 0$,

(iv) $r = \mathrm{char}(\Sigma^*)$,

(v) $\mathrm{supp}(r)$ *is regular.*

Proof: The claims follow from Lemma 5.5 and wellknown undecidability results concerning context-free languages. □

Theorem 5.6 (i) and (ii) can also be deduced from the undecidability of language equivalence for DT0L systems by Example 2.4. This shows that (i) and (ii) are undecidable even if P_i are supposed to be linear and $\omega_i = 0$ $(i = 1, 2)$. However, we do not find it very interesting to translate various undecidability results of language theory to LS series. On the other hand, it is of interest to search classes of LS systems for which new decidability results can be shown. It turns out that these restricted classes often still allow a very large spectre of truly morphic behaviour and do not necessarily restrict the mode of iteration at all. An example is provided by the following theorem.

An LS system $G = (A\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}, P(x_1, \ldots, x_n), \varphi, \omega)$ is *everywhere growing* if for every x_i $(1 \leq i \leq n)$ and $\sigma \in \Sigma$ the length of the shortest word in $\mathrm{supp}(\varphi(x_i)(\sigma))$ is at least two and $\varphi(x_i)(\sigma) \neq 0$.

THEOREM 5.7: *Suppose* $G = (\mathbb{Q}_+\langle\langle\Sigma^*\rangle\rangle, \mathcal{D}_d, P(x_1, \ldots, x_n), \varphi, 0)$ *is an everywhere growing LS system and* $s \in \mathbb{Q}_+\langle\langle\Sigma^*\rangle\rangle$ *is a* $\mathbb{Q}$*-rational series. If* $(P, \lambda) = 0$ *or no term of* P *belongs to* X^+ *then* $S(G)$ *exists and it is decidable whether or not* $S(G) = s$.

Proof: The existence of $S(G)$ is seen inductively. By Theorem 3.4, $S(G)$ is the minimal fixed point of G. Suppose $r' \in \mathbb{Q}_+\langle\langle\Sigma^*\rangle\rangle$ is a fixed point of G such that $(S(G), \lambda) = (r', \lambda)$. It follows inductively that $S(G) = r'$.

Therefore $S(G) = s$ if and only if $(S(G), \lambda) = (s, \lambda)$ and s is a fixed point of G. Hence $S(G) = s$ if and only if

$$(P, \lambda) = (s, \lambda)$$

and

$$P(\varphi(x_1)(s), \ldots, \varphi(x_n)(s)) = s.$$

The decidability of the first condition is clear; the decidability of the second follows by the closure and decidability properties of rational series. □

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