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## CROSSABILITY OF CANCELLATIVE KLEENE SEMIGROUPS (\*)

by C. P. RUPERT <sup>(1)</sup>

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Abstract. – *Every cancellative Kleene semigroup satisfies Eilenberg's theorem.*

Résumé. – *Si  $S$  est un semigroupe simplifiable de type Kleene, alors  $S$  satisfait le théorème d'Eilenberg.*

### INTRODUCTION

A morphism  $\varphi : T \rightarrow S$  of semigroups is called crossable if every rational subset  $R$  of  $T$  contains a rational cross-section  $R_0$  for the restriction of  $\varphi$  to  $R$  or (in other words) if there exists for each rational subset  $R$  of  $T$  another rational subset  $R_0$  of  $T$  satisfying:

- (1)  $R_0 \subset R$ ;
- (2)  $\varphi(R_0) = \varphi(R)$ ; and
- (3)  $\varphi$  is injective on  $R_0$ .

The following classical crossability result is useful in the theory of rational relations.

EILENBERG'S THEOREM [1]: *If  $\Sigma^*$  and  $\Gamma^*$  are finitely generated free monoids, then every morphism  $\varphi : \Sigma^* \rightarrow \Gamma^*$  is crossable.* ■

We say that a semigroup  $S$  satisfies Eilenberg's theorem, or that  $S$  is crossable, if every morphism  $\varphi : \Sigma^+ \rightarrow S$  is crossable for every free semigroup  $\Sigma^+$ .

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Crossability results often have interesting consequences. For example, if  $S$  satisfies Eilenberg's theorem then every rational subset of  $S$  is unambiguously rational. Moreover, an effective proof that  $S$  satisfies Eilenberg's theorem enables us to decide whether a given rational expression over  $S$  is unambiguously rational.

Pelletier [3] introduced a technique for constructing congruences from equivalence relations, used it to produce various counter-examples in the theory of Kleene semigroups, and in this way showed that not all Kleene semigroups satisfy Eilenberg's theorem.

Our major result, Theorem 2 below, proves that every cancellative Kleene semigroup satisfies Eilenberg's theorem, by modifying a method used by Sakarovitch [5] (to prove a special case of Eilenberg's theorem) and by Johnson (to show that every deterministic rational equivalence relation has a rational cross-section, *cf.* Theorem 5.3 in [2]). The method produces rational cross-sections of equivalence relations by lexicographic minimalization, a tactic which does not work in general (*cf.* Theorem 8.2 in [2]) but does work here.

## I. PRELIMINARIES

Recall some definitions and theorems.

A subset  $R$  (of a semigroup  $S$ ), which is saturated by a congruence  $\equiv$  of finite index on  $S$ , is called recognizable.  $\text{Rec}(S)$  denotes the set of recognizable subsets of  $S$ .

**NERODE'S THEOREM:** *A subset  $R$  of a semigroup  $S$  is recognizable iff there are only finitely many different quotient sets  $s^{-1}R := \{t \in S : st \in R\}$ .* ■

**LEMMA 1:** *Let  $R$  be a recognizable subset of a semigroup  $S$ ; for each  $s \in R$ , define the set  $[s]_R := \{t : t^{-1}R = s^{-1}R\}$ . Then there are only finitely many sets  $[s]_R$  and each of these sets is recognizable.* ■

Rational subsets of a semigroup  $S$  are defined as follows: the empty set  $\emptyset$  is rational and so is every singleton  $s \in S$ ; if  $U$  and  $V$  are rational, then so are the union  $U \cup V$ , product  $UV := \{uv : u \in U, v \in V\}$ , and subsemigroup  $U^+ \subset S$  generated by  $U$ .  $\text{Rat}(S)$  denotes the collection of rational subsets of  $S$ .

In an arbitrary semigroup  $S$ ,  $\text{Rec}(S)$  and  $\text{Rat}(S)$  are not closely related. However, the following result holds.

**KLEENE'S THEOREM:** *If  $\Sigma^+$  is a finitely generated free semigroup, then every rational subset of  $\Sigma^+$  is recognizable and conversely.* ■

Motivated by this result, we call a semigroup  $S$  Kleene if  $\text{Rat}(S) = \text{Rec}(S)$ . Clearly, a Kleene semigroup is finitely generated.

By a regulator  $\rho: \Sigma^+ \rightarrow \Sigma^+$ , we mean a rationality-preserving relation: every rational subset  $R \subset \Sigma^+$  has rational  $\rho$ -image  $\rho(R)$ .

**LEMMA 2 [3]:** *A semigroup  $S$  is Kleene iff  $S$  is isomorphic to the quotient  $\Sigma^+/\kappa$  of a finitely-generated free semigroup  $\Sigma^+$  by a congruence  $\kappa$  which is also a regulator.* ■

**LEMMA 3:** *Any relation  $\Sigma^+ \rightarrow \Sigma^+$  which is rational in  $\Sigma^* \times \Sigma^*$  is a regulator.* ■

**LEMMA 4:** *The set of regulators is closed under finite union and under composition. If  $\psi$  is a regulator and if  $P$  and  $Q$  are rational subsets of  $\Sigma^+$  then  $(P \times Q) \cap \psi$  is also a regulator.*

*Proof:* Suppose that  $\psi$  and  $\theta$  are regulators; if  $R \in \text{Rat}(\Sigma^+)$ , then  $(\psi \cup \theta)(R) = \psi(R) \cup \theta(R)$  and  $\psi \circ \theta(R) = \psi(\theta(R))$ ; so the first sentence holds. If  $R$  is rational in  $\Sigma^+$ , then

$$\Delta_R = \{ (r, r) : r \in R \}$$

is a rational relation  $\Sigma^* \rightarrow \Sigma^*$ . Now  $(P \times Q) \cap \psi$  is simply  $\Delta_Q \circ \psi \circ \Delta_P$ ; if  $P$  and  $Q$  are rational, this is a composite of regulators; so the second sentence holds. ■

We also use another closure property of regulators. Given any relations  $\psi: \Sigma^+ \rightarrow \Sigma^+$  and  $\varphi: \Sigma^+ \rightarrow \Sigma^+$ , define the product relation  $\varphi \wedge \psi: \Sigma^+ \rightarrow \Sigma^+$  by

$$\varphi \wedge \psi := \{ (ac, bd) : (a, b) \in \varphi, (c, d) \in \psi \}.$$

**LEMMA 5:** *If  $\psi: \Sigma^+ \rightarrow \Sigma^+$  and  $\varphi: \Sigma^+ \rightarrow \Sigma^+$  are regulators, then the product relation  $\varphi \wedge \psi: \Sigma^+ \rightarrow \Sigma^+$  is also a regulator.*

*Proof:* We begin with the following claim.

*Claim:* For  $R \subset \Sigma^+$ ,  $\varphi \wedge \psi(R) = \bigcup_{x \in \Sigma^+} \varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1}R)$ .

*Explanation:* Suppose  $t \in \varphi \wedge \psi(R)$ . Choose  $(a, b) \in \varphi$ ,  $(c, d) \in \psi$  with  $ac = s \in R$  and  $bd = t$ . Then  $b \in \varphi([a]_R \cap R(\Sigma^+)^{-1})$  (since  $a \in [a]_R$  and  $ac = s \in R$ ), and  $d \in \psi(a^{-1}R)$  (since  $c \in a^{-1}R$ ). So  $t = bd \in \varphi([a]_R \cap R(\Sigma^+)^{-1}) \psi(a^{-1}R)$

and thus

$$\varphi \wedge \psi(R) \subset \bigcup_{x \in \Sigma^+} \varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1}R).$$

For the opposite inclusion, suppose that

$$t \in \bigcup_{x \in \Sigma^+} \varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1}R).$$

Then  $t \in \varphi([p]_R \cap R(\Sigma^+)^{-1}) \psi(p^{-1}R)$  for some  $p \in \Sigma^+$ . So  $t = bd$  for some  $a \in [p]_R \cap R(\Sigma^+)^{-1}$ ,  $b \in \varphi(a)$  and  $d \in \psi(p^{-1}R)$ . Since  $a \in [p]_R$ ,  $[a]_R = [p]_R$  and  $a^{-1}R = p^{-1}R$ ; so  $a \in [a]_R \cap R(\Sigma^+)^{-1}$  and  $d \in \psi(a^{-1}R)$ . Choose  $c \in a^{-1}R$  with  $d \in \psi(c) \subset \psi(a^{-1}R)$ . As  $(a, b) \in \varphi$ ,  $(c, d) \in \psi$ , and  $ac \in R$ , so  $t = bd \in \varphi \wedge \psi(R)$ , and therefore

$$\bigcup_{x \in \Sigma^+} \varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1}R) \subset \varphi \wedge \psi(R),$$

which completes the proof of the claim.  $\square$

We now show that  $\varphi \wedge \psi$  is a regulator. Suppose  $R \in \text{Rat}(\Sigma^+)$ . Then the sets  $[x]_R$ ,  $R(\Sigma^+)^{-1}$ , and  $x^{-1}R$  are also rational; hence so is each set  $\varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1}R)$ . There are but finitely many distinct sets  $x^{-1}R$  and similarly only finitely many sets  $[x]_R$ . It follows that

$$\varphi \wedge \psi(R) = \bigcup_{x \in \Sigma^+} \varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1}R)$$

actually reduces to a finite union of rational sets. Thus,  $\varphi \wedge \psi(R)$  is rational and so  $\varphi \wedge \psi$  is a regulator.  $\blacksquare$

## II. A METHOD OF SAKAROVITCH

By an order on a set  $X$ , we understand a binary relation  $>$  on  $X$  which is asymmetric (no element  $x \in X$  satisfies  $x > x$ ) and transitive. A linear order is an order verifying trichotomy:

$$\forall x \in X \quad \forall y \in X \quad x = y \quad \text{or} \quad x > y \quad \text{or} \quad y > x.$$

If  $>$  is an order on  $X$  and  $R$  is a subset of  $X$ , then by  $a >$ -minimal element of  $R \subset X$  we mean any  $r \in R$  with

$$\{s \in R : r > s\} = \emptyset.$$

When  $\kappa$  is a relation on  $X$ ,  $\Lambda = \Lambda(>, \kappa)$  denotes the relation

$$\kappa \cap >^{-1} = \{(u, v) \in \kappa : v > u\}.$$

If  $\kappa$  is an equivalence relation,  $\text{Min}(R) = \text{Min}(>, \kappa, R)$  denotes the set

$$\{r \in R : r \text{ is } a>\text{-minimal element of } [r]_{\kappa} \cap R\},$$

where  $[r]_{\kappa}$  denotes the  $\kappa$ -class of  $r \in X$ .

Lexicographic orders on a free semigroup  $\Sigma^+$  are constructed as follows: fix a linear order  $>$  on the alphabet  $\Sigma$ ; for distinct words  $u \in \Sigma^+$  and  $v \in \Sigma^+$ ,  $v > u$  means either that  $u$  is a proper prefix of  $v$  or that there exist (possibly empty) words  $w, x$ , and  $y$  over the alphabet  $\Sigma$  and letters  $\sigma, \tau$  in  $\Sigma$  such that  $u = w\tau x$  and  $v = w\sigma y$ . Any lexicographic order is linear.

LEMMA 6: *If  $\kappa$  is a relation and  $>$  a lexicographic order on  $\Sigma^+$  then  $\Lambda(>, \kappa)$  is a union  $\Lambda_1 \cup \Lambda_2$ , where  $\Lambda_1$  denotes the relation*

$$\{(x\sigma u, x\tau v) \in \kappa : x \in \Sigma^*, u \in \Sigma^*, v \in \Sigma^*, \sigma \in \Sigma, \tau \in \Sigma, \tau > \sigma\}$$

and  $\Lambda_2$  denotes the relation

$$\{(x\sigma, x\sigma v) \in \kappa : x \in \Sigma^*, \sigma \in \Sigma, v \in \Sigma^+\}.$$

*Proof:* Obvious from the definition of lexicographic order. ■

Now the method of Sakarovitch [5] is essentially this: for any lexicographic order  $>$  and any morphism  $\pi : \Sigma^+ \rightarrow \Gamma^*$  from the free semigroup  $\Sigma^+$  to the free monoid  $\Gamma^*$ ,  $\Lambda(>, \pi^{-1}\pi)$  is a rational relation  $\Sigma^* \rightarrow \Sigma^*$ , and the set  $\text{Min}(>, \pi^{-1}\pi, R)$  is therefore rational whenever  $R \subset \Sigma^+$  is; when  $\pi$  is non-erasing,  $\text{Min}(>, \pi^{-1}\pi, R)$  is a cross-section for the restriction of  $\pi^{-1}\pi$  to  $R$ , and Eilenberg's theorem follows easily.

The next lemmas isolate some key ideas of this method.

LEMMA 7: *Let  $>$  and  $\kappa$  (respectively) be an order and an equivalence relation on the set  $X$ . Then the following are equivalent:*

(1) *For each  $r \in R$ , there exists at least one  $>$ -minimal element of the set  $[r]_{\kappa} \cap R$ ; and*

(2)  *$\text{Min}(>, \kappa, R)$  intersects every  $\kappa$ -class intersecting  $R$ . If these conditions hold and  $>$  is a linear order, then  $\text{Min}(>, \kappa, R)$  is a cross-section for the restriction of  $\kappa$  to  $R$ .*

*Proof:* (1)  $\Leftrightarrow$  (2) is obvious. If in addition  $>$  is a linear order then each set  $[r]_{\kappa} \cap R$  has a unique  $>$ -minimal element, so  $\text{Min}(R)$  must be a cross-section for the restriction of  $\kappa$  to  $R$ . ■

LEMMA 8: *Suppose that  $>$  and  $\kappa$  (respectively) are an order and an equivalence relation on the finitely generated free semigroup  $\Sigma^+$ , and that  $\Lambda(>, \kappa)$  is a regulator. Then  $\text{Min}(>, \kappa, R)$  is rational for every rational set  $R$ .*

*Proof:* Suppose that  $\Lambda$  is a regulator; let  $\Lambda_0: \Sigma^+ \rightarrow \Sigma^+$  denote the relation  $\Delta_R \circ \Lambda$ , where  $\Delta_R = \{(r, r) : r \in R\}$ . Then

$$\begin{aligned} R \setminus \Lambda_0(R) &= R \setminus \{r \in R : \exists s \in R (s, r) \in \Lambda\} \\ &= \{r \in R : \forall s (s \in [r]_{\kappa} \cap R \Rightarrow \text{not}(r > s))\} = \text{Min}(R). \end{aligned}$$

Whenever  $R$  is rational,  $\Lambda_0$  is a regulator by Lemma 4 and so  $\text{Min}(R)$  is rational. ■

### III. LEXICOGRAPHIC MINIMALIZATION

For the remainder of this article, we fix a finitely generated free semigroup  $\Sigma^+$  and a lexicographic order  $>$  on  $\Sigma^+$ . To generalize Sakarovitch's argument, we show that  $\Lambda(>, \kappa)$  is a regulator when  $\kappa$  is a left-cancellative congruence.

A semigroup  $S$  is called left-cancellative if

$$xy = xz \Rightarrow y = z;$$

the notion right-cancellative is dually defined; cancellative means left- and right-cancellative.

We can now obtain our first result.

THEOREM 1: *If  $\Sigma^+/\kappa$  is a left-cancellative Kleene semigroup, and if  $>$  is a lexicographic order on  $\Sigma^+$ , then  $\Lambda(>, \kappa): \Sigma^+ \rightarrow \Sigma^+$  is a regulator.*

*Proof:* Express  $\Lambda = \Lambda_1 \cup \Lambda_2$  according to Lemma 6. To show that  $\Lambda$  is a regulator, it suffices (according to Lemma 4) to prove that  $\Lambda_1$  and  $\Lambda_2$  are

both regulators. Now  $\Lambda_2$  is the relation

$$\begin{aligned} & \bigcup_{\sigma \in \Sigma} \{ (x \sigma, x \sigma w) \in \kappa : x \in \Sigma^*, w \in \Sigma^+ \} \\ &= \bigcup_{\sigma \in \Sigma} ( \{ \sigma, \sigma w \} \in \kappa : w \in \Sigma^+ ) \cup \Delta \{ (\sigma, \sigma w) \in \kappa : w \in \Sigma^+ \} ) \\ &= \bigcup_{\sigma \in \Sigma} ( (\sigma \times (\sigma \Sigma^+ \cap [\sigma]_\kappa)) \cup \Delta (\sigma \times (\sigma \Sigma^+ \cap [\sigma]_\kappa)) ), \end{aligned}$$

where  $[\sigma]_\kappa$  denotes the  $\kappa$ -class of  $\sigma \in \Sigma$  and  $\Delta$  denotes the diagonal  $\{ (x, x) : x \in \Sigma^+ \}$ ; note that we used the left-cancellativity of  $\kappa$ . Since the semigroup is Kleene, each set  $\sigma \Sigma^+ \cap [\sigma]_\kappa$  is rational; thus,  $\Lambda_2$  is actually a rational relation  $\Sigma^* \rightarrow \Sigma^*$  and hence a regulator.

To show that  $\Lambda_1$  is a regulator, we first observe that each relation  $(\sigma \Sigma^* \times \tau \Sigma^*) \cap \kappa$  (where  $\sigma \in \Sigma, \tau \in \Sigma$ , and  $\tau > \sigma$ ) is a regulator by Lemmas 2 and 4. Thus the union of these relations is another regulator  $\Lambda_3$ . If we show that  $\Lambda_1 = \Lambda_3 \cup \Delta \wedge \Delta_3$  then  $\Lambda_1$  will be a regulator by Lemmas 4 and 5.

Now  $(s, t) \in \Lambda_1$  means  $s \kappa t, (s, t) = (x \sigma u, x \tau v)$  where  $\tau > \sigma$  are letters in  $\Sigma$ , and  $x, u$ , and  $v$  lie in  $\Sigma^*$ . If  $x$  is actually the empty word, then

$$(s, t) = (\sigma u, \tau v) \in (\sigma \Sigma^* \times \tau \Sigma^*) \cap \kappa \subset \Lambda_3.$$

On the other hand, when  $x \in \Sigma^+$  we conclude from  $x \sigma u \kappa x \tau v$  (using left-cancellativity) that

$$(\sigma u, \tau v) \in (\sigma \Sigma^* \times \tau \Sigma^*) \cap \kappa \subset \Lambda_3,$$

whence it is immediate (by the definition of  $\Delta \wedge \Lambda_3$ ) that

$$(s, t) = (x \sigma u, x \tau v) \in \Delta \wedge \Lambda_3.$$

Thus  $\Lambda_1 \subset \Lambda_3 \cup \Delta \wedge \Lambda_3$ ; for the opposite inclusion, read backwards, using the left-compatibility

$$y \kappa x \Rightarrow xy \kappa xz$$

of  $\kappa$  instead of left-cancellativity. ■

Corollaries 1 and 2 below generalize results in [5].

**COROLLARY 1:** *If  $\pi : \Sigma^+ \rightarrow S$  is a morphism from  $\Sigma^+$  to a left-cancellative Kleene semigroup  $S$ , and if  $>$  is a lexicographic order on  $\Sigma^+$ , then  $\text{Min}(>, \pi^{-1} \pi, R)$  is rational for every rational subset  $R \subset \Sigma^+$ .*

*Proof:* Since  $S$  is Kleene,  $\pi^{-1}\pi$  is a regulator by Lemma 2; moreover, a subsemigroup of a left-cancellative semigroup is left-cancellative; hence, the theorem guarantees that  $\Lambda$  is a regulator. The result now follows from Lemma 8. ■

LEMMA 9: *Let  $S$  be a left-cancellative semigroup, every singleton subset of which is recognizable. Then the following conditions are equivalent for any morphism  $\pi: \Sigma^+ \rightarrow S$ :*

- (1) *For each  $s \in \pi(\Sigma^+)$ , the set  $ss^{-1} \cap \pi(\Sigma^+) = \emptyset$ ; and*
- (2) *Each  $\pi^{-1}\pi$ -class is finite.*

*Proof:* (1)  $\Rightarrow$  (2): Suppose that some set  $\pi^{-1}\pi(w)$  is infinite. Then  $\pi^{-1}\pi(w)$  is rational because  $S$  has recognizable singletons and consequently  $\pi^{-1}\pi(w)$  contains an infinite subset  $xy^+z$  by the pumping lemma. From  $\pi(xy^+z) = (\pi(xy^+z))$ , we conclude (by left-cancellativity) that

$$\pi(yz) = \pi(y^2z) = \pi(y)\pi(yz) \text{ so } \pi(y)$$

belongs to the set  $\pi(yz)\pi(yz)^{-1}$ .

(2)  $\Rightarrow$  (1): If  $\pi(v) \in \pi(u)\pi(u)^{-1}$  for some words  $u$  and  $v$ , then  $v^+u$  is an infinite subset of  $\pi^{-1}\pi(u)$ . ■

COROLLARY 2: *If  $S$  is a left-cancellative Kleene semigroup,  $>$  a lexicographic order on  $\Sigma^+$  and  $\pi: \Sigma^+ \rightarrow S$  a morphism such that  $ss^{-1} \cap \pi(\Sigma^+) = \emptyset$  for each  $s \in \pi(\Sigma^+)$ , then  $\text{Min}(>, \pi^{-1}\pi, R)$  is a rational cross-section for the restriction of  $\pi$  to  $R$ , for each rational  $R \subset \Sigma^+$ ; in particular,  $\pi$  is crossable.*

*Proof:* By hypothesis, every  $\pi^{-1}\pi$ -class is finite. Thus  $\text{Min}(R)$  is a rational cross-section by Theorem 1 and Lemmas 7 and 8, regardless of the rational set  $R \subset \Sigma^+$ . ■

For the next application of these ideas, we recall that an equivalence relation  $\kappa_1$  is called locally-finite thinning of the equivalence relation  $\kappa \subset \Sigma^+ \times \Sigma^+$  if  $\kappa_1$  is a restriction of  $\kappa$ , if the domain of  $\kappa_1$  intersects every  $\kappa$ -class, and if each  $\kappa_1$ -class is finite. The following result is due to Johnson.

JOHNSON'S THEOREM [2]: *Every rational equivalence relation has a rational locally-finite thinning.* ■

We also need the following result, which can be restated in various forms (cf. Proposition 1.4.3 in [3]).

CHOFFRUT'S THEOREM: *If the congruence  $\kappa$  on  $\Sigma^+$  is rational as a subset of  $\Sigma^* \times \Sigma^*$  and if  $\kappa$  has a rational cross-section, then the quotient  $\Sigma^+/\kappa$  satisfies Eilenberg's theorem.* ■

**COROLLARY 3:** *Suppose  $\kappa \subset \Sigma^+ \times \Sigma^+$  is a left-cancellative congruence which is rational as a subset of  $\Sigma^* \times \Sigma^*$ . Then  $\Sigma^+/\kappa$  satisfies Eilenberg's theorem.*

*Proof:* According to the Johnson's Theorem, we can find a rational locally-finite thinning  $\kappa_1$  for  $\kappa$ , or (in other words) we can find a rational set  $D \subset \Sigma^+$  such that  $\kappa_1 := \kappa \cap D \times D$  is a locally-finite thinning of  $\kappa$ . Fix any lexicographic order  $>$  on  $\Sigma^+$ . Then  $\text{Min}(>, \kappa, D)$  is a rational cross-section for  $\kappa$  by Theorem 1 and Lemmas 7 and 8. The result now follows by Chofrut's theorem. ■

**IV. CANCELLATIVE KLEENE SEMIGROUPS**

In this section, we show that cancellative Kleene semigroups satisfy Eilenberg's theorem.

**LEMMA 10 [4]:** *Let  $S$  be a semigroup, every singleton subset of which is recognizable. Then every subgroup of  $S$  is finite. If  $S$  has an identity element 1, then every divisor of 1 actually belongs to the group of units of  $S$ .* ■

**LEMMA 11:** *Let  $S$  be a cancellative semigroup, every singleton subset of which is recognizable. Then the following conditions are equivalent for any morphism  $\pi: \Sigma^+ \rightarrow S$ :*

- (1) *For each  $s \in \pi(\Sigma^+)$ , the set  $ss^{-1} \cap \pi(\Sigma^+) = \emptyset$ ;*
- (2)  *$\pi(\Sigma^+)$  does not contain an idempotent;*
- (3) *If  $S$  has an identity element 1, then  $1 \notin \pi(\Sigma^+)$ ; and*
- (4) *For each  $\sigma \in \Sigma$ , the set  $\pi^{-1} \pi(\sigma) \cap \sigma(\sigma^+) = \emptyset$ .*

*Proof:* (4)  $\Rightarrow$  (3): Suppose  $\pi(w)$  is an identity element for  $S$ ; let  $\sigma \in \Sigma$  be any letter appearing in  $w$ ; then, as a divisor of the identity  $\pi(w)$ ,  $\pi(\sigma)$  belongs to the group of units of  $S$ ; moreover, this is a finite group; hence for some  $n > 1$ ,  $\pi(\sigma)^n = \pi(\sigma^n) = \pi(\sigma)$  and therefore  $\pi^{-1} \pi(\sigma) \cap \sigma(\sigma^+) \neq \emptyset$ .

(3)  $\Rightarrow$  (2): An idempotent in a cancellative semigroup must be the identity.

(2)  $\Rightarrow$  (1): If  $\pi(v) \in \pi(u)\pi(u)^{-1}$ , then  $\pi(v^2u) = \pi(vu) = \pi(u)$  and  $\pi(v)$  is idempotent by cancellativity.

(1)  $\Rightarrow$  (4): If  $\sigma \in \Sigma$  and  $n > 1$  satisfy  $\sigma^n \in \pi^{-1} \pi(\sigma)$ , then

$$\pi(\sigma^{n-1}) \in \pi(\sigma)\pi(\sigma)^{-1}. \quad \blacksquare$$

**THEOREM 2:** *Let  $\pi: \Sigma^+ \rightarrow S$  be a morphism from  $\Sigma^+$  to the cancellative Kleene semigroup  $S$ . Then  $\pi$  is crossable.*

*Proof:* Fix a lexicographic order  $>$  on  $\Sigma^+$ . According to part (4) of Lemma 11, we can easily test whether  $\pi(\Sigma^+)$  contains an identity element for  $S$ . Our proof splits according to the outcome of this test; if  $\pi(\Sigma^+)$  does not contain an identity element for  $S$ , and if  $R$  is any rational set, then (by Lemma 11 and Corollary 2)  $\text{Min}(>, \pi^{-1}\pi, R)$  is a rational cross-section for the restriction of  $\pi$  to  $R$ .

On the other hand, if  $S$  is actually a monoid with identity element  $1 \in \pi(\Sigma^+)$ , and if  $R \subset \Sigma^+$  is any rational set, put  $G := \pi^{-1}(1)$ , and define  $\varepsilon: \Sigma^+ \rightarrow \Sigma^+$  by

$$\varepsilon := (\Delta \cup \Theta) * \Theta (\Delta \cup \Theta) *$$

where  $\Delta := \{(x, x) : x \in \Sigma^+\}$  and

$$\Theta := \bigcup_{\sigma \in \Sigma} ((\sigma G \times \sigma) \cup (G \sigma \times \sigma)).$$

Then  $\varepsilon$  is an order which is also a rational relation  $\Sigma^* \rightarrow \Sigma^*$ . If  $(u, v) \in \varepsilon$ , then  $v$  has length strictly less than the length of  $u$ , so there is no infinite chain

$$w_1 \varepsilon w_2 \varepsilon w_3 \varepsilon \dots;$$

hence each  $\pi^{-1}\pi$ -class has an  $\varepsilon$ -minimal element. As  $\varepsilon \subset \pi^{-1}\pi$ , we have  $\Lambda(\varepsilon, \pi^{-1}\pi) = \varepsilon^{-1}$ , which is certainly a regulator. By Lemmas 7 and 8,  $\text{Min}(\varepsilon, \pi^{-1}\pi, R)$  is rational and  $\pi(\text{Min}(\varepsilon, \pi^{-1}\pi, R)) = \pi(R)$ .

We claim no  $\pi^{-1}\pi$ -class contains infinitely many elements of  $R_1 := \text{Min}(\varepsilon, \pi^{-1}\pi, R)$ . If indeed  $R_1 \cap \pi^{-1}\pi(w)$  were infinite, then according to the pumping lemma this rational set would contain an infinite subset  $xy^+z$  with  $y \in \Sigma^+$ , by cancellativity,  $\pi(y)$  is idempotent so  $y \in G$ , which implies that  $(xy^2z, xyz) \in \varepsilon$ ; but this contradicts the fact that  $xy^2z \in R_1 = \text{Min}(\varepsilon, \pi^{-1}\pi, R)$ . By Lemmas 7 and 8,  $\text{Min}(>, \pi^{-1}\pi, R_1)$  is therefore a rational cross-section for the restriction of  $\pi$  to  $R_1$  and even for the restriction of  $\pi$  to  $R$ . ■

We remark that Theorem 2 is effective relative to the given Kleene semi-group  $S$ : if we have an explicit finite generating set for  $S$ , and an algorithm which produces for each  $R \in \text{Rat}(S)$  a congruence of finite index saturating  $R$ , then we can really produce the cross-sections described.

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## REFERENCES

1. S. EILENBERG, Automata, Languages, and Machines: Volume A, *Academic Press*, 1974.
2. J. H. JOHNSON, Do Rational Equivalence Relations have Regular Cross-Sections? Proc. 12th Internat. Conf. on Automata, Languages, and Programming, *Lecture Notes in Comput. Sci.*, 1985, 194, *Springer-Verlag*, pp. 300-309.
3. M. PELLETIER, Descriptions de Semigroupes par Automates, *Thèse de Doctorat*, Université Paris-VI, 1989.
4. C. RUPERT, Equidivisible Kleene Monoids and the Elgot-Mezei Theorem, *Semigroup Forum*, 1990, 40, pp. 129-141.
5. J. SAKAROVITCH, Deux Remarques sur un Théorème de S. Eilenberg, *R.A.I.R.O. Inform. Théor. Appl.*, 1983, 17, pp. 23-48.