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## ON THE AUTOMORPHISM GROUP OF A TOROIDAL HYPERMAP (\*)

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*Abstract.* – We prove that a finite group is (isomorphic to) an automorphism group of a toroidal hypermap if and only if it is abelian of rank at most two or it can be decomposed as the semi-direct product of a cyclic group of small order (namely 2, 3, 4 or 6) and such an abelian group. Moreover for any group arising this way we construct a toroidal hypermap having it as its full automorphism group.

*Résumé.* – Nous démontrons qu'un groupe fini est (isomorphe à) un groupe d'automorphisme d'une hypercarte toroïdale si et seulement si il est abélien de rang au plus deux ou bien il peut être décomposé dans le produit semi-direct d'un groupe cyclique d'ordre 2, 3, 4 ou 6 et d'un groupe abélien de rang au plus deux. De plus nous allons construire pour chacun de ces groupes une hypercarte toroïdale dont il est le groupe d'automorphisme.

### 1. INTRODUCTION

In [4] it is proved that for any (finite) group  $G$  there exists a hypermap  $\mathcal{H}$  such that  $\text{Aut}(\mathcal{H})$ , the full automorphism group of  $\mathcal{H}$ , is (isomorphic to)  $G$ . It is then interesting to ask, for a given non-negative integer  $g$  which groups can arise as the full automorphism group of a hypermap of genus  $g$  and, more generally, which groups can arise as automorphism group of some hypermap of genus  $g$ .

The case  $g=0$  has a complete answer: the groups that can arise as the full automorphism group of a planar (*i. e.*, having genus 0) hypermap are precisely the polyhedral groups [5].

In the case  $g \geq 2$  an important result is the theorem of Hurwitz (*see* [11]) which states

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FACT 1: Let  $\mathcal{H}$  be a hypermap of genus  $g \geq 2$  then:

$$|\text{Aut}(\mathcal{H})| \leq 84(g-1)$$

In then follows that it is possible at least in principle to find all the groups that are the full automorphism group of some hypermap of genus  $g \geq 2$ .

The case  $g=1$ , *i.e.* that of toroidal hypermaps, was settled by the author in [2]. Here we present a condensed form of these results. We will show that the groups that can be automorphism groups of a toroidal hypermap are either finite abelian groups of rank at most two or are decomposable as a semi-direct product of a cyclic group of small order (namely 2, 3, 4 or 6) and such an abelian group. Moreover we will show that each of these groups actually arise as the full automorphism group of some toroidal hypermap.

In proving our theorem we will use a technique introduced by Jones and Singerman [10] and based on Fuchsian groups. The needed facts about those groups will be recalled in the sequel but we will omit proofs.

**2. HYPERMAPS AND AUTOMORPHISMS**

A hypermap  $\mathcal{H}$  is a triple  $(B, \sigma, \alpha)$  where  $B$  is a finite set of *brins* and  $\sigma, \alpha$  are two permutations of  $B$  such that the groupe  $\langle \sigma, \alpha \rangle$  is transitive on  $B$ . In the special case where  $\alpha$  is a fixed point free involution then  $\mathcal{H}$  is a *map*. From the geometrical viewpoint it can be shown that maps and hypermaps give a representation of graphs and hypergraphs, respectively, in an orientable surface ([3], [6]).

This geometrical interpretation of hypermaps explains the reason for calling the cycles of  $\alpha, \sigma$  and  $\sigma\alpha$  the edges, vertex and faces of  $\mathcal{H}$  respectively. Again from the geometrical interpretation of hypermaps we have the following definition

DEFINITION 1: Let  $\mathcal{H} = (B, \sigma, \alpha)$  be a hypermap. The genus of  $\mathcal{H}$  is given by the following Euler's formula:

$$g(\mathcal{H}) := |B| - z(\alpha) - z(\sigma) - z(\alpha\sigma), \tag{1}$$

where for any permutation  $\gamma$ ,  $z(\gamma)$  denotes the number of cycles of  $\gamma$ , including the trivial ones.

It can be proved that  $g(\mathcal{H})$  is the genus of the surface in which  $\mathcal{H}$  defines a embedding; it then follows that  $g(\mathcal{H})$  is a non-negative integer. This last

result can also be proved by combinatorial means [9]. A hypermap is said to be *planar* if  $g(\mathcal{H})=0$  and to be *toroidal* if  $g(\mathcal{H})=1$ .

Geometrically an *automorphism* of a hypermap  $\mathcal{H}=(B, \sigma, \alpha)$  can be defined as an automorphism of the underlying hypergraph preserving the embedding defined by  $\mathcal{H}$ . This can be proved to be equivalent to the following combinatorial definition.

DEFINITION 2: An automorphism of the hypermap  $\mathcal{H}=(B, \sigma, \alpha)$  is a permutation  $\varphi$  of  $B$  commuting with both  $\alpha$  and  $\sigma$ :

$$\varphi\sigma = \sigma\varphi, \quad \varphi\alpha = \alpha\varphi$$

The full automorphism group of  $\mathcal{H}$ , denoted by  $\text{Aut}(\mathcal{H})$ , is nothing else than the centralizer of  $\langle \sigma, \alpha \rangle$  in  $\text{Sym}(B)$ , the symmetric group over  $B$ . The transitivity of  $\langle \sigma, \alpha \rangle$  implies that  $\text{Aut}(\mathcal{H})$  is semiregular. By an *automorphism group* of  $\mathcal{H}$  we mean a subgroup of  $\text{Aut}(\mathcal{H})$ .

PROPOSITION 1: Let  $b_0 \in B$  be a brin of the hypermap  $\mathcal{H}=(B, \sigma, \alpha)$  and let  $G=\langle \sigma, \alpha \rangle$ . Then

$$\text{Aut}(\mathcal{H}) \cong \frac{N_G(G_{b_0})}{G_{b_0}}$$

where  $N_G(G_{b_0})$  is the normalizer in  $G$  of the stabilizer  $G_{b_0}$  of  $b_0 \in B$ .

Proof: As  $G$  is transitive over  $B$ , for each  $b \in B$  there exists  $\gamma_b \in G$  such that  $\gamma_b(b_0)=b$ . Let  $\varphi$  be in  $N_G(G_{b_0})$ . Then

$$\begin{aligned} \hat{\varphi}: B &\rightarrow B \\ b &\rightarrow \gamma_b(\varphi^{-1}(b_0)) \end{aligned}$$

gives a well-defined permutation of  $B$ . Moreover  $\hat{\varphi} \in \text{Aut}(\mathcal{H})$  and  $\varphi\psi = \hat{\varphi}\hat{\psi}$  so that

$$\begin{aligned} \wedge: N_G(G_{b_0}) &\rightarrow \text{Aut}(\mathcal{H}) \\ \varphi &\rightarrow \hat{\varphi} \end{aligned}$$

defines a morphism. As  $\text{Aut}(\mathcal{H})$  is semi-regular the kernel of  $\wedge$  is  $G_{b_0}$ . Conversely let  $\tau \in \text{Aut}(\mathcal{H})$  and let  $\varphi \in G$  be such that  $\varphi^{-1}(b_0)=\tau(b_0)$ . We claim that  $\varphi \in N_G(G_{b_0})$  and that  $\hat{\varphi}=\tau$ . In fact if  $\gamma$  is in  $G_{b_0}$  we have

$$\varphi\gamma\varphi^{-1}(b_0) = \varphi\gamma(\tau(b_0)) = \varphi\tau(\gamma(b_0)) = \varphi(\tau(b_0)) = \varphi(\varphi^{-1}(b_0)) = b_0$$

proving that  $\varphi\gamma\varphi^{-1} \in G_{b_0}$  and so that  $\varphi \in N_G(G_{b_0})$ . Moreover for any brin  $b$  we have

$$\hat{\varphi}(b) = \gamma_b(\varphi^{-1}(b_0)) = \gamma_b(\tau(b_0)) = \tau(\gamma_b(b_0)) = \tau(b_0),$$

proving that  $\hat{\varphi} = \tau$ .

**COROLLARY:** *With the same notation as in Proposition 1 there is a bijection between the automorphism groups of  $\mathcal{H}$  and the subgroups  $N$  of  $G$  such that  $G_{b_0} \trianglelefteq N$ . Namely if  $N \leq G$  is such that  $G_{b_0} \trianglelefteq N$  then  $N/G_{b_0}$  is (isomorphic to) a subgroup of  $\text{Aut}(\mathcal{H})$  and all subgroups of  $\text{Aut}(\mathcal{H})$  arise this way.*

Let us recall the following standard result of group theory

**FACT 2:** *Let  $\pi: \Gamma \rightarrow G$  be a group homomorphism of a group  $\Gamma$  onto a group  $G$ . Let  $H$  be a subgroup of  $G$  and  $T = \pi^{-1}(H)$  is inverse image. Then there is a bijection between the subgroups of  $\Gamma$  containing  $T$  and those of  $G$  containing  $H$  given by*

$$K \rightarrow \pi(K)$$

Moreover  $T \trianglelefteq K$  if and only if  $H \trianglelefteq \pi(K)$  and in this case

$$\frac{K}{T} \cong \frac{\pi(K)}{H}$$

Let now  $\mathcal{H} = (B, \sigma, \alpha)$  be a hypermap and let  $A$  be a subgroup of  $\text{Aut}(\mathcal{H})$ . Clearly  $\langle \sigma, \alpha \rangle$  acts transitively on the set  $B/A$  of all  $A$ -class of brins (that is

$$B/A = \{[b] / b \in B\}$$

where

$$[b] = \{ \varphi(b) / \varphi \in A \}$$

is the set of all brins  $A$ -equivalent to  $b$ ). This means that the triple  $\mathcal{H}' = (B/A, \sigma', \alpha')$  (where  $\sigma'$  and  $\alpha'$  are defined by

$$\sigma'([b]) = [\sigma(b)], \alpha'([b]) = [\alpha(b)]$$

for all  $[b] \in B/A$ ) is a hypermap.  $\mathcal{H}'$  is called the *quotient hypermap of  $\mathcal{H}$  w.r.t. the automorphism group  $A$*  and will be denoted by  $\mathcal{H}/A$ .

The following definition is a generalization of an idea of Jones and Singerman [10].

DEFINITION 3: Let  $l, m$  and  $n$  be three integers such that the orders of  $\alpha, \sigma$ , and  $\sigma\alpha$  divides  $l, m$  and  $n$  respectively. Then the group

$$(0; l, m, n) := \langle x, y, z; x^l = y^m = z^n = x^{-1}y^{-1}z^{-1} = 1 \rangle \tag{2}$$

is a covering of  $\mathcal{H}$ .

Let  $\Gamma = (0; l, m, n)$  be a covering of  $\mathcal{H}$ . Then from the definition of covering and from elementary properties of free groups (see [12], p. 15) it follows that there exists a (unique) group homomorphism  $\pi$  of  $\Gamma$  onto  $\langle \sigma, \alpha \rangle$  defined by

$$\pi(x) = \alpha, \quad \pi(y) = \sigma, \quad \pi(z) = (\sigma\alpha)^{-1}. \tag{3}$$

The projection  $\pi$  defined by (3) will be referred to as the *natural projection* of the covering  $\Gamma$ .

Let now  $b$  be a brin of  $\mathcal{H}$  and let  $G_b$  be the stabilizer of  $b$  in  $\langle \sigma, \alpha \rangle$

$$G_b = \{ \gamma \in \langle \sigma, \alpha \rangle / \gamma(b) = b \}.$$

The subgroup  $H = \pi^{-1}(G_b)$  will be called the *group of  $\mathcal{H}$  w.r.t. the covering  $\Gamma$  and the base point  $b$* , and denoted by  $H = \Gamma_{\mathcal{H}, b}$  or simply by  $H = \Gamma_b$ .

By Fact 2 and the proof of Proposition 1 we obtain the following result

PROPOSITION 2: Let  $\mathcal{H}$  be a hypermap,  $b$  one of its brins and  $\Gamma$  be a covering of  $\mathcal{H}$ . Let  $A$  be an automorphism group of  $\mathcal{H}$ . Then there exists  $N \leq \Gamma$  such that  $\Gamma_b \trianglelefteq N$  and

$$A \cong \frac{N}{\Gamma_b}.$$

Moreover  $\Gamma$  is also a covering of  $\mathcal{H}' = \mathcal{H} / A$  and  $N = \Gamma_{[b], \mathcal{H}'}$ .

As a special case of Proposition 2 we have:

$$\text{Aut}(\mathcal{H}) \cong \frac{N_{\Gamma}(\Gamma_b)}{\Gamma_b}.$$

### 3. FUCHSIAN GROUPS

A group like  $(0; l, m, n)$  is called a *triangular group*. It is a special case of a *Fuchsian group*.

DEFINITION 4: A Fuchsian group of genus  $g \geq 0$  and of periods  $m_1, m_2, \dots, m_t$  ( $t \geq 0, m_i \geq 1$  for  $i = 1, 2, \dots, t$ ) is a group having a presentation of the form

$$\langle x_1, \dots, x_t, a_1, b_1, \dots, a_g, b_g; x_1^{m_1}, \dots, x_t^{m_t}, x_1^{-1} \dots x_t^{-1} [a_1, b_1] \dots [a_g, b_g] \rangle \quad (4)$$

where  $[a, b]$  is the commutator  $a^{-1} b^{-1} ab$ .

Various facts are known about Fuchsian groups. We recall here the most significant for our work. A nice combinatorial proof of Facts 3 to 5 below can be found in ([7, 8]).

FACT 3: Any element of finite order in a Fuchsian group is conjugated to a power of a generator of finite order.

FACT 4: Two Fuchsian groups are isomorphic if and only if they have the same genus and the same periods.

Fact 4 allows us to write  $(g; m_1, \dots, m_t)$  to denote the Fuchsian group of genus  $g$  and periods  $m_1$  to  $m_t$ . Moreover it tells us that the following defines an invariant of a Fuchsian group.

DEFINITION 5: Let  $\Gamma = (g; m_1, \dots, m_t)$  be a Fuchsian group. The measure of  $\Gamma$  is defined by:

$$\mu(\Gamma) = 2g - \sum_{i=1}^t \left(1 - \frac{1}{m_i}\right)$$

FACT 5: Let  $\Gamma$  be as above and let  $H$  be a subgroup of  $\Gamma$  having finite index in  $\Gamma$ . Then the following hold:

1.  $H$  is itself a Fuchsian group  $(g'; m_{1,1}, \dots, m_{1,s_1}, \dots, m_{t,1}, \dots, m_{t,s_t})$ ;
2. the Riemann-Hurwitz formula:

$$\mu(H) = [\Gamma : H] \mu(\Gamma). \quad (5)$$

holds;

3.  $H$  as a presentation

$$\langle x_{i,j} (i=1, \dots, t, j=1, \dots, s_i), a_1, b_1, \dots, a_g, b_g; x_{i,j}^{m_{i,j}}, x_{1,1}^{-1} x_{1,2}^{-1} \dots x_{t,s_t}^{-1} [a_1, b_1] \dots [a_g, b_g] \rangle$$

where

$$x_{i,j} = k^{-1} x_i^{m_i/m_{ij}} k.$$

Note that 2, implies that  $g' \leq g$ .

The following result, which is due to Singerman [13], gives the relationship between Fuchsian groups and permutation groups.

PROPOSITION 3: *Let  $\Gamma$  be the Fuchsian group  $(g; m_1, \dots, m_t)$ . Then  $\Gamma$  has the Fuchsian group  $H = (g'; m_{1,1}, \dots, m_{1,s_1}, \dots, m_{t,1}, \dots, m_{t,s_t})$  as a subgroup of finite index if and only if (5) holds and there exists a transitive permutation group  $G$  of degree  $n$  and a homomorphism  $\pi: \Gamma \rightarrow G$  such that the permutation  $\xi_i = \pi(x_i)$  has exactly  $s_i$  non-trivial cycles of length less than  $m_i$ , these lengths being*

$$\frac{m_i}{m_{i,1}}, \frac{m_i}{m_{i,2}}, \dots, \frac{m_i}{m_{i,s_i}}.$$

*Sketch of the proof* (see [13] for details): If  $\Gamma$  has a subgroup like  $H$ , let act  $\Gamma$  on the set  $\Gamma/H$  of the coset modulo  $H$ . This give us the group we are looking for in the form  $\Gamma/K$  (where  $K$  is the intersection of all subgroup of  $\Gamma$  conjugate to  $H$ ).

Conversely let  $H$  be the inverse image by  $\pi$  of  $G_1$ ; one can then shown that  $H$  has the prescribed presentation.

COROLLARY: *A triangular group  $(0; l, m, n)$  has a subgroup of finite index like  $(g; l_1, \dots, l_p, m_1, \dots, m_q, n_1, \dots, n_r)$  if and only if there exists some hypermap  $\mathcal{H} = (B, \sigma, \alpha)$  of genus  $g$  such that  $\alpha$  (resp.  $\sigma, \sigma\alpha$ ) has  $p$  (resp.  $q, r$ ) cycles of length less than  $l$  (resp. less than  $m, n$ ) these lengths being  $l/l_1, \dots, l/l_p$  (resp.  $m/m_1, \dots, m/m_q; n/n_1, \dots, n/n_r$ ).*

*Proof:* The proof is an application of Proposition 3. A straightforward calculation proves that  $g = g(\mathcal{H})$ .

We shall examine now a special case of Proposition 3 which will be needed in the sequel.

Let  $H \trianglelefteq \Gamma$ . As  $\Gamma$  acts regularly on the quotient group  $\Gamma/H$  we have, for  $i = 1, 2, \dots, t$ ,

$$m_{i,1} = m_{i,2} = \dots = m_{i,s_i} = m'_i$$

and, if  $\pi : \Gamma \rightarrow \Gamma/H$  is the natural projection, the order  $k_i$  of  $\xi_i = \pi(x_i)$  is given by

$$k_i = \frac{m_i}{m'_i}.$$

Moreover, we have

$$s_i = \frac{[\Gamma : H]}{k_i}.$$

With the above notations we obtain

$$\mu(H) = 2g' - 2 - [\Gamma : H] \left( \sum_{i=1}^t \left( 1 - \frac{1}{k_i} \right) - \sum_{i=1}^t \left( 1 - \frac{1}{m_i} \right) \right) \tag{6}$$

and substituting (6) in (5) we get

$$2g' - 2 = [\Gamma : H] \left( 2g - 2 + \sum_{i=1}^t \left( 1 - \frac{1}{k_i} \right) \right) \tag{7}$$

As a final remark, if  $\Gamma$  and  $H$  are as above then

$$\frac{\Gamma}{H} \cong \frac{\Gamma'}{H'}$$

where  $\Gamma'$  is the Fuchsian group  $(g; k_1, \dots, k_t)$  and  $H'$  has no elements of finite order [it then follows from Facts 3 and 5 that  $H$  is the surface group  $(g'; -)$ ].

**4. THE MAIN THEOREM**

Before stating and proving our theorem we need some more result about the relationship between Fuchsian group and hypermaps.

**THEOREM 1:** *Let  $\mathcal{H} = (B, \sigma, \alpha)$  be a hypermap and let  $A \leq \text{Aut}(\mathcal{H})$  be an automorphism group of  $\mathcal{H}$ . If  $g(\mathcal{H}) = g$  and  $g(\mathcal{H}/A) = g'$  then  $A$  can be expressed as a quotient  $\Gamma/H$  where*

$$\Gamma = \langle x_1, \dots, x_t, a_1, b_1, \dots, a_{g'}, b_{g'}; x_i^{k_i} (i=1, \dots, t), x_1^{-1} \dots x_t^{-1} [a_1, b_1] \dots [a_{g'}, b_{g'}] \rangle$$

and  $H$  is a (normal) subgroup of  $\Gamma$  in the form

$$\langle c_1, d_1, \dots, c_{g'}, d_{g'}; [c_1, d_1] \dots [c_{g'}, d_{g'}] \rangle.$$

Moreover the following holds

$$2g - 2 = [\Gamma : \mathbb{H}] \left( 2g' - 2 + \sum_{i=1}^t \left( 1 - \frac{1}{k_i} \right) \right) \tag{8}$$

It should be remarked that formula (8) looks like the formula given in [11]. This is not a coincidence. It can be proved that formula (8) is nothing else than the one in [11] obtained by other means.

Let now  $\mathcal{H} = (B, \sigma, \alpha)$  be toroidal so that in Theorem 1 we have  $g = 1$  and  $g' = 0$  or  $1$ . From (8) we get that  $\mu(\Gamma)$  must be  $0$ . So the first step towards the main theorem is to find all Fuchsian group  $\Gamma$  such that  $\mu(\Gamma) = 0$ .

**PROPOSITION 4:** *Let  $\Gamma$  be a fuchsian group such that  $\mu(\Gamma) = 0$ . Then  $\Gamma$  is one of the following fuchsian groups (see above for notation):  $(1; -)$ ,  $(0; 2, 2, 2, 2)$ ,  $(0; 2, 4, 4)$ ,  $(0; 2, 3, 6)$  or  $(0; 3, 3, 3)$ .*

*Proof:* Straightforward calculation.

**THEOREM 2:** *If  $\mathcal{H} = (B, \sigma, \alpha)$  is a toroidal map and  $A$  an automorphism group of  $\mathcal{H}$  such that  $\mathcal{H}/A$  is still toroidal, then  $A$  is isomorphic to the following group*

$$G(h, q) = C_h \times C_{qh},$$

where  $h$  and  $q$  are integers  $\geq 1$ .

*Proof:* We know that  $A$  is isomorphic to a quotient of  $(1; -)$  so that  $A$  is a finite abelian group of rank at most two. The proposition then follows from the structure theorem for finite abelian groups.

Let  $\Gamma_1 = \langle a, b; [a, b] \rangle$  be a presentation of  $(1; -)$  and let  $h_1, h_2, k_1$  and  $k_2$  be integers. By basic properties of free abelian groups there is a unique endomorphism  $\xi : \Gamma_1 \rightarrow \Gamma_1$  induced by

$$\xi(a) = a^{h_1} b^{k_1}; \quad \xi(b) = a^{h_2} b^{k_2}.$$

Moreover  $\xi$  is an automorphism if and only if

$$\begin{vmatrix} h_1 & k_1 \\ h_2 & k_2 \end{vmatrix} = \pm 1.$$

This means that we can identify  $\text{Aut}(\Gamma_1)$  with  $\text{GL}(2, \mathbb{Z})$  (see [12], p. 169).

The next step is to prove that the groups  $(0; 2, 2, 2, 2)$ ,  $(0; 3, 3, 3)$ ,  $(0; 2, 4, 4)$  and  $(0; 2, 3, 6)$  can be written as a semi-direct product  $C \times_{\varphi} \Gamma_1$  where  $C$  is a cyclic group of order  $2, 3, 4$  and  $6$  respectively. It is well known

that in  $\text{Aut}(\Gamma_1)$  the only elements of finite order have order 2, 3, 4 or 6 so these are the only semi-direct product with cyclic finite group one can form.

Let

$$\begin{aligned} \xi_2 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; & \xi_3 &= \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}; \\ \xi_4 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; & \xi_6 &= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}; \end{aligned}$$

It is immediate to verify that under the identification given above  $\xi_n$  gives an automorphism of  $\Gamma_1$  of order  $n$  ( $n=2, 3, 4, 6$ ).

The following theorem can be found, in a slightly different form, in the classical work of Burnside ([1], p. 410 ff).

**PROPOSITION 5:** *Let  $\Gamma_2, \Gamma_3, \Gamma_4$  and  $\Gamma_6$  the groups  $(0; 2, 2, 2, 2), (0; 3, 3, 3), (0; 2, 4, 4)$  and  $(0; 2, 3, 6)$  respectively. Then*

$$\Gamma_n \cong C_n \times_{\Phi_n} \Gamma_1 \tag{9}$$

where  $C_n = \langle x; x^n \rangle$  and

$$\begin{aligned} \Phi_n: C_n &\rightarrow \text{Aut}(\Gamma_1) \\ x &\rightarrow \xi_n \end{aligned}$$

for  $n=2, 3, 4, 6$ .

*Proof:* Let us start with the case of  $\Gamma_2$ . This group has a presentation of the form

$$\langle x_1, x_2, x_3, x_4; x_i^2, i=1, \dots, 4, x_1 x_2 x_3 x_4 \rangle. \tag{10}$$

As all the relators in (10) have even length we may divide the elements of  $\Gamma_2$  into two subsets: those that can be written as the product of an even number of generators and those which need an odd number of generators. The elements of even length form a subgroup we will denote by  $E$ . As  $[\Gamma_2 : E] = 2$ ,  $E$  is characteristic in  $\Gamma_2$ . It can be proved that  $E$  is generated by the products  $x_1 x_2$  and  $x_2 x_3$  and that

$$\psi(a) = x_1 x_2; \quad \psi(b) = x_2 x_3$$

gives an isomorphism of  $\Gamma_1$  onto  $E$ . To prove this it is enough to remark that

$$[x_1 x_2, x_2 x_3] = (x_1 x_2)^{-1} (x_2 x_3)^{-1} (x_1 x_2) (x_2 x_3) = 1_{\Gamma_2}$$

proving that  $E$  is an abelian Fuchsian group. As  $E$  is of infinite order it must be isomorphic to  $\Gamma_1$ . It can then be proved that  $\Gamma_2$  splits in  $E$  and  $C$ , where  $C$  is the cyclic group of order two generated by  $x_2$ . Moreover as

$$x_2(x_1 x_2)x_2 = x_2 x_1 = (x_1 x_2)^{-1}; \quad x_2(x_2 x_3)x_2 = x_3 x_2 = (x_2 x_3)^{-1}$$

we get (9) for  $n=2$ . (We remark that  $E$  contains all the elements of infinite order of  $\Gamma_2$ . In fact if  $f \notin E$  then  $f = x_2 h$ , some  $h \in E$ , but then

$$f^2 = (x_2 h)(x_2 h) = h^{-1} h = 1_{\Gamma_2}$$

and  $f$  has order 2).

Let us take for  $\Gamma_3, \Gamma_4$  and  $\Gamma_6$  the presentation given by (2) and let  $T_n (n=2, 3, 6)$  be the subgroup of  $\Gamma_n$  whose generators  $a$  and  $b$  are given in Table I below. By a straightforward calculation it can be proved that  $T_n$  is isomorphic to  $\Gamma_1$  under the obvious isomorphism.

TABLE I

$n$	$a$	$b$
3	$xy^{-1}$	$yxy$
4	$xy^2$	$yxy$
6	$y^{-1}z^2$	$zy^{-1}z$

To prove that  $T_n \trianglelefteq \Gamma_n$  one can show that  $T_n$  is formed by all the elements of infinite order of  $\Gamma_n$  and so it is characteristic in  $\Gamma_n$ . To finish the proof of (9) one shows that  $\Gamma_n$  split in  $T_n$  and the cyclic group of order  $n$  generated by  $y$  for  $n=3, 4$  and by  $z$  for  $n=6$ .

Note that, as it follows from the proof of Proposition 5, an element  $\gamma$  of  $\Gamma_n \cong C_n \times_{\phi} \Gamma_1$  is of infinite order if and only if  $\gamma \in \Gamma_1$ . As we are looking for quotients of  $\Gamma_n$  by some normal subgroup  $H$  isomorphic to  $\Gamma_1$ , and so having only elements of infinite order, we are led to look for the subgroups of  $\Gamma_1$  that are

1. of finite index in  $\Gamma_1$ ,
2.  $\xi_n$ -invariant
3. isomorphics to  $\Gamma_1$ .

If  $H \trianglelefteq \Gamma_1$  satisfies 1 to 3 above then we can form the quotient  $\Gamma_n/H$  which turns out to be

$$C_n \times_{\phi} \frac{\Gamma_1}{H}$$

where  $\varphi$  is obtained from  $\Phi_n$  in the obvious way.

LEMMA 1: Let  $T$  be a subgroup of finite index in  $\Gamma_1 = \langle a, b; [a, b] \rangle$ . Let  $n$  be the least positive integer such that  $b^n \in T$  and  $h$  be the least positive integer such that  $a^h b^k \in T$ , some  $k \in \mathbb{Z}$ . Then  $T$  is generated by  $b^n$  and  $a^h b^k$ .

*Proof:* Let  $b^m \in T$ , then  $m = qn$ . Let  $m = qn + r$ , with  $0 \leq r < n$ , we have

$$b^r = b^m (b^n)^{-q} \in T$$

which is impossible by the choice of  $n$  unless  $r = 0$ .

Let now  $a^x b^y \in T$  and let  $x = qh + r$  with  $0 \leq r < h$ . Then

$$a^r b^{y - qk} = (a^x b^y) (a^h b^k)^{-q} \in T$$

which implies  $r = 0$ , so that we get  $b^{y - qk} \in T$  and so  $y - qk = q'n$  which in turn gives us

$$a^x b^y = (a^h b^k)^{-q} (b^n)^{q'}$$

proving that  $a^h b^k$  and  $b^n$  generate  $T$ .

COROLLARY: If  $T$  is a finite index subgroup of  $\Gamma_1$  then  $T$  is isomorphic to  $\Gamma_1$ .

This means that condition 3 above is redundant.

LEMMA 2: Let  $T$  be a subgroup of finite index of  $\Gamma_1$ , then  $T$  is  $\xi_2$ -invariant.

*Proof:* This is straightforward as  $\xi_2$  sends any element of  $\Gamma_1$  into its inverse.

LEMMA 3: Let  $T$  a subgroup of finite index of  $\Gamma_1$  and let  $T$  be generated by  $a^h b^k$  and  $b^n$  as in Lemma 1. Let  $T$  be  $\xi$ -invariant where  $\xi$  is one of  $\xi_3, \xi_4$  or  $\xi_6$ . Then there exist  $q, q' \in \mathbb{Z}$  such that  $k = qh$  and  $n = q'h$ .

*Proof:* Let us prove the lemma for  $\xi = \xi_3$ , the proof for the other cases being similar.

Suppose that  $k = qh + r$ , where  $0 \leq r < h$ . As  $T$  is  $\xi_3$ -invariant we have  $\xi_3(a^h b^k) \in T$  which means that

$$a^r b^{-h - (q-1)k} = (\xi_3(a^h b^k)) (a^h b^k)^{-(q-1)} \in T$$

This can only happen if  $r = 0$ , that is  $k = qh$ .

Let now  $n = q'h + r$ ,  $0 \leq r < h$ . We have

$$a^r b^{-q'k} = (\xi_3(b^n)) (a^h b^k)^{-q'} \in T$$

which implies  $r = 0$  and so  $n = q'h$ .

By Lemma 1 we have that is  $T$  is of finite index in  $\Gamma_1 = \langle a, b; [a, b] \rangle$  then  $G = \Gamma_1/T$  has the following presentation

$$G = \langle a, b; a^h b^k, b^n, [a, b] \rangle.$$

If now  $T$  is  $\xi$ -invariant for  $\xi$  one of  $\xi_3, \xi_4$  or  $\xi_6$  we have that

$$G = \langle b, c; c^h, b^{q^h}, [b, c] \rangle = C_h \times C_{q^h}.$$

where  $c = ab^q$ . Now  $T$  is  $\xi$ -invariant ( $\xi = \xi_3, \xi_4$  or  $\xi_6$ ) if and only if  $\xi$  induces the identity on  $G$ . Table 2 gives the effect of  $\xi_3, \xi_4$  and  $\xi_6$  on the generators  $c$  and  $b$  of  $G$ .

TABLE II

$\xi$	$\xi(c)$	$\xi(b)$
$\xi_3$	$c^{q-1} b^{-(q-1)^2 - q}$	$cb^{-q}$
$\xi_4$	$c^q b^{-(q^2+1)}$	$cb^{-q}$
$\xi_6$	$b$	$cb^{-q^1}$

Then it is simply a matter of calculation to prove that:

$T$  is  $\xi_3$ -invariant if and only if  $q' \mid (q^2 + 1)$ ,

$T$  is  $\xi_4$ -invariant if and only if  $q' \mid (q^2 - q + 1)$ ,

$T$  is  $\xi_6$ -invariant if and only if  $q' \mid (q^2 + q + 1)$ ,

Putting together the results obtained so far we have

**THEOREM 3:** *Let  $T$  be a subgroup of  $\Gamma_n (n=2, 3, 4$  or  $6)$  having finite index in  $\Gamma_n$  and being isomorphic to  $\Gamma_1$ . Let  $G$  be the quotient group  $\Gamma_n/H$ . Then*

1. If  $n=2$  then  $G$  is isomorphic to

$$G(2, h, q) = C_2 \times_{\varphi_2} (C_h \times C_{q^h}),$$

where  $h, q \geq 1$  and  $\varphi_2$  sends the generator of  $C_2$  into the isomorphism  $\xi_2$  of  $C_h \times C_{q^h}$  defined by

$$\xi_2(x) = x^{-1}$$

for all  $x \in C_h \times C_{q^h}$ .

2. If  $n=3$  then  $G$  is isomorphic to

$$G(3, h, q, q') = C_3 \times_{\varphi_{3,q}} (C_h \times C_{q^h}),$$

where  $q' \mid (q^2 + 1)$  and  $\varphi_{3,q}$  sends the generator of  $C_3$  into the automorphism of  $C_h \times C_{q^h}$  defined by first row of Table I.

3. If  $n=4$   $G$  is isomorphic to

$$G(4, h, q, q') = C_4 \times_{\varphi_{4,q}} (C_h \times C_{q'h}),$$

where  $q' | (q^2 - q + 1)$  and  $\varphi_{4,q}$  sends the generators of  $C_4$  into the automorphism of  $C_h \times C_{q'h}$  defined by second row of Table I.

4. Finally, if  $n=6$   $G$  is isomorphic to

$$G(6, h, q, q') = C_6 \times_{\varphi_{6,q}} (C_h \times C_{q'h}),$$

where  $q' | (q^2 + q + 1)$  and  $\varphi_{6,q}$  sends the generators of  $C_6$  into the automorphism of  $C_h \times C_{q'h}$  defined by third row of Table I.

**THEOREM 4:** Let  $\mathcal{H}$  be a toroidal hypermap and  $A \leq \text{Aut}(\mathcal{H})$ . Then  $A$  is (isomorphic to) one of  $G(h, q)$ ,  $G(2, h, q)$ ,  $G(3, h, q, q')$ ,  $G(4, h, q, q')$  or  $G(6, h, q, q')$ .

*Proof:* This follows immediately from Theorems 1, 2 and 3 and Proposition 2 above.

Notice that Theorem 4 only gives a necessary condition for a finite group to be the full automorphism group of a toroidal hypermap. We still have to prove that these groups actually arise as the full automorphism group of some toroidal hypermap. This will be done in the following section.

## 5. THE HYPERMAPS

In the present section we will show that for each of the groups of Theorem 4 it can be constructed a hypermap having it as its full automorphism group. This will complete the proof of the following theorem:

**THEOREM 5:** Let  $A$  be a finite group. There exists a toroidal hypermap  $\mathcal{H}$  such that  $A$  is (isomorphic to) a subgroup of  $\text{Aut}(\mathcal{H})$  if and only if  $A$  is (isomorphic to) one of  $G(h, q)$ ,  $G(2, h, q)$ ,  $G(3, h, q, q')$ ,  $G(4, h, q, q')$  or  $G(6, h, q, q')$ . Moreover if  $A$  is one of the groups listed above then there exists a toroidal hypermap whose full automorphism group is (isomorphic to)  $A$ .

*Proof:* First of all note that  $\Gamma_3$ ,  $\Gamma_4$  and  $\Gamma_6$  all are triangular group and that  $G(n, h, q, q')$  is obtained as a quotient of  $\Gamma_n$  by a normal subgroup  $H$  of finite index isomorphic to  $\Gamma_1$ . It follows from Proposition 2 and from the corollary to Proposition 3 that setting  $B = \Gamma_n/H$  and letting  $\alpha$  (resp.  $\sigma$ ) be the permutation of  $B$  induced by the generator  $x$  (resp.  $y$ ) of  $\Gamma_n$  gives us a

hypermap  $\mathcal{H}(n, h, q, q')$  such that

$$\text{Aut}(\mathcal{H}(n, h, q, q')) = \frac{\Gamma_n}{H} = G(n, h, q, q').$$

setting the problem for the last three groups.

The proof of the theorem for the remaining groups is a straightforward consequence of Lemmas 5 and 6 below.

LEMMA 4: *Let  $\mathcal{H} = (B, \sigma, \alpha)$  be a hypermap and  $A$  an automorphism group of  $\mathcal{H}$ . The cardinality of  $B$  is a multiple of that of  $A$ .*

*Proof:* This is obvious since  $A$  acts semi-regularly on  $B$  with orbit length  $|A|$ .

PROPOSITION 6: *With the same notations as in Lemma 4 if the ratio  $|B|/|A|$  is a prime number then either  $A = \text{Aut}(\mathcal{H})$  or  $\text{Aut}(\mathcal{H})$  is transitive on  $B$ .*

For any  $h, q \geq 1$  define  $\mathcal{H}(h, q)$  as follows: the set of brins is

$$B = \mathbb{Z}_h \times \mathbb{Z}_{qh} \times \mathbb{Z}_5;$$

the vertices are given by

$$\sigma((i, j, k)) = (i, j, k + 1)$$

and the edges by

$$\alpha((i, j, k)) = \begin{cases} (i, j, 1) & \text{if } k=0, \\ (i, j+1, 3) & \text{if } k=1, \\ (i+1, j, 4) & \text{if } k=2, \\ (i, j-1, 0) & \text{if } k=3, \\ (i, -1, j, 2) & \text{if } k=4. \end{cases}$$

LEMMA 5: *For any  $h, q \geq 1$  we have*

$$\text{Aut}(\mathcal{H}(h, q)) \cong G(h, q).$$

*Proof:* Let  $\tau$  and  $\rho$  be defined as follows for any  $(i, j, k) \in B$ :

$$\rho((i, j, k)) = (i+1, j, k), \quad \tau((i, j, k)) = (i, j+1, k). \tag{11}$$

It's easy to see that  $\tau, \rho \in \text{Aut}(\mathcal{H}(h, q))$  and that  $[\tau, \rho] = \text{id}_B$ . Moreover for any  $x, y$  ( $0 \leq x < h, 0 \leq y < qh$ ) we have

$$\rho^x \tau^y((i, j, k)) = (i+x, j+y, k) \neq (i, j, k),$$

proving that  $\rho^x \tau^y \neq \text{id}_B$ . Also the order of  $\rho$  is  $h$  and that of  $\tau$  is  $qh$  so that  $a \rightarrow \rho, b \rightarrow \tau$  gives an isomorphism of  $G(h, q) = \langle a, b; a^h, b^{qh}, [a, b] \rangle$  into  $\text{Aut}(\mathcal{H}(h, q))$ . To prove that this isomorphism is onto let  $A$  be the subgroup

of  $\text{Aut}(\mathcal{H})$  generated by  $\rho$  and  $\tau$ . We have  $|B|/|A|=5$  so that, by Proposition 6 either  $A$  is all of  $\text{Aut}(\mathcal{H}(h, q))$  or  $\text{Aut}(\mathcal{H}(h, q))$  acts transitively on  $B$ . Now  $\mathcal{H}(h, q)$  has  $h^2q$  edges of length 3 and the same number of edges of length 2 and no automorphism can send a brin belonging to an edge of length 2 into a brin belonging to an edge of length 3. So  $\text{Aut}(\mathcal{H}(h, q))$  can't be transitive on  $B$  and the lemma is proved.

Let  $h, q$  still denote two positive integers. We can define a hypermap  $\mathcal{H}(2, h, q)$ , whose automorphism group will turn out to be  $G(2, h, q)$ , as follows: the brin set is

$$B = \mathbb{Z}_h \times \mathbb{Z}_{qh} \times \mathbb{Z}_6;$$

the vertices are given by

$$\sigma((i, j, k)) = (i, j, k + 1)$$

and the edges by

$$\alpha((i, j, k)) = \begin{cases} (i, j, 1) & \text{if } k=0, \\ (i, j+1, 3) & \text{if } k=1, \\ (i+1, j, 5) & \text{if } k=2, \\ (i, j, 4) & \text{if } k=3, \\ (i, j-1, 0) & \text{if } k=4, \\ (i, j, 2) & \text{if } k=5. \end{cases}$$

LEMMA 6: For any  $h, q \geq 1$  we have

$$\text{Aut}(\mathcal{H}(2, h, q)) \cong G(2, h, q).$$

*Proof:* As in the proof of Lemma 5 the permutations of  $B$  defined by (11) form a subgroup of  $\text{Aut}(\mathcal{H}(2, h, q))$  isomorphic to  $G(h, q)$ . Let now  $\xi$  be defined by

$$\xi((i, j, k)) = (-i, -j, k + 3).$$

Then  $\xi$  is an automorphism of  $\mathcal{H}(2, h, q)$  of order two. Moreover if  $x, y$  are integers such that  $0 \leq x < h$  and  $0 \leq y < qh$  we have

$$\xi \rho^x \tau^y((i, j, k)) = (-(i+x), -(j+y), k+3) \neq (i, j, k).$$

Finally  $\xi \rho \xi = \rho^{-1}$  and  $\xi \tau \xi = \tau^{-1}$  proving that  $a \rightarrow \rho, b \rightarrow \tau$  and  $x \rightarrow \xi$  define an isomorphism of  $G(2, h, q) = \langle x, a, b; x^2, a^h, b^{qh}, [a, b], (xa)^2, (xb)^2 \rangle$  into  $\text{Aut}(\mathcal{H}(2, h, q))$ . To prove that this isomorphism is onto let  $A$  be the subgroup of  $\text{Aut}(\mathcal{H}(2, h, q))$  generated by  $\xi, \rho$  and  $\tau$ , then  $|B|/|A|=3$  so that we are left with two possibility: either  $A$  is all of  $\text{Aut}(\mathcal{H}(2, h, q))$  or

$\text{Aut}(\mathcal{H}(2, h, q))$  is transitive on  $B$ . Now if  $h \neq 2$  there cannot exist an automorphism carrying  $(i, j, 0)$  into  $(i, j, 2)$  as the edge containing the former has length 4 and the one containing the latter has length  $2h \neq 4$ . On the other side if  $h = 2$  then it is impossible to find an automorphism carrying  $(i, j, 0)$  into  $(i, j, 1)$  as the face containing the former has length 4 and the one containing the latter has length 2.

## 6. CONCLUSIONS

We have proved that the finite groups that can arise as the automorphism group of some toroidal hypermap are of a really simple structure. Moreover there exist infinitely many such groups, like in the planar case and unlike the case of genus greater than 1 (see Fact 1).

Some work is left. For instance can it be proved that the five families of groups described above are disjoint (except for triviality like  $G(2, 1, 1)$  being isomorphic to  $G(1, 2)$ )?

Another interesting question is to find a proof of our theorem by elementary means, that is by arguments like the ones used in [5] to settle the planar case.

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