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THE LOCAL CATENATIVITY OF DOL-SEQUENCES IN FREE COMMUTATIVE MONOIDS IS DECIDABLE IN THE BINARY CASE (*)

by J. L. LAMBERT ⁽¹⁾

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Abstract. – Given a matrix $A \in \mathbb{Z}^{2 \times 2}$ and a vector $V_0 \in \mathbb{Z}^2$ we determine if there exists an integer m and m positive integers $a_{m-1} \dots a_0$ such that $A^m V_0 = \sum_{i=0}^{m-1} a_i A^i V_0$. When such an m exists, we compute the smallest one and m positive integers $a_{m-1} \dots a_0$ that satisfy the relation.

Keywords : DOL-Sequences; Commutative monoids.

Résumé. – Étant donné une matrice $A \in \mathbb{Z}^{2 \times 2}$ et un vecteur $V_0 \in \mathbb{Z}^2$ on détermine l'existence d'un entier m et de m autres entiers positifs $a_{m-1} \dots a_0$ tels que $A^m V_0 = \sum_{i=0}^{m-1} a_i A^i V_0$. Quand un tel m existe, on calcule le plus petit ainsi que les entiers $a_{m-1} \dots a_0$ qui satisfont la relation.

INTRODUCTION

The DOL sequences were introduced by Lindenmayer [3]. They are defined in a free monoid Σ^* (Σ being a finite alphabet) by a morphism $h: \Sigma^* \rightarrow \Sigma^*$ and an axiom $w \in \Sigma^*$; the DOL sequences is then the sequence $w, h(w), h^2(w), \dots, h^n(w), \dots$. Such a sequence in Σ^* is said locally catenative if there exists an integer m and some positive integers $i_0 \dots i_r$, smaller than m such that:

$$h^m(w) = h^{m-i_0}(w) \dots h^{m-i_r}(w)$$

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C. Choffrut proved in [1] that when $\text{card}(\Sigma)=2$, then the local catenativity of DOL sequences is decidable since it is equivalent to $h^3(w) \in \{w, h(w), h^2(w)\}^*$. The problem of deciding if a given DOL sequence is catenative or not is still open.

In a free commutative monoid, the definition is easily extended. The morphism is given by a matrix $A \in \mathbb{N}^{n \times n}$ and the axiom is a vector $x \in \mathbb{N}^n$. The problem is then to determine if there exist an $m \in \mathbb{N}$ and m positive integers a_{m-1}, \dots, a_0 such that

$$A^m x = \sum_{i=0}^{m-1} a_i A^i x$$

The problem is then a decidability result concerning matrices with entries in \mathbb{N} ([4]).

In this article we will prove the decidability of this property when $n=2$ (the binary case). We will actually solve the more general problem:

Problem 1:

Instance: $V_0 \in \mathbb{Z}^2, A \in \mathbb{Z}^{2 \times 2}$

Question: Do there exist an integer $m \in \mathbb{N}$ and m positive integers a_{m-1}, \dots, a_0 such that

$$A^m V_0 = \sum_{i=0}^{m-1} a_i A^i V_0$$

We will prove that this problem is decidable and will compute the smallest m for which some integers a_{m-1}, \dots, a_0 satisfying the property exist.

1. TURNING THE PROBLEM INTO A PROBLEM CONCERNING POLYNOMIALS

In this section, we will express our initial problem 1 into a more suitable form and study it in any dimension. It is first clear that we can rewrite and

generalize the problem under the following form:

Problem 2:

Instance: $V_0 \in \mathbb{Z}^n$, $A \in \mathbb{Z}^{n \times n}$

Question: Does there exist a polynomial $P \in \mathbb{Z}[X]$ of degree m such that $P = X^m - \sum_{i=0}^{m-1} a_i X^i$ where $a_i \in \mathbb{N}$ for $0 \leq i \leq m-1$ and $P(A) V_0 = 0$?

We just recall that for a polynomial $P(X) = \sum_{i=0}^m a_i X^i$ and a matrix A , $P(A)$ is the matrix given by:

$$P(A) = \sum_{i=0}^m a_i A^i$$

A polynomial P of degree m such that $a_m = 1$ is said **monic**. We will denote by $\mathbb{Z}_1[X]$ the set of monic polynomials with coefficients in \mathbb{Z} .

We prove here that the monic polynomials of $\mathbb{Z}[X]$ that satisfy $P(A) V_0 = 0$ are the multiples of a computable monic polynomial P_0 of degree at most n . This property is a consequence of the classical Gauss' lemma on integer polynomials and Hamilton-Cayley theorem ([2]).

LEMMA 1 (Gauss' Lemma): *Let P and Q be two polynomials in $\mathbb{Z}[X]$ and denote by $C(P)$ the GCD of the coefficients of P then*

$$C(P \cdot Q) = C(P) \cdot C(Q)$$

LEMMA 2 (Hamilton-Cayley Theorem): *Let $A \in \mathbb{Z}^{n \times n}$ and let $K_A(X)$ be the characteristic polynomial of A :*

$$K_A(X) = \text{Det}(A - XI)$$

then $K_A(A) = 0$.

We will use Gauss' lemma under the following more convenient form:

LEMMA 3: *Let $P \in \mathbb{Z}[X]$ a monic polynomial. Then if $P = Q \cdot R$ where Q and R are monic polynomials of $\mathbb{Q}[X]$ then $Q \in \mathbb{Z}[X]$ and $R \in \mathbb{Z}[X]$.*

Proof. — Let $Q' = \lambda Q$ and $R' = \mu R$ where λ and μ are the least positive integers such that Q' and R' have integer coefficients. Then since Q is monic, $C(Q')$ divides λ (λ is the highest degree coefficient of Q') and then $C(Q') = 1$

since λ is minimal. Similarly, $C(R') = 1$. Now

$$C(\lambda\mu P) = \lambda\mu = C(Q')C(R') = 1$$

Thus $\lambda = \mu = 1$ and $Q = Q'$, $R = R'$. \square

We are in position to prove the first proposition:

PROPOSITION 1: *Let $V_0 \in \mathbb{Z}^n$, $A \in \mathbb{Z}^{n \times n}$. Define the set*

$$I = \{ P \in \mathbb{Z}_1[X] / P(A)V_0 = 0 \}$$

Then we can compute a monic polynomial of degree at most n : $P_0 \in \mathbb{Z}[X]$. Such that:

$$I = P_0 \cdot \mathbb{Z}_1[X]$$

Proof: — Let $I' = \{ P \in \mathbb{Q}[X] / P(A)V_0 = 0 \}$. Then I' is an ideal over $\mathbb{Q}[X]$ which is a principal ring then there exists a monic polynomial $P_0 \in \mathbb{Q}[X]$ such that $I' = P_0 \mathbb{Q}[X]$.

By lemma 2, the monic polynomial $(-1)^n K_A(X)$ is in I' thus

$$(-1)^n K_A(X) = P_0 Q$$

which implies by lemma 3 that $P_0 \in \mathbb{Z}_1[X]$. Since P_0 is a divisor of $K_A(X)$, it is clear its degree is at most n and that there exist only a finite set of values for P_0 which can easily be computed.

Finally, Let $P \in I$, since $I \subset I'$:

$$P = P_0 Q$$

but since P is monic, $Q \in \mathbb{Z}_1[X]$ and $I = P_0 \mathbb{Z}_1[X]$. \square

With the help of proposition 1, one can see that problem 2 is decidable if the following one is:

Problem 3:

Instance: A polynomial $P_0 \in \mathbb{Z}[X]$ of degree at most n

Question: Does there exist a polynomial $Q \in \mathbb{Z}_1[X]$ such that

$$P_0 \cdot Q = X^m - \sum_{i=0}^{m-1} a_i X^i \quad \text{where } a_i \in \mathbb{N}$$

We now solve problem 3 in the case $n = 2$.

2. SOLVING PROBLEM 3 FOR $n=2$, THE EASY CASES

Except in one case which is the most interesting one, problem 3 when $n=2$ is easy to solve. In this case the polynomial P_0 of proposition 3 has degree 1 or:

$$P_0(X) = X^2 - \text{Tr}(A)X + \text{Det}(A)$$

In this latter case the discussion will concern the signs of $\text{Tr}(A)$ and $\text{Det}(A)$.

In the remainder of the paper a and b are positive integers.

1st case: P_0 has degree 1:

This means that V_0 is an eigenvalue of A .

If $P_0 = X - a$, let $Q = 1$ else if $P_0 = X + a$, then $Q = X - a$ works.

2nd case: $P_0 = X^2 - aX - b$:

$Q = 1$ is clearly suitable

3rd case: $P_0 = X^2 + aX + b, a \neq 0$:

Then for any large enough $\lambda \in \mathbb{N}$:

$$(X - \lambda)P_0 = X^3 + (a - \lambda)X^2 + (b - a\lambda)X - b\lambda$$

has the convenient form. Thus $Q = X - \lambda$ is suitable.

4th case: $P_0 = X^2 + aX - b, a \neq 0, b \neq 0$:

PROPOSITION 2: Let $P_0 = X^2 + aX - b$ where $(a, b) \in (\mathbb{N} - \{0\})^2$. Then there exists no polynomial $Q \in \mathbb{Z}_1[X]$ such that

$$P_0 \cdot Q = X^n - \sum_{i=0}^{m-1} \lambda_i X^i \quad \text{where } \lambda_i \in \mathbb{N}$$

Proof. - Let $Q = X^n + a_{n-1}X^{n-1} + \dots + a_0$. Then

$$\begin{aligned} P_0 \cdot Q &= (X^2 + aX - b)(X^n + a_{n-1}X^{n-1} + \dots + a_0) \\ &= X^{n+2} + (a + a_{n-1})X^{n+1} + (a_{n-2} + aa_{n-1} - b)X^n + \sum_{i=0}^{n-3} (a_i + aa_{i+1} - ba_{i+2})X^{i+2} \\ &\quad + (aa_0 - ba_1)X - ba_0 \end{aligned}$$

Now we want

$$\begin{aligned} -ba_0 &\leq 0 \\ aa_0 - ba_1 &\leq 0 \\ a_i + aa_{i+1} - ba_{i+2} &\leq 0 \quad \text{for } i=0, \dots, n-3 \end{aligned}$$

This implies directly that $a_0 \geq 0$ (since $b > 0$) and then $a_1 \geq 0$. By induction one has:

$$a_i \geq 0 \text{ and } a_{i+1} \geq 0 \Rightarrow a_{i+2} \geq 0$$

and thus $a_{n-1} \geq 0$ which gives $a + a_{n-1} \geq a > 0$. A contradiction. \square

We now deal with the interesting case $P_0 = X^2 - aX + b$.

3. PROBLEM 3: THE CASE $P_0 = X^2 - aX + b$, $b \neq 0$

3.1. The main result

PROPOSITION 3: *Let $P_0 = X^2 - aX + b$, $a \in \mathbb{N}$, $b \in \mathbb{N} - \{0\}$. Then there exists $Q \in \mathbb{Z}_1[X]$ such that*

$$P_0 \cdot Q = X^n - \sum_{i=0}^{n-1} \lambda_i X^i$$

with $\lambda_i \in \mathbb{N}$ iff P_0 has no root in \mathbb{R} .

Proof. — We first prove that the problem is equivalent to the existence of a non fully negative solution to a certain regular system of inequations. To check this existence, we use an easy criteria concerning the signs of the coefficients of a matrix. We compute that matrix and show that those coefficients are given by a linear recursion formula which the discussion is based on.

Let us look at a polynomial $Q = X^n + a_{n-1}X^{n-1} + \dots + a_0$ then

$$\begin{aligned} P_0 \cdot Q &= (X^2 - aX + b)(X^n + a_{n-1}X^{n-1} + \dots + a_0) \\ &= X^{n+2} + (a_{n-1} - a)X^{n+1} + (a_{n-2} - aa_{n-1} + b) + \sum_{i=0}^{n-3} (a_j - aa_{i+1} + ba_{i+2})X^{i+2} \\ &\quad + (ba_1 - aa_0)X + ba_0 \end{aligned}$$

and

$$a_{n_0-1} - a \leq x_{n_0-1} \leq 0$$

all inequations of (I) are satisfied. This states the claim.

We will thus solve the latter problem. Let us define

$$A_n = \begin{pmatrix} b & & & & \\ -a & b & & & \\ 1-a & b & & & \\ & & \dots & & \\ & & & 1-a & b \end{pmatrix} \in \mathbb{Z}^{(n+1) \times (n+1)}$$

and for two vectors x, y in \mathbb{Z}^{n+1} :

$$x \leq y \Leftrightarrow \forall i, \quad 0 \leq i \leq n, \quad x_i \leq y_i$$

We have to determine if

$$\forall n \in \mathbb{N}, \quad \forall x \in \mathbb{Z}^{n+1}, \quad A_n x \leq 0 \Rightarrow x \leq 0$$

But this is clearly equivalent to

$$\forall n \in \mathbb{N}, A_n^{-1} \text{ is positive}$$

Now we easily compute that

$$A_n^{-1} = \begin{pmatrix} \frac{\alpha_0}{b} & & & & \\ \frac{\alpha_1}{b^2} & \frac{\alpha_0}{b} & & & \\ & & \dots & & \\ \frac{\alpha_n}{b^{n+1}} & \dots & \frac{\alpha_1}{b^2} & \frac{\alpha_0}{b} \end{pmatrix}$$

where $\alpha_0, \dots, \alpha_n$ is an integer sequence given by $\alpha_0 = 1, \alpha_1 = a$ and the recursion formula:

$$\alpha_n = a \alpha_{n-1} - b \alpha_{n-2}$$

The characteristic equation of the recursion is $P_0(X) = 0$.

We now have three cases.

1st case: P_0 has two distinct real roots. Let $\lambda_1 > \lambda_2 > 0$ be these roots. An elementary computation leads to the formula:

$$\alpha_n = \frac{(\lambda_1)^{n+1} - (\lambda_2)^{n+1}}{\sqrt{\Delta}} \quad (\Delta = a^2 - 4b)$$

and $\alpha_n > 0$ for every n , the problem has no solution.

2nd case: P_0 has an unique double root $\lambda = a/2$. The new formula is

$$\alpha_n = (n+1)(a/2)^n$$

then $\alpha_n > 0$ for every $n \in \mathbb{N}$, there is no solution either.

3rd case: P_0 has no root in \mathbb{R} then P_0 has two roots in \mathbb{C} : λ and $\bar{\lambda}$, we get the new formula:

$$\alpha_n = \frac{(\lambda)^{n+1} - (\bar{\lambda})^{n+1}}{i\sqrt{-\Delta}}$$

Let $\lambda = \rho e^{i\theta}$ then $\rho = \sqrt{b}$ and $\theta = \text{Arctan}(\sqrt{(4b/a^2) - 1})$ (if $a = 0$ then $\theta = \pi/2$) and

$$\alpha_n = \frac{2\sqrt{b}^{n+1}}{\sqrt{-\Delta}} \sin(n+1)\theta$$

and $\alpha_n < 0$ as soon as $(n+1)\theta > \pi$ then for $n = [\pi/\text{Arctan}(\sqrt{(4b/a^2) - 1})]$ (where $[x]$ is the integer part of x) the problem has a solution of degree n . \square

3.2. Some examples

Let A be a 2×2 integer matrix such that $\text{Det}(A) > 0$ and $\text{Tr}(A) > 0$. When the matrix is positive, there is no solution since the characteristic equation has real roots. If we do not restrict the matrix A to be positive it is notable that even for matrices with coefficients of small size then the polynomial Q of proposition 3 can have a relatively high degree and surprisingly large size coefficients.

Before introducing some explicit examples, we note that it is easy to compute a polynomial Q satisfying the conclusion of proposition 3. Let X

and $a_{n-1} - a = -\alpha_{n-1} b - a$. Thus we get:

$$P_0 \cdot Q = X^{n+2} - (a + b \alpha_{n-1}) X^{n+1} + (1 + \alpha_n) b X^n - b^{n+1}$$

Note that the computation of that polynomial is reduced to the computation of α_n and α_{n-1} by a simple linear recursion formula!

Now we can give some examples. We note that for matrices of the form:

$$A = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}$$

The characteristic polynomial is $X^2 - 2aX + a^2 + 1$ which discriminant is $\Delta = -4$. Thus the degree of the polynomial Q of lowest degree is:

$$n = \left\lceil \frac{\pi}{\text{Arctan}(\sqrt{1/a^2})} \right\rceil \geq [\pi a]$$

For example, if $a = 1$, the degree of Q is 4 since:

$$\alpha_0 = 1, \alpha_1 = 2, \alpha_2 = 2, \alpha_3 = 0, \alpha_4 = -4$$

The polynomial $Q = X^4 - 8X^2 - 16X - 16$ and

$$(X^4 - 8X^2 - 16X - 16)(X^2 - 2X + 2) = X^6 - 2X^5 - 6X^4 - 32$$

The size of the polynomial grows rapidly with a . For $a = 3$ for example, the formula gives $n \geq 9$. In fact $n = 9$ is the lowest degree as one can see by computing the sequence:

$$\begin{aligned} \alpha_0 &= 1 \\ \alpha_1 &= 6 \\ \alpha_2 &= 26 \\ \alpha_3 &= 96 \\ \alpha_4 &= 316 \\ \alpha_5 &= 936 \\ \alpha_6 &= 2456 \\ \alpha_7 &= 5376 \\ \alpha_8 &= 7696 \\ \alpha_9 &= -7584 \end{aligned}$$

Q is then a polynomial of surprisingly large size while $P_0 = X^2 - 6X + 10$:

$$Q = X^9 - 76\,960 X^8 - 537\,600 X^7 - 2\,456\,000 X^6 - 9\,360\,000 X^5 - 31\,600\,000 X^4 \\ - 96\,000\,000 X^3 - 260\,000\,000 X^2 - 600\,000\,000 X - 1\,000\,000\,000$$

and we get by the proved formula:

$$(X^2 - 6X + 10) \cdot Q = X^{11} - 76\,966 X^{10} - 75\,830 X^9 - 10\,000\,000\,000$$

The size of the obtained polynomials is the most surprising fact. If $a = 5$ then the degree of the polynomial is at least 15 with very big coefficients, for the following polynomial

$$X^2 - 200X + 10\,001 \quad (a = 100)$$

We must search for a polynomial of degree at least 314!

4. CONCLUSIONS

We now easily express the results of section 3 in terms of problem 1.

THEOREM 1: *Problem 1 has a solution except if V_0 is not an eigenvector of A and*

i) $\text{Tr}(A) < 0$, $\text{Det}(A) > 0$

or

ii) $\text{Tr}(A) > 0$, $\text{Det}(A) > 0$ and A has eigenvalues in \mathbb{R} .

Moreover the smallest value m satisfying the property can be determined in each case. If V_0 is an eigenvector of A then it is 1 if the eigenvalue is positive, 2 otherwise. If V_0 is not an eigenvector then the results are summed up in the following tableau:

	$\text{Det}(A) > 0$	$\text{Det}(A) < 0$	$\text{Det}(A) = 0$
$\text{Tr}(A) > 0$	n or imp.	2	2
$\text{Tr}(A) < 0$	3	imp.	3
$\text{Tr}(A) = 0$	4	2	2

$$\text{where } n = \left\lceil \frac{\pi}{\text{Arctan}(\sqrt{(4 \det(A)/\text{Tr}(A)^2 - 1})}) \right\rceil + 2$$

In the case where A and V_0 are positive, the result is greatly simplified:

THEOREM 2: *In a commutative free monoid, a Dol sequence given by a matrix A and an axiom V_0 is locally catenative iff V_0 is an eigenvector of A or $\text{Det}(A) \leq 0$. In both cases one has:*

$$A^2 V_0 \in \{A V_0, V_0\}^*$$

CONCLUSION

The technique involved to solve the problem in the binary case can be in part generalized to solve the other cases. But the analysis of the sequence α_n will not be possible in the same way (but perhaps the decidability will remain true) and principally the particularity of sign that permits to deduce from a suitable non monic polynomial another monic polynomial with the same quality will not remain valid. The generalization is then not so clear.

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