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## PRESCRIBED ULTRAMETRICS (\*)

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**Abstract.** – Let  $G=(S, E)$  be a subgraph of  $K_n=(S, F)$ , the complete graph on  $n$  vertices. Let  $v$  be a function from  $E$  to  $R^+$ . We prove two theorems on the extensibility of  $v$ . Every function  $v$  extends to a metric on  $F$  iff  $G$  is a forest. The function  $v$  extends to an ultrametric on  $F$  if and only if for all non-trivial cycles  $p$  in  $G$ ,  $\text{mult}(p) > 1$ , where  $\text{mult}(p)$  depends on the values of  $v$  on paths.

**Résumé.** – Soit  $G=(S, E)$  un sous-graphe de  $K_n=(S, F)$ , le graphe complet sur  $n$  sommets. Soit  $v$  une fonction de  $E$  dans  $R^+$ . Nous prouvons deux théorèmes sur le prolongement de  $v$ . Toute fonction  $v$  se prolonge en une métrique sur  $F$  si et seulement si  $G$  est une forêt. La fonction  $v$  se prolonge en une ultramétrique sur  $F$  si et seulement si pour tout cycle non trivial  $p$  dans  $G$ , on a  $\text{mult}(p) > 1$ , où  $\text{mult}(p)$  dépend des valeurs de  $v$  sur les chemins.

### INTRODUCTION

Let  $S$  be a set of points and  $u$  a non-negative real-valued function on  $S \times S$ . The function  $u$  is called a *metric* if

1.  $u(x, y) \geq 0$ ;
2.  $u(x, y) = 0$ ;
3.  $u(x, y) = u(y, x)$ ;
4.  $u(x, y) \leq u(x, z) + u(z, y)$ .

If for all  $z$  in  $S$ ,  $u$  also satisfies

5.  $u(x, y) \leq \max \{u(x, z), u(z, y)\}$ ,

then  $u$  is called an *ultrametric*.

Ultrametrics satisfy more than the triangle inequality; inequality (5) prevents scalene triangles; that is, for any three points  $x, y, z$  of  $S$ , it is

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impossible that  $u(x, y) < u(y, z) < u(x, z)$ . To see why, note that (5) implies  $u(x, z) \leq \max\{u(x, y), u(y, z)\} = u(y, z)$ , a contradiction. Thus, any three points in an ultrametric space determine either an *isosceles* triangle or an *equilateral* triangle.

Ultrametrics arise in the context of  $p$ -adic evaluations on infinite fields [5]. There is interest in creating arbitrary ultrametrics on finite sets, in particular, on  $K_n$ , the complete graph on  $n$  points [1 to 4]. Since many ultrametric extensions are known to be NP-complete [3], it is most interesting that one extension can be done in a polynomial number of steps.

**THEOREM 1:** *Let  $G=(S, E)$  be a subgraph of the complete graph  $K_n=(S, F)$  and let  $v$  be an arbitrary function from  $E$  to  $R^+$ . If  $G$  is a forest, then  $v$  extends to an ultrametric on  $F$  in at most  $O(n^2)$  steps.*

*Proof:* Extend  $G$  to a spanning tree  $Q$  for  $K_n$ . Extend  $v$  to the edges of  $Q-G$  by assigning arbitrary positive number to each such edge. We use induction on  $n$  to extend  $v$  to an ultrametric  $u$  on all edges of  $K_n$  in at most  $(n+1)(n-2)/2$  additional steps.

*Basis:* There is nothing to prove for  $n=1$  or  $n=2$ . The case of  $n=3$  is the so called isosceles restriction of an ultrametric. Namely, we define the ultrametric  $u$  on the missing edge to be the maximum of  $v$  on the other two sides. This extension takes one additional step.

Assume the result for  $n$  and consider the case  $n+1$ . There exists an end  $x$  of the tree  $Q$ . Let  $U=S-\{x\}$ . Let  $T$  be the restriction of  $Q$  to  $U$ . By induction, in at most  $(n+1)(n-2)/2$  additional steps, we can find an ultrametric extension  $u$  to  $U$  of the restriction of  $v$  to  $T$ . As  $x$  is an end, there exists a unique  $y$  in  $U$  with  $(x, y)$  in  $Q$ . Let  $w=v(x, y)$ . For each  $z$  in  $U-\{y\}$ , set  $u(x, z)=\max\{w, u(y, z)\}$ . The number of steps to create this extension is at most  $n+((n+1)(n+2)/2)=(n+2)(n+1)/2$  as claimed.

To check that our extension  $u$  is an ultrametric, we need only verify  $u(a, b) \leq \max\{u(a, c), u(b, c)\}$  for all choices of distinct  $a, b, c$  in  $S$ . There are two cases: (1)  $x$  is not in  $\{a, b, c\}$ . (2)  $x$  is in  $\{a, b, c\}$ . In case (1), the inequality holds as  $u$  is an ultrametric on  $U$ . In case (2), there are two subcases: (I)  $y$  is in  $\{a, b, c\}$ , (II)  $y$  is not in  $\{a, b, c\}$ . In case (I), the inequality holds by construction. In case (II), there are three subcases: (A)  $x=a$ , (B)  $x=b$ , (C)  $x=c$ . Since  $y$  is not in  $\{a, b, c\}$ , each of these three verifications is straightforward. This concludes the proof of theorem 1.

**THEOREM 2:** *Let  $G=(S, E)$  be a subgraph of the complete graph  $K_n=(S, F)$ . Then the following are equivalent:*

- (a) *Every function  $v : E \rightarrow R^+$  extends to a metric on  $F$ ;*
- (b)  *$G$  is a forest.*

*Proof:* Theorem 1 proves that (1 b) implies (1 a). To show (1 a) implies (1 b) it suffices to prove that if  $G$  is not a forest, then there exists a function  $v$  from  $E$  to  $R^+$  that does not extend to a metric on  $F$ . If  $G$  is not a forest, then  $G$  contains a (simple) cycle  $e_1, e_2, \dots, e_k, k>2$ . Define  $v$  on  $e_i, 1 \leq i < k$ , to be arbitrary positive numbers. Define  $v$  on the edge  $e_k$  to be any number greater than the sum of  $v(e_i), 1 \leq i < k$ . Since  $v$  fails to satisfy the triangle inequality on the edge  $e_k$ , no extension of  $v$  can be a metric on  $F$ . This concludes the proof of theorem 2.

We now extend theorem 2 to ultrametrics. We will see that whether a particular function  $v : S \rightarrow R^+$  has an ultrametric extension depends on the behaviour of  $v$  on non-trivial cycles of  $G$ . A cycle is any sequence of edge connected vertices  $v_0 \dots v_n, v_0=v_n$ , allowing repeated vertices and repeated edges. A cycle is trivial, by definition, if it is a cycle with only two edges.

Let  $p$  be a (not necessarily simple) path in  $G$ . Let  $\max(p)$  denote the largest value of  $v$  on  $p$ . Let  $\text{mult}(p)$  denote the number of times  $v$  attains  $\max(p)$  on  $p$ . Clearly, for all paths  $p, \text{mult}(p) \geq 1$ .

We require two preliminary lemmas.

**LEMMA 3:** *A symmetric function  $u : S \times S - \{(s, s) : s \text{ is in } S\} \rightarrow R^+$  is an ultrametric if and only if for each triple  $x, y,$  and  $z$  of distinct members of  $S, \text{mult}(xyzx) > 1$ .*

*Proof:* If  $u$  is an ultrametric, then as remarked at the start of the paper, every triangle is either isosceles or equilateral, that is,  $\text{mult}(xyzx) > 1$ . Conversely, to show that  $u$  must be an ultrametric when  $\text{mult}(xyzx) > 1$  on all triangles, it suffices to observe that (5) always holds.

**LEMMA 4:** *Let  $G=(S, E)$  be a subgraph of the complete graph  $K_n=(S, F)$ . Let  $x$  and  $y$  belong to  $S$ . Let  $v$  be an arbitrary function from  $E$  to  $R^+$ . Let  $Q$  be the set of all paths from  $x$  to  $y$  in  $G$ . Let  $P$  be the set of all paths  $p$  in  $Q$  such that  $\text{mult}(p)=1$ . If all non-trivial cycles  $p$  in  $G$  satisfy  $\text{mult}(p) > 1$ , then*

- (1) *For any  $p_1$  and  $p_2$  in  $P, \max(p_1)=\max(p_2)$ .*
- (2) *For each  $q$  in  $Q$  and each  $p$  in  $P, \max(q) \geq \max(p)$ .*

*Proof:* We prove (1) by contradiction. Suppose there were elements  $p_1$  and  $p_2$  of  $P$  with  $\max(p_1) < \max(p_2)$ . Since  $c = p_1 p_2^{-1}$  is a non-trivial cycle in  $G$ , we have by hypothesis  $\text{mult}(c) > 1$ . Thus, there are at least two places that  $p_2$  takes on its max, contrary to  $p_2$  belonging to  $P$ . This proves (1). Similar proof holds for (2).

**THEOREM 3:** *Let  $G=(S, E)$  be a subgraph of the complete graph  $K_n=(S, F)$ . A function  $v : E \rightarrow R^+$  extends to an ultrametric on  $F$  if and only if*

( $\star$ ) *for all non-trivial cycles  $p$  in  $G$ ,  $\text{mult}(p) > 1$ .*

*Proof:* First assume that  $v$  extends to an ultrametric on  $F$ , but that ( $\star$ ) fails for some non-trivial cycle  $p = x_0 \dots x_n$ . Of all cycles  $p$  with  $\text{mult}(p) = 1$ , choose one whose length,  $n$ , is minimal. By lemma 3,  $\text{mult}(p) > 1$  on all 3-edged cycles. Therefore,  $n$  must be  $> 3$ . Without loss of generality, let  $w = \max(p) = v(x_0, x_1)$ . Since  $\text{mult}(p) = 1$ ,  $v(x_1, x_2)$  must be strictly less than  $w$ . Applying lemma 3 to  $x_0 x_1 x_2 x_0$ , and knowing that  $v(x_0, x_1) = w$  and  $v(x_1, x_2) < w$ , we conclude that  $v(x_0, x_2)$  must also be  $w$ . Now form the cycle  $q = x_0 x_2 \dots x_n$  of length  $n-1$ . Since  $\text{mult}(q) = 1$  we have obtained a contradiction to the choice of  $n$ .

Conversely, suppose that ( $\star$ ) holds. To prove that  $v$  extends to an ultrametric, we consider two cases:  $G$  is complete,  $G$  is not complete. If  $G$  is complete, and ( $\star$ ) holds for all triangles of  $G$ , then by lemma 3,  $v$  must be an ultrametric on  $S$ . On the other hand, if  $G$  is not complete, then there are  $x$  and  $y$  in  $S$  for which  $(x, y)$  is not in  $E$ . Let  $J$  be the union of  $E$  and the edge  $(x, y)$  and let  $H=(S, J)$ . Proceeding by induction on the cardinality of  $E$ , it suffices to show that  $H$  satisfies ( $\star$ ).

Let  $Q$  be the set of paths  $p$  from  $x$  to  $y$  in  $G$ . Let  $P$  be the set of paths in  $Q$  such that  $\text{mult}(p) = 1$ . By lemma 4,

(1) for any  $p_1$  and  $p_2$  in  $P$ ,  $\max(p_1) = \max(p_2)$ ;

(2) for all  $q$  in  $Q$  and all  $p$  in  $P$ ,  $\max(q) \geq \max(p)$ .

Define  $v$  on the edge  $(x, y)$  to be  $\min \{ \max(q) : q \text{ in } Q \}$ . We need only show that the extension  $v$  from  $J$  to  $R^+$  still satisfies ( $\star$ ).

Let  $s = x_0 \dots x_n$  be a non-trivial cycle in  $H$ . Since  $G$  satisfies ( $\star$ ) there is nothing to prove unless the edge  $(x, y)$  belongs to the cycle  $s$ . Therefore, without loss of generality, we may take  $y = x_0$  and  $x = x_1$ . Thus,  $q = x_1 \dots x_n$ , a path  $x$  to  $y$ , belongs to  $Q$ . By the definition of  $v(x, y)$  and the choice of  $w$ ,  $v(x, y) = w \leq \max(q)$ . There are two possibilities:  $\text{mult}(q) > 1$ ,  $\text{mult}(q) = 1$ . If

$\text{mult}(q) > 1$ , then  $\text{mult}(s) > 1$  and we are done. If  $\text{mult}(q) = 1$ , then  $q$  belongs to  $P$ . By (2) and the construction,  $\max(q)$  must itself be  $w$ . Since  $v(x_0, x_1)$  is also  $w$ , we can conclude in this case also that  $\text{mult}(s) > 1$ . This completes the proof of theorem 3.

Theorem 2 and 3 differ significantly in computational requirements. Testing for a forest can be done in a polynomial number of steps; testing  $(\star)$  for all cycles may require a factorial number of steps. For example, consider the complete graph on  $n$  vertices with a few edges removed. Such a graph has more than  $n!$  non-trivial cycles.

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