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THE HAMILTON CIRCUIT PROBLEM ON GRIDS (*)

by FOTO AFRATI ⁽¹⁾

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Abstract. – This paper investigates the Hamilton circuit problem on grid graphs. For general grid graphs it is known to be \mathcal{NP} -complete. We consider a non-trivial subclass of grid graphs and present a linear algorithm for finding a Hamilton circuit. Moreover, we show that our algorithm can be optimally parallelized, hence the problem belongs to \mathcal{NC} .

Résumé. – Cet article étudie le problème des circuits Hamiltoniens sur les graphes qui sont des grilles. Dans le cas général des grilles, on sait que le problème est \mathcal{NP} -complet. Nous considérons une sous-classe non triviale de grilles pour laquelle nous proposons un algorithme linéaire pour trouver un circuit Hamiltonien. De plus, nous montrons que notre algorithme peut être parallélisé de façon optimale et par conséquent que le problème appartient à la classe \mathcal{NC} .

1. INTRODUCTION

The *Hamilton path problem* on a graph G is to decide whether there is a Hamilton path between two given vertices of G . The *Hamilton circuit (or Hamilton cycle) problem* on G is to decide whether there is a Hamilton cycle in G [1]. Both problem have been long known to be \mathcal{NP} -complete on general graphs [2]. Their many applications, though, have caused a large body of research to be directed both towards finding efficient heuristics and towards trying to single out classes of graphs on which the Hamilton cycle problem is polynomially solvable: There are classes of graphs that all the members are Hamiltonian (*i.e.*, they have a Hamilton cycle). One such class is the tournaments (complete directed graphs) [9], another class is the dense graphs [6], [7]. Moreover, the Hamilton cycle problem is studied on the context of parallel computation; in [10] they present an optimal polylogarithmic parallel algorithm that computes a Hamilton cycle on dense graphs. Among others,

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probabilistic algorithms that compute the Hamilton cycle problem on both sequential and parallel machines are given in [8] and [5]. In this paper, we single out another natural class of graphs on which the Hamilton cycle problem is computable in polynomial time; moreover our polynomial-time algorithm is optimally parallelizable, so that we obtain a polylogarithmic parallel algorithm, as well.

Consider an undirected graph, $G^\infty(V, E)$, where V consists of all points of the plane with integer coordinates and there is an edge connecting two vertices if and only if the Euclidean distance between them is equal to 1. A *grid* is a finite node-induced subgraph of G^∞ . A grid G is *solid* if it does not have “holes”, *i.e.* both G and $G^\infty - G$ are connected. It is proven [3] that the Hamilton circuit problem is \mathcal{NP} -complete on grid graphs. Problems related to grid graphs appear in various fields, including packet radio networks (the Manhattan networks are grids, in fact solid grids are mostly of interest there) [11].

In this paper, we consider solid grids like the one shown in Figure 1; *i.e.* one “vertical boundary”, one “horizontal boundary” and a “distorted ladder boundary”. In order to describe such a grid, it suffices to describe the “distorted ladder boundary” as a sequence of integers that denote the length of each horizontal stripe of the grid. We call this class, the class of *restricted grids* (*see* Section 2 for a rigorous definition).

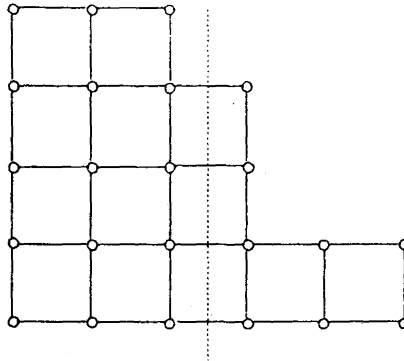


Figure 1. – Example of a restricted grid; a vertical cut is shown.

A *rectangular graph* is a restricted grid in which all horizontal stripes have the same length. In [3], necessary and sufficient conditions are given for the Hamilton path problem to have a solution on rectangular graphs. In this paper, we give necessary and sufficient conditions for the Hamilton

circuit problem to have a solution on restricted grids, thus solving partially an open problem of [3]. Further on, the constructive proof of this result leads to a linear sequential algorithm and to an optimal parallel algorithm that solve the Hamilton circuit problem, proving, thus, this problem to be in the parallel complexity class \mathcal{NC} .

2. BASIC DEFINITIONS AND PRELIMINARIES

We define a *restricted grid* by a sequence of positive integers which denote the lengths of all “horizontal stripes” of the grid; *i.e.*, the grid of Figure 1 can be fully described by the sequence of positive integers: 5, 3, 3, 2, meaning that there is a stripe of length 5 in the bottom, a stripe of length 3 on top of it, a stripe of length 3 on top of the last one and so on. In general, the sequence of positive integers a_1, a_2, \dots, a_n where $a_1 \geq a_2 \geq \dots \geq a_n$, which describes a grid in this class, means that the grid is comprised of horizontal stripes so that: a) their beginnings match and b) their lengths are, from bottom to top: a_1, a_2, \dots, a_n .

Hereon, we refer only to restricted grids. Consider a certain embedding of a grid in the plane like the one shown in Figure 1, so as we can view the set of its edges as partitioned into two sets, namely the *horizontal* edges and the *vertical* edges. We define a *vertical cut*, a cut $T = (V_1, V_2)$ such that the edges with endpoints to both sets V_1 and V_2 are solely horizontal edges (*see* Figure 1); similarly we define a *horizontal cut*. Sometimes, if confusion does not arise, we will be defining a cut by the set of edges that have exactly one endpoint in V_1 and exactly one endpoint in V_2 . Hereafter, whenever we consider a grid, its corresponding convenient embedding in the plane is also considered, so as to be able to refer, without confusion, to horizontal and vertical edges and cuts and to expressions like: “the set of nodes that lie above cut T ” or “the part of the grid that lies on the right hand side of cut T ”.

DEFINITION 2.1: We say that a restricted grid defined by the sequence a_1, a_2, \dots, a_n contains a *ladder* if there is a subsequence a_i, a_{i+1}, \dots, a_j such that $a_{i+1} = a_i$, $a_{i+2} = a_{i+1} - 1$, $a_{i+3} = a_{i+2} - 1, \dots, a_j = a_{j-1} - 1$ and, either $a_{j+1} < a_j - 1$, or $j = n$ and $a_n \geq 2$; the *length* of the ladder equals $j - i$. ■

Suppose a ladder is defined by the subsequence a_i, a_{i+1}, \dots, a_j ; then, if $i = 1$ we say that the ladder is in the bottom and, if $j = n$ we say that the ladder is at the top.

For example, the grid in Figure 2(a) contains a ladder of length 5 while the grid in Figure 2(b) does not contain a ladder (the sequence that describes the grid is 7, 5, 5, 4, 3, 2, 2, 1; and, for an example, for $i = 2$ and $j = 5$ all conditions for the existence of a ladder hold except the one that says $a_{j+1} < a_j - 1$). Moreover, to clarify the definition, observe that the grid $(i, i - 1, \dots, 1)$ does not have a ladder.

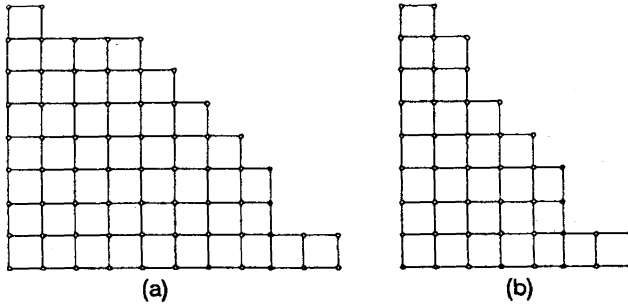


Figure 2. - Example of a grid with ladder (a) and a grid without a ladder (b).

LEMMA 2.2: Consider any restricted grid G described by the sequence a_1, a_2, \dots, a_n . Then, either of the following cases happen: a) $a_2 < a_1$ or $a_n = 1$, b) there is a ladder.

Proof: We prove it by induction on n . For $n = 1$, it is trivial. Suppose the lemma is true for $n = k$; we shall prove that it is true for $n = k + 1$ too. Let G be a grid given by the sequence $(a_1, a_2, \dots, a_{k+1})$, and suppose that (a) of lemma is not the case for G . Consider grid G' given by the sequence (a_1, a_2, \dots, a_k) . According to the inductive hypothesis, G' has a ladder. If the ladder of G' is not at the top, we deduce, in a straightforward way, that grid G has the same ladder. Suppose grid G' has a ladder at the top, i.e., $(a_j, a_{j+1}, \dots, a_k)$ is a ladder (for some j). Then, we have three cases: a) $a_{k+1} < a_k - 1$; then $(a_j, a_{j+1}, \dots, a_k)$ is a ladder of G' too. b) $a_{k+1} = a_k - 1$; then (since (a) is not the case) $(a_j, a_{j+1}, \dots, a_k, a_{k+1})$ is a ladder of G . c) $a_{k+1} = a_k$; then there is a ladder of length one at the top of G . ■

DEFINITION 2.3: We say that a graph G can be factored into cycles if there is a partition of the set of vertices into subsets, so that, each subset induces a Hamiltonian subgraph. ■

Let $G = (V, E)$ be a grid. We can color the nodes with two colors (say color “x” and color “o”) such that no two adjacent nodes have the same color.

Hereafter, by *colored grid*, we shall mean a grid colored in this particular way. The proof of the following two lemmata is straightforward:

LEMMA 2.4: *Consider a colored grid.*

a) *Any path from a node colored x to a node colored o contains as many x -nodes as o -nodes. In any path from an x -node to an x -node, the number of x -nodes, equals the number of o -nodes plus one.*

b) *A cycle contains as many x -nodes as o -nodes.*

LEMMA 2.5: *Consider a colored grid that can be factored into cycles. Then, the number of x -nodes equals the number of o -nodes.*

Hereafter, we consider only colored grids.

DEFINITION 2.6: A grid is called *balanced* if the number of x -nodes equals the number of o -nodes. ■

DEFINITION 2.7: Consider a colored grid, denote V its set of nodes and let $V_1 \subset V$. We denote by $d_{V_1}^x$ (we often refer to it as the d^x -number of V_1) (by $d_{V_1}^o$ resp.) the number of x -nodes (o -nodes resp.) in V_1 minus the number of o -nodes (x -nodes resp.) in V_1 . ■

Note that the above definition implies that $d_{V_1}^x + d_{V_1}^o = 0$.

DEFINITION 2.8: Consider a cut (V_1, V_2) of a colored grid. Let $B_1 \subseteq V_1$ be the *boundary* of V_1 , i.e. those nodes of V_1 that are adjacent to nodes of V_2 . We denote by $b_{V_1}^x$ (we often refer to it as the b^x -number of V_1) the number of x -nodes in B_1 . In similar fashion, we define $b_{V_1}^o$.

A horizontal or vertical cut (V_1, V_2) is called *proper* if either i) $d_{V_1}^x \geq 0$ and $b_{V_1}^x \geq 2d_{V_1}^x$, or ii) $d_{V_1}^o \geq 0$ and $b_{V_1}^o \geq 2d_{V_1}^o$. ■

LEMMA 2.9: *Consider a balanced grid and let (V_1, V_2) be a horizontal or vertical cut. Then, it holds: If cut (V_1, V_2) is proper, then cut (V_2, V_1) is proper too.*

Proof: To prove, suppose $d_{V_1}^x \geq 0$ and $b_{V_1}^x \geq 2d_{V_1}^x$; then $d_{V_2}^o = d_{V_1}^x \geq 0$ and $b_{V_2}^o = b_{V_1}^x \geq 2d_{V_1}^x = 2d_{V_2}^o$. ■

We often refer to a pair (d, b) , where $d \geq 0$ and $b \geq 2d$ as a *proper pair*.

LEMMA 2.10: *If a grid G can be factored into cycles, then G is balanced and, every vertical cut and every horizontal cut of G is proper.*

Proof: The proof that G is balanced is a straightforward consequence of lemma 2.5. To prove the rest, consider a cut (V_1, V_2) . Consider the part of the cycles contained in V_1 . This is comprised of a) some cycles, b) some paths from x-nodes to o-nodes, c) l_x paths from x-nodes to x-nodes and d) l_o paths from o-nodes to o-nodes; all these four categories span all the nodes of V_1 ; also the end-nodes of the paths belong to the boundary of V_1 . Let $d_{V_1}^x$ and $b_{V_1}^x$ be the numbers defined above. Without loss of generality, suppose that $d_{V_1}^x \geq 0$. Because of the existence of the four categories of paths and cycles and, according to lemma 2.4, we have:

$$d_{V_1}^x = l_x - l_o$$

and, obviously

$$b_{V_1}^x \geq 2l_x$$

Thus, the inequality $b_{V_1}^x \geq 2d_{V_1}^x$. ■

3. HAMILTON GRIDS

In the following theorem we state the main result of this paper:

THEOREM 1: A restricted grid has a Hamilton cycle iff i) it is balanced and ii) all vertical and horizontal cuts are proper.

Proof: The “only if” direction is a consequence of lemmata 2.4 (b) and 2.10.

The “if” direction is proved inductively on the number, N , of nodes. The basis step is trivial; just observe that for $N = 4$ the theorem is true. Suppose the theorem is true for every restricted grid with $N < k$. Let G be a restricted grid, with k nodes, which is balanced and such that all vertical and all horizontal cuts are proper. We shall prove that G has a Hamilton cycle.

To prove, we need to consider five cases, which are listed in the following lemma:

LEMMA 3.1: Consider any restricted grid G described by the sequence a_1, a_2, \dots, a_n , which is balanced and such that every vertical and every horizontal cut is proper. Then, either of the following cases happen:

- a) $a_2 < a_1$ or $a_n = 1$.
- b) Either there is a ladder of length 2 at the bottom or (its symmetrical) there is a ladder of length 2 at the top.
- c) There is a ladder of length 1.

- d) *There is a ladder of length 2, which is neither at the top nor at the bottom.*
- e) *There is a ladder of length ≥ 3 , which is neither at the top nor at the bottom.*

Proof: Recall lemma 2.2. Case (a) of the present lemma is identical to case (a) in lemma 2.2. Cases (b) through (e) define subcases of the case when there is a ladder. Finally, cases (a) through (e) do not include grids that have a ladder of length ≥ 3 either at the top or at the bottom. These grids, though, are such that there is at least one non-proper either vertical or horizontal cut (in both cases it is the fourth smallest cut; it is horizontal if the ladder is at the top and vertical if the ladder is at the bottom). ■

The five cases of lemma 3.1 are illustrated in Figure 3; in each case, only the region of the grid around the ladder (or around the considered peculiarity, in case (a)) is drawn. Also, notice that, in Figure 3, G is the grid induced by both the white nodes and the bold nodes; moreover, the bold broken-line edges, in the figure, represent edges that belong to G , but they do not belong to the Hamiltonian cycle of G .

Now, back to the proof of the main theorem. For each case, we will prove that we can construct a grid G' by deleting some nodes from grid G , such that G' is balanced and all its vertical and horizontal cuts are proper too. According to the inductive hypothesis, G' has a Hamilton cycle. Then we will show that from any Hamilton cycle of G' we can easily construct a Hamilton cycle of G . Thus G will be proven Hamiltonian too.

Consider any of the five cases in Figure 3. The way by which we obtain G' from G is fully described in this figure: G' is the grid induced by the white nodes or, in other words, G' is obtained from G after deleting the bold nodes.

It is easy to prove that G' is balanced. It is, also, easy to show how we can obtain a Hamilton cycle of G from a Hamilton cycle of G' : Consider, again, any case of Figure 3. Observe that any Hamilton cycle of G' contains the dotted edge in Figure 3. Thus, given any Hamilton cycle of G' , we obtain a Hamilton cycle of G by deleting (from the Hamilton cycle of G') the dotted edge and adding the bold-edges path.

The difficult part is to prove that the resulting G' is such that all vertical and horizontal cuts are proper. The rest of this proof is dealing with this part. In fact we prove it in five lemmata, each lemma treating one of the five cases. We present, though, the explicit proof only for one of the five cases, namely for lemma 3.2, which treats case (e) of lemma 3.1; we regard

this to be the hardest case, the other cases using only part of the tricks and arguments used in the explicitly treated case.

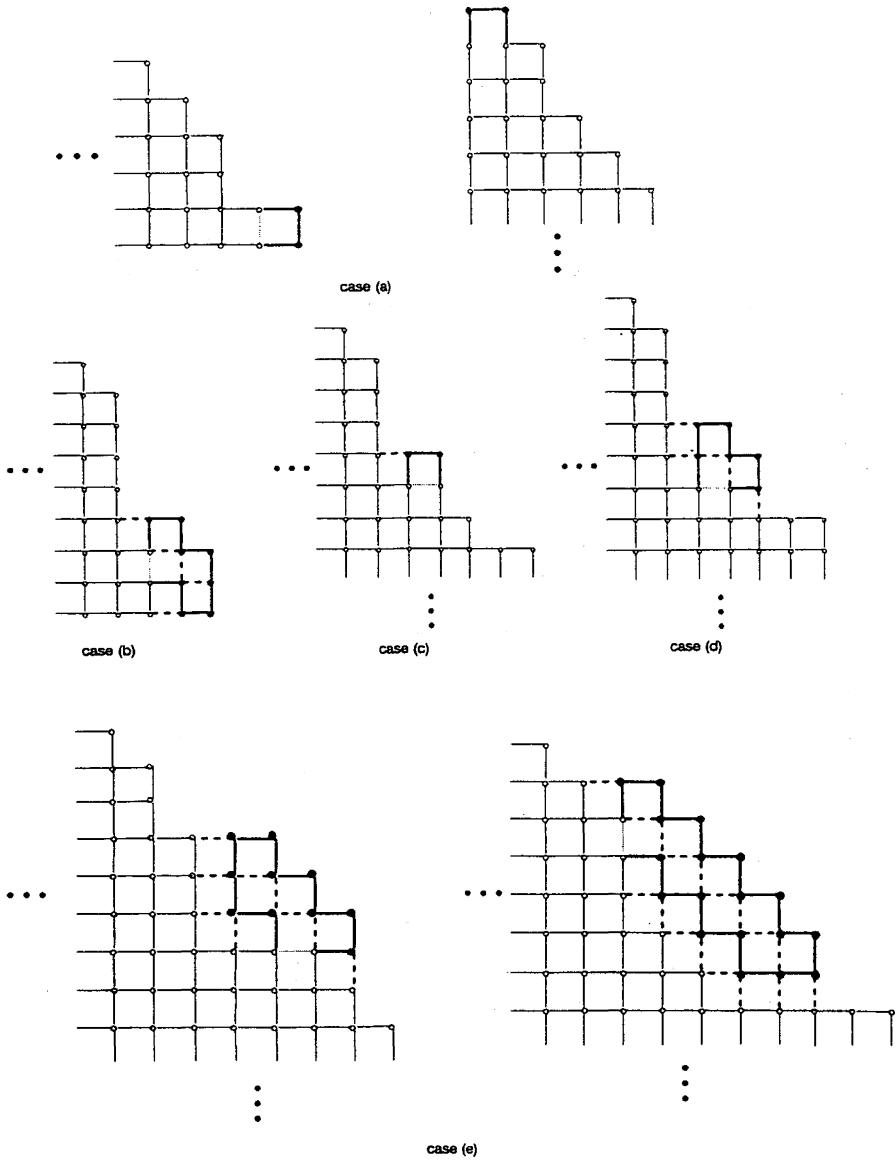


Figure 3. - The five cases for the proof of the main theorem.

LEMMA 3.2: Consider a balance grid, G , with all its vertical and horizontal cuts being proper. Moreover G satisfies case (e) of lemma 3.1. Let G' be the grid constructed from G as described above and illustrated in Figure 3 (e). Then all vertical and horizontal cuts of G' are proper.

Proof: The arguments in this proof refer to Figure 4. Figure 4 is drawn with the same codification as Figure 3. Thus, recall that grid G , in Figure 4, is the grid *with* the bold nodes and grid G' is the grid *without* the bold nodes. For convenience, we suppose that the “outer” nodes of the ladder are colored with color “x”. Let $\lambda + 1$ be the length of the ladder and since $\lambda + 1 \geq 3$, we have $\lambda > 1$.

Denote by $T_0^v, T_1^v, \dots, T_{\lambda+1}^v, T_{\lambda+2}^v$ the vertical cuts and by $T_0^h, T_1^h, \dots, T_{\lambda+1}^h, T_{\lambda+2}^h$ the horizontal cuts, whose relation with the ladder under consideration is shown explicitly in Figure 4.

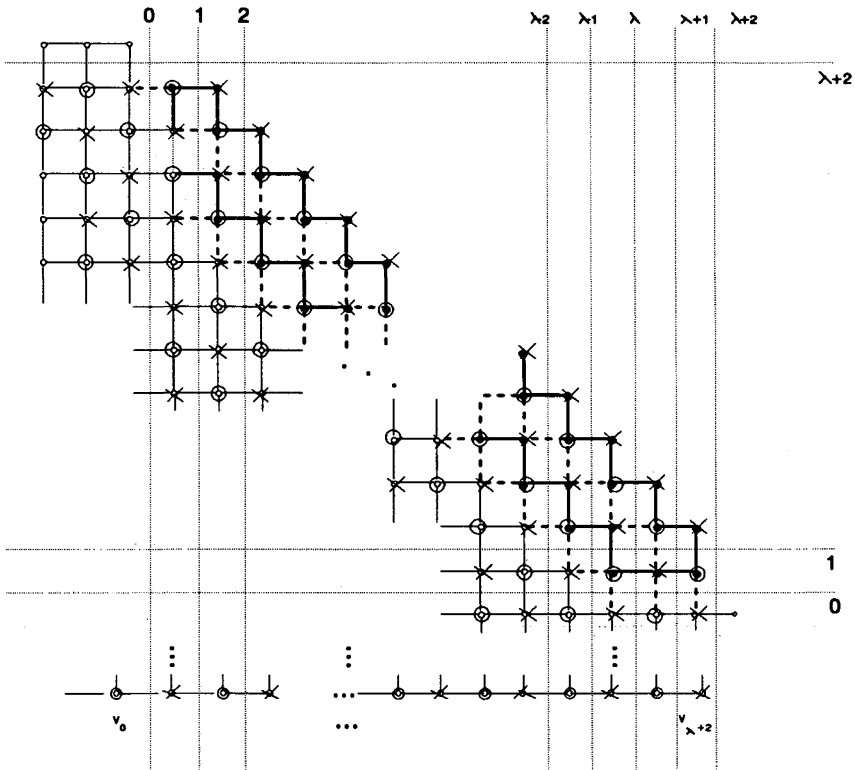


Figure 4. – The region of a grid around a ladder. The vertical cuts labelled $0, 1, \dots, \lambda + 1, \lambda + 2$ correspond to cuts $T_0^v, T_1^v, \dots, T_{\lambda+1}^v, T_{\lambda+2}^v$ and the horizontal cuts labelled $0, 1, \dots, \lambda + 2$ correspond to cuts $T_0^h, T_1^h, \dots, T_{\lambda+2}^h$.

Remark 3.3: Consider graph G and consider the cuts $T_{\lambda+2}^h$ and $T_{\lambda+2}^v$; they separate the graph in three parts: (1) the part “above” $T_{\lambda+2}^h$, (2) the part “on the right” of $T_{\lambda+2}^v$ and (3) the part inbetween. The d^x -number of part (3) is (i) $\frac{\lambda + 1}{2}$, for λ odd and (ii) either $\frac{\lambda}{2}$ or $\frac{\lambda}{2} + 1$ for λ even; that is the d^x -number of part (3) is always ≥ 1 . Since the grid is balanced, the d^x -number of either part (1) or part (2) must be ≤ -1 . We assume, without loss of generality, that the d^x -number of part (1) is ≤ -1 . ■

Taking the above remark into account, and the fact that the vertical and horizontal cuts of G are proper, we shall prove that the vertical and the horizontal cuts of grid G' (without the bold nodes) are proper too.

Proof for vertical cuts:

The affected by the modification vertical cuts are $T_0^v, T_1^v, \dots, T_\lambda^v, T_{\lambda+1}^v$. We examine these cuts for “properness” w.r.t. the nodes that lie on the right hand side of the cut in Figure 4 (recall lemma 2.9); thus, for convenience, we shall refer to the d^x -number of cut T_i^v and we shall mean the d^x -number of the set of nodes on the right hand side of the cut; similarly for the b^x number of a cut.

For the part of the proof that refers to the vertical cuts, we drop superscripts conveniently; thus a) by T_i we denote the vertical cut T_i^v of G , by T_i' we denote the corresponding vertical cut $T_i^{v'}$ of G' , and b) the corresponding d^x -numbers and b^x -numbers are denoted d_i and b_i (as referring to cut T_i) and d_i' and b_i' (as referring to cut T_i'). In the few instances where d^o 's and b^o 's are mentioned, superscripts appear to denote it explicitly.

We have four cases depending on the coloring of the nodes in the base of the grid and on whether λ is odd or even. In detail, they are defined as follows: Consider the nodes in the base of the grid $v_0, v_1, \dots, v_{\lambda+2}$, where node v_i is the base node “between” the vertical cut T_{i-1}^v and the vertical cut T_i^v (the placement, with respect to the ladder, of base nodes v_0 and $v_{\lambda+2}$ is shown in Figure 4). There are four different colorings of this sequence of nodes and these colorings define the four cases:

subcase (a): λ is odd and the coloring of the sequence of nodes $v_0, v_1, v_2, \dots, v_{\lambda+2}$ is (o, x, o, \dots, o, x) ; this subcase is illustrated in Figure 4.

subcase (b): λ is even and the coloring of the sequence of nodes $v_0, v_1, v_2, \dots, v_{\lambda+2}$ is (x, o, x, \dots, o, x) .

subcase (c): λ is even and the coloring of the sequence of nodes $v_0, v_1, v_2, \dots, v_{\lambda+2}$ is (o, x, o, \dots, x, o) .

subcase (d): λ is odd and the coloring of the sequence of nodes $v_0, v_1, v_2, \dots, v_{\lambda+2}$ is (x, o, x, \dots, x, o) .

Subcase (a): In this case we have: $d_0 = d_1 = d_2 = \frac{\lambda + 1}{2} + d_{\lambda+2}$.

Observe that the d^x -number of cut $T_{\lambda+2}^h$ (which is the d^x -number of part (1) in Remark 3.3) plus the d^x -number of cut T_0^v sum up to zero. Thus, (yielding from the remark) we have that $d_0 \geq 1$.

For $i = 4, 6, 8, \dots, \lambda + 1$, it holds:

$$d_i = d_{i-1} = d_{i-2} - 1 \quad \text{and} \quad d_{\lambda+2} = d_{\lambda+1} - 1 \quad (1)$$

and consequently $d_3 = d_4 \geq 0$. Thus, we have three cases:

i) There exists some $j \geq 3$ such that for $2j - 1 \leq k \leq \lambda + 2$ it is the case that $d_k < 0$. In this case, we refer to the d^o -numbers to prove properness.

For cuts $T'_{\lambda-1}, T'_\lambda$ and $T'_{\lambda+1}$ we have:

$$\begin{cases} d_{\lambda-1}^{o'} = d_{\lambda+1}^{o'}, & d_\lambda^{o'} = d_{\lambda+2}^{o'}, & d_{\lambda+1}^{o'} = d_{\lambda+2}^{o'} - 1 \\ b_{\lambda-1}^{o'} = b_{\lambda+1}^{o'}, & b_\lambda^{o'} = b_{\lambda+2}^{o'}, & b_{\lambda+1}^{o'} = b_{\lambda+2}^{o'} - 1 \end{cases} \quad (2)$$

From these equations properness of cuts $T'_{\lambda-1}, T'_\lambda$ and $T'_{\lambda+1}$ is proved easily as a consequence of properness of cut $T_{\lambda+2}$.

Along the sequence of cuts $T'_{2j-1}, T'_{2j}, \dots, T'_{\lambda-1}$ d^o -numbers increase ($d_{2j-1}^{o'} \leq d_{2j}^{o'} \leq \dots \leq d_{\lambda-1}^{o'}$) and b^o -numbers decrease ($b_{2j-1}^{o'} \geq b_{2j}^{o'} \geq \dots \geq b_{\lambda-1}^{o'}$). Thus, properness of all cuts in the sequence follows from properness of cut $T'_{\lambda-1}$.

The proof for cuts with non-negative d^x -numbers is the same as the proof for the same cuts as presented in case (iii).

ii) $d_{\lambda+2} = 0$. Then, we have $d'_{\lambda+1} = 1$ and $b'_{\lambda+1} \geq 2$ (if $b'_{\lambda+1} = 1$ then $T_{\lambda-1}^v$ would not be proper).

The proof for cuts with non-negative d^x -numbers is the same as the proof for the same cuts as presented in case (iii).

iii) $d_0, d_1, \dots, d_{\lambda+2} \geq 0$. Then, the argument is a little more complicated:

$$\begin{cases} d'_0 = d_0, & d'_1 = d_1 - 1, & d'_2 = d_2 - 1, \\ d'_i = d_i - 1, & i = 2, 3, \dots, \lambda - 1 \\ b'_0 = b_0, & b'_1 = b_1 - 2, & b'_2 = b_2 - 2, \\ b'_i = b_i - 2, & i = 2, 3, \dots, \lambda - 1 \end{cases} \quad (3)$$

So far equations 3 guarantee that the cuts $T'_0, \dots, T'_{\lambda-1}$ are proper. For the rest two affected cuts T'_λ and $T'_{\lambda+1}$ we compute their d 's and b 's in terms of $d_{\lambda+2}$. It is easy to see that

$$d'_\lambda = d_{\lambda+2} \quad \text{and} \quad d'_{\lambda+1} = d_{\lambda+2} + 1 \quad (4)$$

and that

$$b'_\lambda = b'_{\lambda-1} - 1 \quad \text{and} \quad b'_{\lambda+1} = b'_{\lambda-1} \quad (5)$$

Now, because $T'_{\lambda-1}$ is proper, we have

$$b'_{\lambda-1} \geq 2d'_{\lambda-1}$$

and since

$$d'_{\lambda-1} = d_{\lambda+2} + 1$$

we conclude that

$$b'_{\lambda-1} \geq 2d'_{\lambda+2} + 2$$

Substituting the last inequality in equations 5 we have

$$b'_\lambda \geq 2d_{\lambda+2} + 1 \quad \text{and} \quad b'_{\lambda+1} \geq 2d_{\lambda+2} + 2. \quad (6)$$

Equations 4 and 6 yield that cuts T'_λ and $T'_{\lambda+1}$ are proper too.

Subcase (b): $d_0 = \frac{\lambda}{2} + d_{\lambda+2}$ and $d_1 = \frac{\lambda}{2} + 1 + d_{\lambda+2}$. The argument is exactly the same as in (a) (and the computed relevant numbers).

Subcase (c): $d_0 = d_1 = d_2 = \frac{\lambda}{2} + d_{\lambda+2}$. The argument for $T'_0, \dots, T'_{\lambda-1}$ is the same as in subcase (a).

$d'_{\lambda-1} = d_{\lambda+2}$ and since $T_{\lambda-1}$ is proper $b'_{\lambda-1} \geq 2d_{\lambda+2}$. Again, by counting and taking into account the inequality

$$\begin{aligned} d'_\lambda = d_{\lambda+2} & \quad \text{and} \quad d'_{\lambda+1} = d_{\lambda+2} \\ b'_\lambda \geq 2d_{\lambda+2} & \quad \text{and} \quad b'_{\lambda+1} \geq 2d_{\lambda+2} \end{aligned}$$

Hence, all vertical cuts are proper.

Subcase (d): Similar to (c).

Proof for horizontal cuts: For the horizontal cuts the argument is easier. The horizontal cuts $T'_{\lambda+2}, T'_{\lambda+1}, \dots, T'_\mu$, with non-negative d^x -numbers are proper trivially (along the sequence the d^o -numbers decrease and the b^o -numbers increase, thus properness of all cuts in the sequence is proved easily as a consequence of properness of cut $T_{\lambda+2}$). The argument for the rest is very similar (in fact simpler) to the one for vertical cuts. Note that the two cases are not treated in a completely symmetric way due to the assumption made in Remark 3.3; to clarify this point, observe that the parts (1) and (2) of the grid (referred in remark 3.3) are symmetric wrto the ladder up until

the point where the assumption is made that the d_x -number of part (1) of the grid is ≤ -1 (combine this assumption with the assumption, made earlier, that the “outer” nodes of the ladder are colored “x”). ■

LEMMA 3.4: *Consider a balanced grid, G , with all its vertical and horizontal cuts being proper. Moreover G satisfies case (a) of lemma 3.1. Let G' be the grid constructed from G as described above and illustrated in Figure 3 (a). Then all vertical and horizontal cuts of G' are proper.*

LEMMA 3.5: *Consider a balanced grid, G , with all its vertical and horizontal cuts being proper. Moreover G satisfies case (b) of lemma 3.1. Let G' be the grid constructed from G as described above and illustrated in Figure 3 (b). Then all vertical and horizontal cuts of G' are proper.*

LEMMA 3.6: *Consider a balanced grid, G , with all its vertical and horizontal cuts being proper. Moreover G satisfies case (c) of lemma 3.1. Let G' be the grid constructed from G as described above and illustrated in Figure 3 (c). Then all vertical and horizontal cuts of G' are proper.*

LEMMA 3.7: *Consider a balanced grid, G , with all its vertical and horizontal cuts being proper. Moreover G satisfies case (d) of lemma 3.1. Let G' be the grid constructed from G as described above and illustrated in Figure 3 (d). Then all vertical and horizontal cuts of G' are proper.*

The four lemmata listed above are proved by using exactly the same proof techniques as the ones used in the proof of lemma 3.2; in fact the tricks used are simpler. ■

An easy corollary of the theorem of this section is that any restricted grid which can be factored into cycles is Hamiltonian too.

4. ALGORITHMS

The proof of theorem 1 suggests a straightforward sequential linear algorithm that decides whether the Hamilton circuit problem has a solution on restricted grids; in the case that there is a solution, it also provides a linear algorithm that finds the Hamilton cycle. The algorithm sorts out a ladder considering the boundary nodes (*i.e.* nodes with degree less than 4) and updates the set of the new boundary nodes in time proportional to the number of nodes in the ladder.

In the rest of this section, we shall list an optimal parallel algorithm for the decision problem. The algorithm consists of three steps: 1) We associate with each face of the grid its four edges. 2) We find all vertical and horizontal

cuts and their corresponding b -number. 3) We find the d -number of each vertical and each horizontal cut.

It is rather easy to implement in parallel step (1); we only need Brent's scheduling principle [4], according to which, if there is an algorithm on a PRAM that runs in $O(T(n))$ time and uses $O(p(n))$ processors, then there is, also, an algorithm on a PRAM that runs in $O(kT(n))$ time and uses $O(p(n)/k)$ processors. Steps (2) and (3) are more involved and make use of the following results: It is known that, we can find all the chains of a graph (a chain is a connected graph with all nodes of degree 2 except two nodes that have degree 1), with their lengths and assign the nodes that belong to each chain in $O(\log n)$ parallel time and using $O(n/\log n)$ processors on a EREW PRAM [4]. The same algorithm [3] can be, also, used to characterize a node on a chain either odd or even (*i.e.*, if we imagine the vertices on a chain as linearly ordered, we can set, arbitrarily, one endpoint to zero and the rest accordingly). Finally, we, also, make use of the result, according to which, we can solve the prefix-sums problem (*i.e.*, given a sequence of integers, find, for all i , the sum of the i first terms) in $O(\log n)$ parallel time using $O(n/\log n)$ processors [4].

We parallelize step (1) as follows: We consider a node u and its four adjacent nodes v_1, v_2, v_3, v_4 ; if v_1 and v_2 have a common node, w , adjacent to both of them, then the edges $(u, v_1), (u, v_2), (w, v_1)$ and (w, v_2) define a face in the grid. Thus, we can find the edges in each face of a grid in constant parallel time using linear number of processors on a EREW Parallel Random Access Machine (PRAM); and, by Brent's scheduling principle [4], we can do that also with $O(n/\log n)$ processors in $O(\log n)$ parallel time.

For step (2), our algorithm proceeds as follows: It marks the edges that connect either i) two nodes of degree 3 or ii) a node of degree 3 with a node of degree 2. In general, the marked edges form chains; we consider those chains with both endpoints of degree 2; there are at most 4 such chains. Finally, we choose the two longest adjacent chains amongst them (we break ties arbitrarily); note that the two chosen chains correspond to the vertical and horizontal bound of the grid. We label the edges of one chain "vertical" and we label the edges of the other chain "horizontal". Now, we form an auxiliary graph $A(V_A, E_A)$; each node of A corresponds to exactly one edge of G , and two nodes of A are connected by an edge if the corresponding edges in G belong to the same face in the planar embedding and they are not adjacent. Observe that A consists solely of chains and each chain has exactly one labelled ("vertical" or "horizontal") end-node (*i.e.* the corresponding to

the end-node edge of G is labelled by the previous procedure). We label all the nodes of each chain after the label of that node. Note that each chain corresponds to either exactly one horizontal cut or exactly one vertical cut; thus the length of each chain gives the b -number of the cut.

In order to find the d -number (step (3)), we, first color the restricted grid with two colors as follows: Consider a boundary of a vertical cut and consider all vertices on the boundary; the induced subgraph is a *vertical chain*. Form an auxiliary graph A as follows: The vertices of A are all vertical chains of G and there is an edge in A if the corresponding chains are adjacent in G (i.e., there are at least two nodes one in each chain that are connected by an edge in G); note that A is a single chain. We label odd and even vertices on A ; thus we label odd and even vertical chains in G . The first node of an even vertical chains is colored with color x and the first node of an odd vertical chain is colored with color o . To finish, we consider each vertical chain and label its nodes odd and even; an even node of an even chain is colored x , an even node of an odd chain is colored o , an odd node of an even chain is colored o , an odd node of an odd chain is colored x . Finally, to each vertical chain assign one of the numbers 1, 0, -1, if the x 's outnumber the o 's in this chain, if their numbers are equal or if the o 's outnumber the x 's respectively. Thus, finding the d -number for each vertical cut corresponds to the prefix-sums problem. Therefore, we need $O(\log n)$ parallel time and $O(n/\log n)$ processors to solve on a EREW PRAM the Hamilton circuit problem for restricted grids.

Open problems: We note that the results of this paper leave open the question whether the Hamilton circuit problem has a polynomial algorithm for a wider class of restricted grids; our conjecture is that theorem 1 holds for wider classes of grids (e.g. consider a grid that is created by identifying the vertical boundaries of two restricted grids). We have proved that the Hamilton circuit problem belongs to \mathcal{NC} ; but we do not know how to solve in \mathcal{NC} the problem of finding a Hamilton cycle if there exists one; the polynomial algorithm we give here for this problem does not seem to be efficiently parallelizable. Another open question is whether the Hamilton circuit problem is \mathcal{P} -hard for general solid grids.

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