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A NOTE ON THE PAPER "DISJUNCTIVE LANGUAGES AND COMPATIBLE ORDERS" (*)

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Abstract – *In a previous paper we have dealt with m-disjunctive languages, s-disjunctive languages over an alphabet X and related compatible partial orders on X* However, in our former paper, the cardinality of X has been assumed to be greater than one In this note, we will deal with the case X an alphabet with one letter We show that every disjunctive language over an alphabet with one letter is s-disjunctive*

Résumé – *Dans un article antérieur, nous avons étudié les langages m-disjonctifs et s-disjonctifs sur un alphabet X ainsi que les relations d'ordre partiel compatibles qui leur sont associées Cependant, nous n'avions considéré jusqu'à présent que des alphabets d'au moins deux lettres Dans cette note, nous traitons le cas d'un alphabet à une lettre et nous montrons que tout langage disjonctif sur un alphabet à une lettre est s-disjonctif*

Let X be a finite nonempty alphabet and let L be a language over X. For $u \in X^*$, let $L..u=\{(x, y) \mid xuy \in L\}$. The language L is called *disjunctive* if $L..u=L..v$ implies $u=v$. If L is disjunctive, then the relation $\leqq_L$ defined on X^* by $u \leqq_L v$ if and only if $L..u \subseteqq L..v$ is a compatible partial order. In the case that $\leqq_L$ is the identity relation, then L is said to be *s-disjunctive;* otherwise L is called m-disjunctive. The purpose of this note is to prove that every disjunctive language over an alphabet with one letter is s-disjunctive. One may think that this result has already been shown by Proposition 2.2 in [1]. The statement of Proposition 2.2 is as follows: *If $L \subseteqq X^*$ is a semi-discrete language, then L is s-disjunctive.* However, as the proof of Proposition 2.2 depends on the fact $|X| \geqq 2$, we must show it independently for the case $|X|=1$.

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LEMMA 1: *Let X be a finite nonempty alphabet, let $L \subseteqq X^*$ be an m-disjunctive language and let $u \in X^+$. If $L..u^i \subseteqq L..u^j$ and $L..u^m \subseteqq L..u^n$ for some $i, j, m, n \geqq 0$, then $(j-i)(n-m) \geqq 0$.*

Proof: Suppose that $i<j$ and $n<m$. Let r, s be positive integers such that $r(j-i)=s(m-n)$. Then $L..u^{ri} \subseteqq L..u^{rj}$ and $L..u^{sm} \subseteqq L..u^{sn}$. Therefore $L..u^{ri+sm} \subseteqq L..u^{rj+sm} \subseteqq L..u^{rj+sn}$. Since L is disjunctive, we have $u^{ri+sm}=u^{rj+sm}$, *i.e.* $i=j$, a contradiction. This completes the proof of the lemma.

LEMMA 2: *Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be positive integers whose greatest common divisor equals γ. Then there exists a positive integer s satisfying the following condition:*

For every $i \geqq 0$, there exist nonnegative integers

$$\alpha_1, \alpha_2, \ldots, \alpha_k \quad \text{such that} \quad (s+i)\gamma=\alpha_1\gamma_1+\alpha_2\gamma_2+\ldots+\alpha_k\gamma_k.$$

Proof: This follows from the definition of the greatest common divisor.

LEMMA 3: *Let $L \subseteqq X^*$ be an m-disjunctive language, let $u \in X^+$ and let $L..u^{m_i} \subseteqq L..u^{n_i}$ with $n_i > m_i$ and $1 \leqq i \leqq k$. Let $\gamma_i=n_i-m_i$ and let $t > \max\{m_1, m_2, \ldots, m_k\}$. If $u^t \in L$, then $u^{t+\beta_1\gamma_1+\beta_2\gamma_2+\ldots+\beta_k\gamma_k} \in L$ for every $\beta_1, \beta_2, \ldots, \beta_k \geqq 0$.*

Proof: Since $u^t=u^{t-m_i}u^{m_i}$, $t>m_i$ and $L..u^{m_i} \subseteqq L..u^{n_i}$, we have $u^{t-m_i}u^{n_i}=u^{t+\gamma_i} \in L$. The lemma follows by induction.

In the following lemma, L is assumed to be an m-disjunctive language satisfying the conditions in Lemma 3. Moreover, s, γ, t and u are assumed to be the values and the element in X^+ appearing in the preceding lemmas.

LEMMA 4: *Let $t < t'$ and let $t \equiv t' \pmod{s\gamma}$. If $u^t \in L$, then*

$$u^{t'} \cup u^{t'} u^{s\gamma}(u^{\gamma})^* \subseteqq u^t \cup u^t u^{s\gamma}(u^{\gamma})^* \subseteqq L.$$

Proof: This follows immediately from the preceding lemmas.

Now we are ready to prove that every disjunctive language over an alphabet with one letter is s-disjunctive.

PROPOSITION 1: *Let X be an alphabet with only one letter and let $L \subseteqq X^*$ be a disjunctive language. Then L is s-disjunctive.*

Proof: Let $X=\{a\}$. Suppose that L is not s-disjunctive. Let $H=\{n-m \mid n \neq m, L..a^m \subseteqq L..a^n\}$. Since L is m-disjunctive, H is not empty. From Lemma 1, we have the following two cases.

Case 1: For every $h \in H$, $h > 0$. Let γ be the greatest common divisor of the numbers from H. Then there exist $\gamma_1, \gamma_2, \ldots, \gamma_k$ such that $\gamma = g.c.d. \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ where $\gamma_1, \gamma_2, \ldots, \gamma_k \in H$ and $\gamma_i = n_i - m_i$ with $L..a^{m_i} \subseteqq L..a^{n_i}$. For every i, $1 \leqq i \leqq s\gamma$, let t_i be the least integer such that $\max\{m_1, m_2, \ldots, m_k\} \leqq t_i$, $t_i \equiv i \pmod{\gamma s}$ and $a^{t_i} \in L$. Let

$$I = \{i \mid 1 \leqq i \leqq s\gamma \text{ and } t_i \text{ exists}\}.$$

Then by Lemma 4, we have $L = (\bigcup_{i \in I} (a^{t_i} \cup a^{t_i} a^{s\gamma} (a^{\gamma})^*)) \cup F$ where $F \subseteqq \bigcup_{0 \leqq j \leqq T} a^j$ with

$$T = \max\{m_1, m_2, \ldots, m_k\}.$$

Clearly L is regular, a contradiction.

Case 2: For every $h \in H$, $h < 0$. Note that $L..a^m \subseteqq L..a^n$ implies $(X^* \setminus L)..a^n \subseteqq (X^* \setminus L)..a^m$. As in Case 1, it can be shown that the language $X^* \setminus L$ is regular. Hence L is regular, a contradiction. Therefore, L is s-disjunctive.

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REFERENCES

1 M. Ito, H. J. Shyr, G. Thierrin, Disjunctive Languages and Compatible Orders, *R.A.I.R.O. Inform. Théor. Appl.*, 1989, **23**, pp 149-163