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REPETITIONS IN THE FIBONACCI INFINITE WORD (*)

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Abstract. – Let φ be the golden number; we prove that the Fibonacci infinite word contains no fractional power with exponent greater than $2 + \varphi$ and we prove that for any real number $\varepsilon > 0$ the Fibonacci infinite word contains a fractional power with exponent greater than $2 + \varphi - \varepsilon$.

Résumé. – Soit φ le nombre d'or; nous prouvons que le mot infini de Fibonacci ne contient pas la puissance fractionnaire d'exposant supérieur à $2 + \varphi$, et nous prouvons qu'il contient des puissances d'exposant supérieur à $2 + \varphi - \varepsilon$, quel que soit le nombre réel $\varepsilon > 0$.

INTRODUCTION

Many papers are concerned with the existence of integer powers in “long enough” words or in infinite words; a classical combinatorial property is whether a given infinite word is k power-free or not, with k natural number.

No word on a two letters alphabet can avoid a square but it is well known that the Thue infinite word t on a two letter alphabet does not contain cubes and that the Thue infinite word m on a three letter alphabet does not contain squares (see [9], [10]).

The notion of overlap-free word and more generally the notion of fractional power are considered in many papers (see for instance [4], [7], [9], [10]).

In this paper we prove that the Fibonacci infinite word contains no fractional power with exponent greater than $2 + ((\sqrt{5} + 1)/2)$ and that for any real number $\varepsilon > 0$ the Fibonacci infinite word contains a fractional power with exponent greater than $2 + ((\sqrt{5} + 1)/2) - \varepsilon$.

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To our knowledge this is the first time that this property for a non rational value is looked for in a given infinite word.

DEFINITIONS AND PRELIMINARY RESULTS

We refer to [6] for the terminology.

Let A be an alphabet. We denote by A^* the *free monoid* on A . The elements of A^* are called *words* and the elements of A are called *letters*. We denote by 1 the empty word which is the identity of A^* ; we also denote by $|v|$ the length of a word v .

A word v is a *factor* of a word w if there exist $u, u' \in A^*$ such that

$$w = uvu'$$

and we say that v is a *left factor* of w if u is the empty word.

If a word w is of the form

$$w = v \dots v = v^k$$

with $u \neq 1$, we say that w is a *k-power* of v ; k is called the *exponent* of the power and v is the *base* of the power.

If a word w is of the form

$$w = v \dots vu = v^k u$$

with $u \neq 1$, $k \geq 1$ and u left factor of v , we say that w is a *fractional power* of v of exponent $e = |w|/|v|$ and v is the base of the power.

An infinite word s on an alphabet A is a map from the set of positive integers into A ; we denote by A^ω the set of all infinite words on the alphabet A .

A word $v \in A^*$ is a factor of the infinite word s if there exist $u \in A^*$, $s' \in A^\omega$ such that $s = uvs'$. If u is the empty word then v is a left factor of s .

The Fibonacci infinite word \mathbf{f} on the alphabet $A = \{a, b\}$ is obtained by iterating the morphism $\psi : \{a, b\} \rightarrow \{a, b\}$ given by

$$\psi(a) = ab, \quad \psi(b) = a$$

starting with the letter a (see [1]). Therefore

$$\mathbf{f} = abaababaabaabab\dots$$

We define the sequence of the finite Fibonacci words by the rule:

$$\begin{aligned} \mathbf{f}_0 &= b, \\ \mathbf{f}_{n+1} &= \psi(\mathbf{f}_n). \end{aligned}$$

It is easy to see that $\mathbf{f}_{n+2} = \mathbf{f}_{n+1} \mathbf{f}_n$ and, consequently, the sequence $|\mathbf{f}_n|$, $n \in \mathbb{N}$ is the sequence of Fibonacci numbers; moreover for any $n \geq 1$, \mathbf{f}_n is a left factor of \mathbf{f}_{n+1} and of \mathbf{f} .

For $n \geq 2$ we denote by \mathbf{g}_n the word $\mathbf{f}_{n-2} \mathbf{f}_{n-1}$. It is easy to see that for each $n \geq 2$ there exists a word \mathbf{v}_n such that $\mathbf{f}_n = \mathbf{v}_n xy$ and $\mathbf{g}_n = \mathbf{v}_n yx$ with $x, y \in \{a, b\}$ and $x \neq y$ and also that $\mathbf{f}_{n+2} = \mathbf{f}_n \mathbf{f}_n \mathbf{g}_{n-1}$.

The following fact is straightforward

Fact. — If u is a left factor of \mathbf{f}_n and also of \mathbf{g}_{n-1} then u is a left factor of \mathbf{v}_{n-1} and, consequently

$$|u| \leq |\mathbf{v}_{n-1}| = |\mathbf{g}_{n-1}| - 2 = |\mathbf{f}_{n-1}| - 2.$$

In the sequel we will use the following results.

PROPOSITION 1 (Karhumäki [4]): *The Fibonacci infinite word \mathbf{f} contains no 4-power.*

PROPOSITION 2 (Séebold [8]): *Let $v \neq 1$; if v^2 is a factor of the Fibonacci infinite word \mathbf{f} then there exists n such that $|v| = |\mathbf{f}_n|$; more precisely $v = wz$ with $zw = \mathbf{f}_n$ for some words z and w , $|w| > 0$, i. e. v is a conjugate of \mathbf{f}_n .*

Now let $u \neq 1$, $u \in A^*$ and let $u = x_1 \dots x_n$, $x_i \in A$; we denote by \hat{u} the mirror image of u , that is $x_n \dots x_1$.

We say that a factor u of \mathbf{f} is *special* if ua and ub are both factors of \mathbf{f} .

PROPOSITION 3 (Berstel [1]): *If u is a special factor of the Fibonacci infinite word \mathbf{f} then \hat{u} is a left factor of \mathbf{f} .*

Since the sequence $|\mathbf{f}_n|$, $n \in \mathbb{N}$, is the sequence of Fibonacci numbers, we have the following proposition.

PROPOSITION 4 (Hardy and Wright [5]): *For any $n > 1$*

$$\frac{|\mathbf{f}_{n+1}| - 2}{|\mathbf{f}_n|} = \frac{|\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2}{|\mathbf{f}_n|} < \frac{\sqrt{5} + 1}{2}$$

and

$$\lim_{n \rightarrow \infty} \frac{|\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2}{|\mathbf{f}_n|} = \frac{\sqrt{5} + 1}{2}.$$

PROPOSITION 5 (de Luca [2]): *For each i the word \mathbf{f}_i is primitive; therefore for each i the conjugates of \mathbf{f}_i are distinct.*

RESULTS AND PROOFS

Let us prove the following lemma.

LEMMA: *No fractional power with exponent greater than $1 + (\sqrt{5} + 1)/2$ can be a left factor of the Fibonacci infinite word \mathbf{f} . More precisely, if vvu is a fractional power which is a left factor of \mathbf{f} then $v = \mathbf{f}_n$ for some n and $|vvu| \leq |\mathbf{f}_n| + |\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2$.*

Proof: Let vvu be a fractional power which is a left factor of \mathbf{f} .

By using Proposition 2 we have that $|v| = |\mathbf{f}_n|$ for some n , and, consequently vv is a left factor of \mathbf{f} with length $2|\mathbf{f}_n|$. By inspection one can easily see that n is greater than or equal to 3.

As \mathbf{f}_n is a left factor of \mathbf{f} we have that $v = \mathbf{f}_n$ for some $n \geq 3$. Thus $vvu = \mathbf{f}_n \mathbf{f}_n u$ and either u is a left factor of \mathbf{f}_n or \mathbf{f}_n is a left factor of u .

But for $n \geq 3$ $\mathbf{f}_{n+2} = \mathbf{f}_n \mathbf{f}_n \mathbf{g}_{n-1}$ is a left factor of \mathbf{f} .

Hence, since \mathbf{g}_{n-1} is not a left factor of \mathbf{f}_n , we have that u is necessarily a left factor of \mathbf{g}_{n-1} ; by the fact

$$|u| \leq |\mathbf{f}_{n-1}| - 2.$$

Thus $|vvu| \leq |\mathbf{f}_n| + |\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2$ and, by Proposition 4,

$$\frac{|vvu|}{|v|} \leq \frac{|\mathbf{f}_n| + |\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2}{|\mathbf{f}_n|} < 1 + \frac{\sqrt{5} + 1}{2}, \quad \square$$

We are now ready to prove our main result.

PROPOSITION 6: *The Fibonacci infinite word \mathbf{f} contains no fractional power with exponent greater than $2 + ((\sqrt{5} + 1)/2)$ and, for any real number $\varepsilon > 0$, it contains a fractional power with exponent greater than $2 + ((\sqrt{5} + 1)/2) - \varepsilon$.*

Proof: Let $vvvu$ be a fractional power factor of \mathbf{f} . As in \mathbf{f} there are no 4 powers (Proposition 1) one can find in \mathbf{f} a factor

$$u'xu''u'xu''u'xu''u'y$$

where $u'xu''=v$, u is a left factor of u' , $u'' \in \{a, b\}^*$ and $x, y \in \{a, b\}$ with $x \neq y$.

It follows that $u'xu''u'xu''u'$ is a special factor of \mathbf{f} . By Proposition 3, $\hat{u}'\hat{u}''x\hat{u}'\hat{u}''x\hat{u}'$ is a left factor of \mathbf{f} . From the Lemma

$$\frac{|\hat{u}'\hat{u}''x\hat{u}'\hat{u}''x\hat{u}'|}{|\hat{u}'\hat{u}''x|} = \frac{|vvvu'|}{|v|} < 1 + \frac{\sqrt{5}+1}{2},$$

and, consequently,

$$\frac{|vvvu|}{|v|} \leq \frac{|vvvu'|}{|v|} < 2 + \frac{\sqrt{5}+1}{2}.$$

At last, for $n \geq 3$, $\mathbf{f}_{n+4} = \mathbf{f}_{n+1} \mathbf{f}_n \mathbf{f}_n \mathbf{g}_{n-1} \mathbf{f}_{n-1} \mathbf{f}_n$.

Hence, for $n \geq 3$, $\mathbf{f}_n \mathbf{f}_n \mathbf{f}_n \mathbf{v}_{n-1}$ is always a factor of \mathbf{f} .

Since

$$\frac{|\mathbf{f}_n \mathbf{f}_n \mathbf{f}_n \mathbf{v}_{n-1}|}{|\mathbf{f}_n|} = 2 + \frac{|\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2}{|\mathbf{f}_n|},$$

the second part of the proposition follows from Proposition 4. \square

In the proof of the above proposition we used the fact that for $n \geq 3$, $\mathbf{f}_n \mathbf{f}_n \mathbf{f}_n \mathbf{v}_{n-1}$ is a factor of \mathbf{f} . As a consequence all words of the form $wzwwz$ with $zw = \mathbf{f}_n$ and $|z| \leq |\mathbf{v}_{n-1}|$ are factors of \mathbf{f} ; by Proposition 5 all these words are distinct. Since $0 \leq |z| \leq |\mathbf{v}_{n-1}|$, the number of these words is $|\mathbf{v}_{n-1}| + 1$.

Let us suppose that vvv is a factor of \mathbf{f} and that $|v| = |\mathbf{f}_n|$ for some $n \geq 3$. By proposition 2, $v = wz$, $|w| > 0$, and $zw = \mathbf{f}_n$.

Suppose that $|z| > |\mathbf{v}_{n-1}|$; since $\mathbf{f}_n = \mathbf{f}_{n-1} \mathbf{f}_{n-2} = \mathbf{v}_{n-1} yx \mathbf{f}_{n-2}$ with $x, y \in \{a, b\}$ and $x \neq y$, we can write $\mathbf{f}_n = \mathbf{v}_{n-1} yuw$ with $z = \mathbf{v}_{n-1} yu$ and, consequently, $vvv = w \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} yu$.

We know that $\mathbf{f}_n \mathbf{f}_n \mathbf{f}_n \mathbf{g}_{n-1} = \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} xy$ is a factor of \mathbf{f} ; thus $w \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} = w \mathbf{v}_{n-1} (yuw \mathbf{v}_{n-1})^2$ is a special factor and by Proposition 3 its mirror image must be a prefix of \mathbf{f} . This is impossible by the Lemma because $|w| > 0$.

Hence we have proved the following proposition.

PROPOSITION 7: For $n \geq 3$ the number of distinct factors v of \mathbf{f} with length $|\mathbf{f}_n|$ such that vvv is also a factor of \mathbf{f} is exactly $|\mathbf{v}_{n-1}| + 1$. More precisely they are all the words of the form wz with $zw = \mathbf{f}_n$ and $|z| \leq |\mathbf{v}_{n-1}|$.

OBSERVATION: As $2 + ((\sqrt{5} + 1)/2)$ is an irrational number it cannot exist a fractional power with exponent equal to it.

In the Thue infinite word \mathbf{t} on a two letters alphabet A there are clearly squares but there are no overlaps (that is factors like $xvxy$, $x \in A$, $v \in A^*$). On the contrary it is easy to see that, for any $\varepsilon > 0$, in the Thue infinite word \mathbf{m} on a three letters alphabet there exists a fractional power with exponent greater than $2 - \varepsilon$ but it is a classical result that \mathbf{m} is square free.

Remark: Proposition 6 and 7 were firstly proved by using techniques of Sturmian words. Following the suggestion of P. Séébold we tried to find a simpler proof; actually our proof is simpler than the previous one and use only elementary properties of the Fibonacci infinite word.

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