

M. ITO

**A note on the paper “disjunctive languages  
and compatible orders”**

*RAIRO. Informatique théorique et applications*, tome 25, n° 5 (1991),  
p. 419-421

[http://www.numdam.org/item?id=ITA\\_1991\\_\\_25\\_5\\_419\\_0](http://www.numdam.org/item?id=ITA_1991__25_5_419_0)

© AFCET, 1991, tous droits réservés.

L'accès aux archives de la revue « RAIRO. Informatique théorique et applications » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

**A NOTE ON THE PAPER  
"DISJUNCTIVE LANGUAGES  
AND COMPATIBLE ORDERS" (\*)**

by M. ITO (<sup>1</sup>)

Communicated by J-E PIN

---

*Abstract – In a previous paper we have dealt with  $m$ -disjunctive languages,  $s$ -disjunctive languages over an alphabet  $X$  and related compatible partial orders on  $X^*$ . However, in our former paper, the cardinality of  $X$  has been assumed to be greater than one. In this note, we will deal with the case  $X$  an alphabet with one letter. We show that every disjunctive language over an alphabet with one letter is  $s$ -disjunctive.*

*Résumé – Dans un article antérieur, nous avons étudié les langages  $m$ -disjonctifs et  $s$ -disjonctifs sur un alphabet  $X$  ainsi que les relations d'ordre partiel compatibles qui leur sont associées. Cependant, nous n'avons considéré jusqu'à présent que des alphabets d'au moins deux lettres. Dans cette note, nous traitons le cas d'un alphabet à une lettre et nous montrons que tout langage disjonctif sur un alphabet à une lettre est  $s$ -disjonctif.*

Let  $X$  be a finite nonempty alphabet and let  $L$  be a language over  $X$ . For  $u \in X^*$ , let  $L \cdot u = \{(x, y) \mid xuy \in L\}$ . The language  $L$  is called *disjunctive* if  $L \cdot u = L \cdot v$  implies  $u = v$ . If  $L$  is disjunctive, then the relation  $\leq_L$  defined on  $X^*$  by  $u \leq_L v$  if and only if  $L \cdot u \subseteq L \cdot v$  is a compatible partial order. In the case that  $\leq_L$  is the identity relation, then  $L$  is said to be  *$s$ -disjunctive*; otherwise  $L$  is called  *$m$ -disjunctive*. The purpose of this note is to prove that every disjunctive language over an alphabet with one letter is  $s$ -disjunctive. One may think that this result has already been shown by Proposition 2.2 in [1]. The statement of Proposition 2.2 is as follows: *If  $L \subseteq X^*$  is a semi-discrete language, then  $L$  is  $s$ -disjunctive.* However, as the proof of Proposition 2.2 depends on the fact  $|X| \geq 2$ , we must show it independently for the case  $|X| = 1$ .

---

(\*) Received November 1989, revised July 1990.

(<sup>1</sup>) Faculty of Science, Kyoto Sangyo University, Kyoto 603, Japan

LEMMA 1: Let  $X$  be a finite nonempty alphabet, let  $L \subseteq X^*$  be an  $m$ -disjunctive language and let  $u \in X^+$ . If  $L \cdot u^i \subseteq L \cdot u^j$  and  $L \cdot u^m \subseteq L \cdot u^n$  for some  $i, j, m, n \geq 0$ , then  $(j-i)(n-m) \geq 0$ .

*Proof:* Suppose that  $i < j$  and  $n < m$ . Let  $r, s$  be positive integers such that  $r(j-i) = s(m-n)$ . Then  $L \cdot u^{ri} \subseteq L \cdot u^{rj}$  and  $L \cdot u^{sm} \subseteq L \cdot u^{sn}$ . Therefore  $L \cdot u^{ri+sm} \subseteq L \cdot u^{rj+sm} \subseteq L \cdot u^{rj+sn}$ . Since  $L$  is disjunctive, we have  $u^{ri+sm} = u^{rj+sn}$ , i. e.  $i=j$ , a contradiction. This completes the proof of the lemma.

LEMMA 2: Let  $\gamma_1, \gamma_2, \dots, \gamma_k$  be positive integers whose greatest common divisor equals  $\gamma$ . Then there exists a positive integer  $s$  satisfying the following condition:

For every  $i \geq 0$ , there exist nonnegative integers

$$\alpha_1, \alpha_2, \dots, \alpha_k \quad \text{such that} \quad (s+i)\gamma = \alpha_1\gamma_1 + \alpha_2\gamma_2 + \dots + \alpha_k\gamma_k.$$

*Proof:* This follows from the definition of the greatest common divisor.

LEMMA 3: Let  $L \subseteq X^*$  be an  $m$ -disjunctive language, let  $u \in X^+$  and let  $L \cdot u^{m_i} \subseteq L \cdot u^{n_i}$  with  $n_i > m_i$  and  $1 \leq i \leq k$ . Let  $\gamma_i = n_i - m_i$  and let  $t > \max \{m_1, m_2, \dots, m_k\}$ . If  $u^t \in L$ , then  $u^{t+\beta_1\gamma_1+\beta_2\gamma_2+\dots+\beta_k\gamma_k} \in L$  for every  $\beta_1, \beta_2, \dots, \beta_k \geq 0$ .

*Proof:* Since  $u^t = u^{t-m_i}u^{m_i}$ ,  $t > m_i$  and  $L \cdot u^{m_i} \subseteq L \cdot u^{n_i}$ , we have  $u^{t-m_i}u^{m_i} = u^{t+\gamma_i} \in L$ . The lemma follows by induction.

In the following lemma,  $L$  is assumed to be an  $m$ -disjunctive language satisfying the conditions in Lemma 3. Moreover,  $s, \gamma, t$  and  $u$  are assumed to be the values and the element in  $X^+$  appearing in the preceding lemmas.

LEMMA 4: Let  $t < t'$  and let  $t \equiv t' \pmod{s\gamma}$ . If  $u^t \in L$ , then

$$u^{t'} \cup u^{t'}u^{s\gamma}(u^\gamma)^* \subseteq u^t \cup u^t u^{s\gamma}(u^\gamma)^* \subseteq L.$$

*Proof:* This follows immediately from the preceding lemmas.

Now we are ready to prove that every disjunctive language over an alphabet with one letter is  $s$ -disjunctive.

PROPOSITION 1: Let  $X$  be an alphabet with only one letter and let  $L \subseteq X^*$  be a disjunctive language. Then  $L$  is  $s$ -disjunctive.

*Proof:* Let  $X = \{a\}$ . Suppose that  $L$  is not  $s$ -disjunctive. Let  $H = \{n-m \mid n \neq m, L \cdot a^m \subseteq L \cdot a^n\}$ . Since  $L$  is  $m$ -disjunctive,  $H$  is not empty. From Lemma 1, we have the following two cases.

Case 1: For every  $h \in H$ ,  $h > 0$ . Let  $\gamma$  be the greatest common divisor of the numbers from  $H$ . Then there exist  $\gamma_1, \gamma_2, \dots, \gamma_k$  such that  $\gamma = g.c.d. \{ \gamma_1, \gamma_2, \dots, \gamma_k \}$  where  $\gamma_1, \gamma_2, \dots, \gamma_k \in H$  and  $\gamma_i = n_i - m_i$  with  $L..a^{m_i} \subseteq L..a^{n_i}$ . For every  $i$ ,  $1 \leq i \leq s\gamma$ , let  $t_i$  be the least integer such that  $\max \{ m_1, m_2, \dots, m_k \} \leq t_i$ ,  $t_i \equiv i \pmod{\gamma s}$  and  $a^{t_i} \in L$ . Let

$$I = \{ i \mid 1 \leq i \leq s\gamma \text{ and } t_i \text{ exists} \}.$$

Then by Lemma 4, we have  $L = (\bigcup_{i \in I} (a^{t_i} \cup a^{t_i} a^{s\gamma} (a^\gamma)^*)) \cup F$  where

$$F \subseteq \bigcup_{0 \leq j \leq T} a^j \text{ with}$$

$$T = \max \{ m_1, m_2, \dots, m_k \}.$$

Clearly  $L$  is regular, a contradiction.

Case 2: For every  $h \in H$ ,  $h < 0$ . Note that  $L..a^m \subseteq L..a^n$  implies  $(X^* \setminus L)..a^n \subseteq (X^* \setminus L)..a^m$ . As in Case 1, it can be shown that the language  $X^* \setminus L$  is regular. Hence  $L$  is regular, a contradiction. Therefore,  $L$  is  $s$ -disjunctive.

#### ACKNOWLEDGEMENTS

This work was done during the author's stay at The University of Western Ontario. The author would like to thank Prof G. Thierrin for his invitation and useful discussion. The author would like also to thank the referee for his valuable comments.

#### REFERENCES

- 1 M. ITO, H. J. SHYR, G. THIERRIN, Disjunctive Languages and Compatible Orders, *R.A.I.R.O. Inform. Théor. Appl.*, 1989, **23**, pp 149-163