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ON A SUBCLASS OF CONTEXT-FREE GROUPS (*)

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Abstract. – *We investigate those groups which are finite or finite extensions of* $\mathbb{Z}$. *We give several characterizations of this class in terms of formal languages and automata theory, and show that, from a formal language point of view, this is the most important class of groups between context-free and finite groups.*

Résumé. – *Nous examinons les groupes finis, ou qui sont extensions finies du groupe* $\mathbb{Z}$. *Nous donnons plusieurs caractérisations de cette classe, soit en termes de langages formels, soit en termes de la théorie des automates, et nous montrons que, du point de vue des langages formels, ces groupes forment la classe la plus importante entre les groupes context-free et les groupes finis.*

1. INTRODUCTION

Following Muller and Schupp [19] we define the word problem of a group G in a given presentation to be the set of words which are equivalent to the unit of G. This definition links in a natural way the theory of formal languages and the theory of groups. An easy lemma shows that under some mild assumption the complexity of the word problem is independent of the presentation and thus an invariant of the group. Therefore, every family of languages closed under inverse homomorphism defines a class of groups. It is an interesting task to give a group theoretical description of those groups which belong to a given family of languages closed under inverse homomorphism. Such description is known for regular and context-free languages: From Kleene's theorem together with the observation that the unit of a group is disjunctive we can conclude that the class of groups with a regular word problem is equal to the class of finite groups (a different proof of this

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statement was given in [2]). From the results in [19, 11] we can deduce that the class of groups with a context-free word problem is equal to the class of groups having a free subgroup of finite rank and finite index. In this paper we determine for any cone which is a subfamily of the context-free languages the corresponding class of groups. It is an interesting fact that, besides finite and context-free groups, only one further class of groups does occur. We call these groups one counter groups, since their word problem is a one counter language. One counter groups prove to be exactly those groups which have a free subgroup of finite index and rank at most 1. Examples are the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$ or any abelian group of rank at most 1. Moreover we give several further characterizations of one counter groups which are based on their combinatorial structure or on theorems similar to that of Kleene which hold exactly in one counter groups.

Another subclass of context-free groups, the so-called plain groups, was investigated in [15]. But the definition of this class of groups is not independent of the chosen presentation. The plain groups are incomparable with one counter groups.

This paper is organized as follows: in the next section we give the basic definitions so that we can state our main theorem in section 3. This theorem presents seven different characterizations of one counter groups. Furthermore, we discuss some of the results. The proof of the main theorem is given at the end of section 5. It is based on several propositions which we state and prove in the sections 4 and 5. In section 6 we point out the importance of one counter groups by the following fact: given a proper subcone $\mathfrak{C}$ of the family of context-free languages the class of groups whose word problem is in $\mathfrak{C}$ is either the class of finite groups or of one counter groups. From this point of view one counter groups prove to be the most important class of groups between context-free and finite groups. Furthermore, we show that given the word problem of a context-free group G it is decidable whether G is one counter. Finally, we are able to prove that the deterministic context-free groups are precisely the thin groups. This result is a special case of a conjecture of Sakarovitch stated in [21, 23].

2. PRELIMINARIES

In this paper X denotes always an alphabet, that is a finite nonempty set, and X^* the free monoid generated by X. In general, if T is a subset of a monoid M, T^* denotes the submonoid generated by T in M. If $T=\{t\}$ is a singleton, we write t^* and omit the brackets. If M is a group, $\langle T \rangle$ denotes the subgroup generated by T in M.

Let M be a monoid and A, B subsets of M. The *right quotients* AB^{-1} *of* A *by* B is the set $\{m \in M \mid \exists b \in B : mb \in A\}$. In groups the quotient and the inverse have the same meaning, that is $\{1\} B^{-1}$ is the set of inverses of elements of B. $\mathbb{N}$ is the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. $\mathbb{Z}$ denotes the set of integers.

Given a monoid $M \mathfrak{P}(M)$ is the powerset of M.

A *cone* is a family of languages closed under homomorphism, inverse homomorphism, and intersection with regular languages. $\mathfrak{L}_3$ is the family of regular languages and $\mathfrak{L}_2$ the family of context-free languages.

LEMMA 2.1 (*cf.* [21, 22]): *Let* M *be a finitely generated monoid,* Y *an alphabet,* $\varphi : X^* \to M$ *a homomorphism, and* $\psi : Y^* \to M$ *a surjective homomorphism. Then there is a homomorphism* $h : X^* \to Y^*$ *such that* $\psi h = \varphi$ *holds.*

Proof: Let $h(x) := w_x$ with $\varphi(x) = \psi(w_x)$ for all x in X.

The claim is an easy consequence of this definition then.

Lemma 2.1 gives the frame of this paper: Since for every $T \subseteq M$ we have $h^{-1}(\psi^{-1}(T)) = \varphi^{-1}(T)$, and for every $L \subseteq X^*$ such that $\varphi(L) = T$ holds we have $\psi(h(L)) = T$, the lemma above links in a natural way subsets of monoids and families of languages. This leads to

DEFINITION 2.2: Let M be a finitely generated monoid.

(*a*) Let $\mathfrak{C}$ be a family of languages closed under inverse homomorphism.

$$\mathfrak{C}^{\forall}(M) := \{T \subseteq M \mid \text{there is a surjective homomorphism } \varphi : X^* \to M \text{ and } \varphi^{-1}(T) \in \mathfrak{C}\}.$$

(*b*) Let $\mathfrak{C}$ be a family of languages closed under homomorphism.

$$\mathfrak{C}^{\exists}(M) := \{T \subseteq M \mid \text{there is a surjective homomorphism } \varphi : X^* \to M \text{ and } L \in \mathfrak{C} \text{ with } \varphi(L) = \mathrm{T}\}.$$

Lemma 2.1 asserts that the definitions above are independent of the alphabet and the homomorphism.

Next we introduce for some classes of subsets which will frequently occur in this paper a more comfortable notation.

DEFINITION 2.3: Let M be a finitely generated monoid.

(*a*) $Rec(M) := \mathfrak{L}_3^{\forall}(M)$.

(*b*) $Rat(M) := \mathfrak{L}_3^{\exists}(M)$.

(*c*) $CF(M) := \mathfrak{L}_2^{\forall}(M)$.

(*d*) $Alg(M) := \mathfrak{L}_2^{\exists}(M)$.

An element of Rat(M) is called *rational* and an element of Alg(M) *algebraic*.

The wellknown definitions of the classes Rec, Rat, and Alg which can be found for example in [12, 6, 7] are different from those definitions we have given here. But it is easy to see that the definitions are equivalent in finitely generated monoids. The advantage of our approach is the possibility to relate classes of subsets not only with regular or context-free languages but with any family of languages which is closed under homomorphism (inverse homomorphism resp.).

It is clear that the rational subsets of X^* are the regular languages.

If M is a free partially commutative monoid and the alphabet X is fixed, the definitions above are common in the theory of traces (*cf.* [1] for example). From the inclusion $\mathfrak{L}_3 \subseteqq \mathfrak{L}_2$ and the definitions we can conclude for every finitely generated monoid M:

$$\mathrm{Rec}(M) \subseteqq CF(M) \subseteqq \mathrm{Alg}(M)$$

and

$$\mathrm{Rec}(M) \subseteqq \mathrm{Rat}(M) \subseteqq \mathrm{Alg}(M).$$

The inclusions above need not be proper. For example in finite monoids we have the identity and in free monoids X^* we have $\mathfrak{C}^{\forall}(X^*) = \mathfrak{C}^{\exists}(X^*)$ for every family of languages $\mathfrak{C}$ which is closed under inverse homomorphism and homomorphism.

This paper deals with groups and we are able to characterize those groups G for which $CF(G) = \mathrm{Alg}(G)$ ($CF(G) = \mathrm{Rat}(G)$ resp.) holds. One way to do this is to define a family of languages which contains the inverse images of the unit elements of the groups.

DEFINITION 2.4: *Ocl* (or family of one counter languages) are those languages which are accepted by one counter pushdown automata as defined in [6]. In the usual way we can define a deterministic version of a one counter pushdown automaton. This leads to *Docl* (or family of deterministic one counter languages) which are those languages accepted by deterministic one counter pushdown automata.

It is shown in [6] that Ocl is a cone and a proper subfamily of $\mathfrak{L}_2$. Docl is closed under inverse homomorphism. To verify this, one may use the standard proof which shows the closure of context-free languages under inverse homomorphism (*cf.* [16] for example) in this special case. Thus we can define in any finitely generated monoid the classes $\mathrm{Ocl}^{\forall}(M)$ and $\mathrm{Docl}^{\forall}(M)$. For the sake of brevity we will write in this paper just $\mathrm{Ocl}(M)$ and $\mathrm{Docl}(M)$.

DEFINITION 2.5: Let G be a finitely generated group.

(*a*) G is a *context-free group* if $\{1\} \in CF(G)$.

(*b*) G is a *one counter group* if $\{1\} \in \mathrm{Ocl}(G)$.

Context-free groups were studied by Anisimov in [2]. Among others he showed that every finitely generated free group is context-free and that the abelian context-free groups are exactly the abelian groups with rank less or equal 1. Muller and Schupp gave in [19] a group theoretical description of context-free groups. A survey of the so far known results can be found in [4].

The next two definitions are from [21].

DEFINITION 2.6: Let M be a monoid and $T \subseteq M$. T is called *thin* if T is a finite union of subsets $uv^* w$ with $u, v, w \in M$.

M is called *thin* if M is a thin subset of itself.

DEFINITION 2.7: A monoid M is called *deterministic* if every context-free language whose syntactic monoid is isomorphic to M is deterministic context-free, and there is at least one of those.

Sakarovitch conjectured in [21, 23] that the thin syntactic monoids are precisely the deterministic monoids and proved this conjecture for abelian groups. (Recall that a monoid M is syntactic if there is a language L such that the syntactic monoid of L is isomorphic to M.) Certainly, every deterministic monoid is syntactic. But there are thin monoids which are not syntactic, since there are even finite monoids which are not syntactic (*cf.* [26]). In this paper we are able to prove this conjecture in case the monoid is a context-free group. In order to do so we show that thin groups are exactly the one counter groups and therefore have always a deterministic context-free word problem. This solves an open problem stated in [20].

3. THE MAIN THEOREM

THEOREM 3.1: *Let G be a finitely generated group. The following statements are equivalent:*

(*a*) *G is finite, or $\mathbb{Z}$ is subgroup of G with finite index.*

(*b*) *G is a one counter group.*

(*c*) $\{1\} \in \text{Docl}(G)$.

(*d*) $CF(G) = \text{Alg}(G)$.

(*e*) $\text{Ocl}(G) = \text{Alg}(G)$.

(*f*) $CF(G) = \text{Rat}(G)$.

(*g*) *G is thin.*

Theorem 3.1 provides several characterizations of one counter groups. We think that the statements (*c*) and (*d*) deserve some more comments.

Although in general the deterministic version of a type of automata is weaker than the nondeterministic one, for some types of automata the class of groups whose word problem is recognized by these automata does not depend on the determinism of the automata. This is the case for pushdown automata (*cf.* [19]) and one counter pushdown automata. The latter will be proven in this paper.

An example of a different behaviour is the supercounter machine which was investigated in [9]. Nondeterministic supercounter machines recognize all context-free groups, but from [9, prop. 4] we can conclude that there is no deterministic supercounter machine which recognizes the word problem of the free group generated by two elements.

The famous theorem of Kleene can be formulated as follows: In every finitely generated free monoid X^* we have $\text{Rat}(X^*) = \text{Rec}(X^*)$. For that reason we call a monoid M in which $\text{Rat}(M) = \text{Rec}(M)$ holds a *Kleene monoid.* There are monoids which aren't Kleene monoids, for example every infinite group. Therefore, it is an interesting task to characterize all Kleene monoids. A survey concerning this matter is given in [7].

Similar to the classical case we have $\text{Alg}(X^*) = CF(X^*)$ in finitely generated free monoids X^*. We call a monoid in which this identity holds *algebraic.* Every finite monoid is a Kleene monoid as well as algebraic. $\mathbb{Z}$ is not a Kleene monoid but is algebraic. The question which groups are algebraic monoids is answered by our main theorem.

Finally, we should remark that a group G in which $\text{Rat}(G) = \text{Alg}(G)$ holds need not be a one counter group. Take $\mathbb{Z}^2$ as a counterexample. From the results presented in this paper we can easily derive that $\text{Rat}(G) = \text{Alg}(G)$ holds in every group G which has a finitely generated abelian subgroup of finite index. But we don't know whether this condition is sufficient, too.

4. SOME BASIC OBSERVATIONS

LEMMA 4.1: *Let G be a finitely generated group, $R \in \mathrm{Rat}(G)$. Let $\mathfrak{C}$ be a cone.*

(*a*) $T \in \mathfrak{C}^{\exists}(G) \Rightarrow TR \in \mathfrak{C}^{\exists}(G)$.

(*b*) $T \in \mathfrak{C}^{\forall}(G) \Rightarrow TR \in \mathfrak{C}^{\forall}(G)$.

(*c*) $T \in \mathfrak{C}^{\forall}(G) \Rightarrow RT \in \mathfrak{C}^{\forall}(G)$.

Proof: Let $\varphi : X^* \to G$ be a surjective homomorphism.

(*a*) By definition there exists $L \subseteq X^*$, $L \in \mathfrak{C}$, such that $\varphi(L) = T$ holds, and a regular language $R' \subseteq X^*$ such that $\varphi(R') = R$. It follows $\varphi(LR') = TR$. $LR' \in \mathfrak{C}$, since cones are closed under concatenation by regular languages (*cf.* [6]).

(*b*) It is easy to show that $R^{-1} := \{ r^{-1} \in G \mid r \in R \} \in \mathrm{Rat}(G)$. By definition there exists a regular language $R' \subseteq X^*$ such that $\varphi(R') = R^{-1}$. Next note $\varphi^{-1}(TR) = \varphi^{-1}(T) R'^{-1}$. The claim is a consequence of the closure of cones under right quotients by regular languages (*cf.* [16]).

(*c*) is similar to (*b*).

From part (*b*) or (*c*) above we can draw the following conclusion.

LEMMA 4.2: *In a context-free group G we have* $\mathrm{Rat}(G) \subseteq CF(G)$.

Next we note a lemma which is an easy consequence of Lemma 2.1 (or of a lemma stated in [22]) but from which we can draw an important conclusion.

LEMMA 4.3: *Let M', M be finitely generated monoids and $\tau : M' \to M$ any homomorphism, $T \subseteq M$, $T' := \tau^{-1}(T) \subseteq M'$, Y an alphabet, and φ, ψ surjective homomorphisms, $\psi : Y^* \to M$, $\varphi : X^* \to M'$, $L_1 := \varphi^{-1}(T')$, $L_2 := \psi^{-1}(T)$.*

Then there exists a homomorphism $h : X^* \to Y^*$ such that $L_1 = h^{-1}(L_2)$ holds.

Proof: The following diagram explains the situation.

$$\begin{array}{ccc} L_1 \subseteq X^* & \xrightarrow{h} & Y^* \supseteq L_2 \\ \varphi \downarrow & & \downarrow \psi \\ T' \subseteq M' & \xrightarrow{\tau} & M \supseteq T \end{array}$$

The claim follows from Lemma 2.1, since the composition of φ and τ yields a homomorphism from X^* to M.

Considering the natural embedding leads to the following corollary.

COROLLARY 4.4: *Let G be a finitely generated group and U a finitely generated subgroup of G, $T \subseteq G$. Then the following holds:*

(*a*) $T \in \mathrm{Ocl}(G) \Rightarrow T \cap U \in \mathrm{Ocl}(U)$.

(*b*) $T \in CF(G) \Rightarrow T \cap U \in CF(U)$.

Corollary 4.4 plays an important role in the proof of the main theorem. We only stated it in a form we shall need in the proof. But certainly it can be generalized to monoids or other families of languages. The converse of this corollary isn't true in general: $\{1\} \in CF(\mathbb{Z})$, but $\{1\} \notin CF(\mathbb{Z}^2)$. In the next chapter we shall be able to prove a partial converse by imposing some restrictions on the subgroup U.

The last thing we want to do in this chapter is to take a look at free groups. In the following F_n denotes always the free group generated by n elements, $Z_n := \{x_1, x_2, \ldots, x_n, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}$, and φ the natural homomorphism from Z_n^* onto F_n.

It is a classical result (*cf.* [18]) that for all w in Z_n^* there is exactly one reduced word $\bar{w}$ with the following properties: $\varphi(w) = \varphi(\bar{w})$ and $\bar{w}$ contains no subword $x_i x_i^{-1}$ or $x_i^{-1} x_i$ for $1 \leqq i \leqq n$.

DEFINITION 4.5: *Let $L \subseteq Z_n^*$.*

$$Red(L) := \{\bar{w} \in Z_n^* \mid \bar{w} \text{ is reduced and there exists } w \in L \text{ such that } \varphi(w) = \varphi(\bar{w})\}.$$

LEMMA 4.6: *Let $T \subseteq F_n$.*

$T \in CF(F_n) \Leftrightarrow \mathrm{Red}(\varphi^{-1}(T))$ is context-free.

Proof: One direction follows from the fact that $\mathrm{Red}(Z_n^*)$ is a regular language and the other from [8, Theorem 2.2].

The result of Benois [5] that regular languages are closed under the Red-operator in connection with Lemma 4.6 leads to:

COROLLARY 4.7: *Let $T \in CF(F_n)$, $R \in \mathrm{Rat}(F_n)$. Then $T \cap R \in CF(F_n)$.*

From Lemma 4.2 we can deduce that $\mathrm{Rat}(F_n) \subseteq CF(F_n)$. An improvement is the following:

PROPOSITION 4.8: *Let $n \in \mathbb{N}$, $n \geqq 2$.*

Then $\mathrm{Rat}(F_n) \subset CF(F_n) \subset \mathrm{Alg}(F_n)$.

Proof: The first inclusion is shown in [21] and the second in [13]. We shall give new proofs which seem easier to us.

Let $T := \{ x_1^n x_2^n \mid n \in \mathbb{N} \}$.

By Lemma 4.6 we have $T \in CF(F_n)$. The rationality of T would imply by Benois' theorem the regularity of $\mathrm{Red}(\varphi^{-1}(T))$.

In order to show the second inclusion we use an example of [17].

Let $G = (\{ S, A \}, Z_n, P, S)$ be a grammar and

$$\begin{aligned} P = \{ & S \to x_1 S x_1^{-1}, \\ & S \to A, \\ & A \to x_1^{-1} A A x_1, \\ & A \to x_2 \}. \end{aligned}$$

$T := \varphi(L(G))$. Obviously $T \in \mathrm{Alg}(F_n)$. Assume $T \in CF(F_n)$. From Corollary 4.7 we obtain $T \cap x_2^* \in CF(F_n)$. But $T \cap x_2^* = \{ x_2^{2^n} \mid n \in \mathbb{N}_0 \}$, a contradiction with Lemma 4.6.

5. PROOF OF THE MAIN THEOREM

It is easy to see that a set which is rational or algebraic in a submonoid U of a monoid M has the same property in M if we assume U and M to be finitely generated. In the first part of this chapter we shall investigate the converse (or at least a partial converse) of this fact. We start with a lemma which we quote from [3].

LEMMA 5.1: *Let G be a group, $A \subseteq G$, $A = x_1 T_1^* x_2 T_2^* \ldots x_n T_n^* x_{n+1}$, where $x_i \in G$, $T_j \subseteq G$, $1 \leq i \leq n+1$, $1 \leq j \leq n$.*

Let $y_i := x_1 x_2 \ldots x_i$, then

$$\langle A \rangle = \langle y_{n+1}, y_1 T_1 y_1^{-1}, \ldots, y_n T_n y_n^{-1} \rangle.$$

PROPOSITION 5.2 (*cf.* [7, 13]): *Let G be a group, U a subgroup of G, $T \in \mathrm{Rat}(G)$, and $T \subseteq U$. Then $T \in \mathrm{Rat}(U)$.*

Proof: It is convenient for this proposition to assume that a rational subset is defined by a rational expressions (*cf.* [6] for example). We already mentioned that in finitely generated monoids this definition is equivalent to the definition given here.

Assume that there is a subgroup $U \subseteq H$, $T \in \mathrm{Rat}(G)$ defined by a rational expression with minimal starheight h, such that $T \notin \mathrm{Rat}(U)$.

Certainly $h>0$.

Without loss of generality we may assume $T=x_1 T_1^* x_2 T_2^* \dots x_n T_n^* x_{n+1}$ where $x_i \in G$, $1 \leqq i \leqq n+1$, $T_j \in \mathrm{Rat}(G)$, and T_j has starheight at most $h-1$ for every $1 \leqq j \leqq n$.

$T \subseteqq U$ implies $\langle T \rangle \subseteqq U$.

$S_i := x_1 \dots x_i T_i x_i^{-1} \dots x_1^{-1}$.

Using Lemma 5.1 we obtain $\langle T \rangle = \langle x_1 x_2 \dots x_{n+1}, S_1, S_2, \dots, S_n \rangle$.

Each S_i has starheight at most $h-1$. Therefore, we can apply Proposition 5.2 to S_i, and we have $S_i \in \mathrm{Rat}(U)$, $1 \leqq i \leqq n$. Furthermore, $x_1 x_2 \dots x_{n+1} \in U$ and $x_1 x_2 \dots x_{n+1} \in \mathrm{Rat}(U)$. Thus we conclude $T = S_1^* S_2^* \dots S_n^* x_1 x_2 \dots x_{n+1} \in \mathrm{Rat}(U)$ which contradicts our assumption.

We failed in proving the statement that arises if one replaces in the last proposition Rat by Alg. Luckily, for the proof of the main theorem a weaker version is sufficient, which can be deduced from the following lemma.

LEMMA 5.3: *Let G be a finitely generated group, N a normal subgroup of G with finite index, $\varphi : X^* \to N$ a surjective homomorphism, $\{1, d_2, \dots, d_n\}$ a set of coset representatives of N, $D := \{d_2, \dots, d_n\}$,* and $\psi : (X \cup D)^* \to G$ *the obvious surjective homomorphism. Then there exists a subsequential function* (cf. [6]) τ *from $(X \cup D)^*$ into itself with the following two properties:*

- $\psi(w) = \psi(\tau(w))$ *for all w in $(X \cup D)^*$,*
- $\tau(w) \in X^* \cup X^* D$ *for all w in $(X \cup D)^*$.*

Proof: N is normal in G. Hence there are $u_{i,j} \in X^*$ such that

$$\psi(d_i x_j) = \psi(u_{i,j} d_i) \tag{1}$$

and $z_{i,j} \in X^*$, $d_{i,j} \in D \cup \{\varepsilon\}$ such that

$$\psi(d_i d_j) = \psi(z_{i,j} d_{i,j}) \tag{2}$$

for all d_i, d_j in D, x_j in X.

The function τ maps every word to a kind of normal form which has the same value in the group, and which contains, at most, one letter of D at the right side. The equations above show that τ can be realized by a subsequential transducer S (*cf.* [6]) such that the set of states of S are the cosets of N, the next state function and the output function are defined by (1), (2), and the partial function ρ is the identity.

PROPOSITION 5.4: *Let G be a finitely generated group, N a normal subgroup of G with finite index, $\mathfrak{C}$ a cone, and $T \in \mathfrak{C}^{\exists}(G)$.*

Then $T \cap N \in \mathfrak{C}^{\exists}(N)$.

Proof: The notations are the same as in the proof of the last lemma.

Let $T \in \mathfrak{C}^{\exists}(G)$. Hence there exists $L \subseteq (X \cup D)^*$, $L \in \mathfrak{C}$, such that $\psi(L) = T$. It follows from Lemma 5.3 that $\psi(\tau(L) \cap X^*) = T \cap N$. Since cones are closed under subsequential functions [6] and intersection with regular languages, this implies the claim.

PROPOSITION 5.5: *Let G be a finitely generated group, N a normal subgroup of G with finite index, and $T \subseteq N$.*

(*a*) *Let $\mathfrak{C}$ be a cone and $T \in \mathfrak{C}^{\forall}(N)$. Then $T \in \mathfrak{C}^{\forall}(G)$.*

(*b*) $T \in \mathrm{Docl}(N) \Rightarrow T \in \mathrm{Docl}(G)$.

Proof: The notations are the same as in the proof of Lemma 5.3.

Clearly $\tau^{-1}(\varphi^{-1}(T)) = \psi^{-1}(T)$ holds. Thus it is sufficient to show that cones (the family of deterministic one counter languages resp.) are closed under the inverse of τ.

This is obvious for the first case, since τ^{-1} is a rational transduction (as well as τ) and cones are closed under rational transductions [6].

The second case cannot be treated in the same way, since deterministic one counter languages are not closed under homomorphism. But the underlying subsequential transducer of τ can be simulated in the finite control of a one counter automaton.

Let A be such automaton recognizing $\varphi^{-1}(T)$. We can construct a new automaton A' which recognizes $\tau^{-1}(\varphi^{-1}(T))$. A' carries out the rewriting process described by the equations (1) and (2) of Lemma 5.3 and simulates A. Certainly, if A was a deterministic one counter automaton, so is A'.

Now we are ready to prove our main theorem. The implications $(a) \Rightarrow (c)$, (e), (f) may be deduced from [22], but we give a different treatment here.

Proof (of Theorem 3.1): If G is finite, the claim is trivial. Therefore, assume G to be infinite. Let $\mathbb{Z}$ be a subgroup of G with finite index. It follows from a wellknown theorem in group theory that there is a subgroup Z' of $\mathbb{Z}$ that is normal in G and has finite index in G. Hence we can assume that $\mathbb{Z}$ is a normal subgroup of G with finite index.

$(a) \Rightarrow (c)$: Obviously $\{1\} \in \mathrm{Docl}(\mathbb{Z})$.

Then, by Proposition 5.5, $\{1\} \in \mathrm{Docl}(G)$.

$(c) \Rightarrow (b)$: trivial.

$(b) \Rightarrow (a)$: $\{1\} \in \mathrm{Ocl}(G)$. Thus, G is a context-free group. The results of [19, 11] imply that there exists a finitely generated free normal subgroup N of G with finite index. It follows from Corollary 4.4 that $\{1\} \in \mathrm{Ocl}(N)$. Thus $\mathrm{rank}(N)=1$, since otherwise the cone generated by the word problem of N would be the family of context-free languages.

$(a) \Rightarrow (d)$: $CF(G) \subseteq \mathrm{Alg}(G)$ follows from the definitions.

Let $T \in \mathrm{Alg}(G)$. $\mathbb{Z} \in \mathrm{Rec}(G)$, since the syntactic monoid of $\mathbb{Z}$ is finite. The same holds for every coset of $\mathbb{Z}$.

Hence $T \cap \mathbb{Z} g \in \mathrm{Alg}(G)$ for all g in G. This gives a partition of T into finitely many disjunct subsets $T_1, T_2, \ldots, T_k$ according to the finitely many cosets of $\mathbb{Z}$ and $T_i \in \mathrm{Alg}(G)$, $1 \leqq i \leqq k$. To prove the claim it is sufficient to show $T_i \in CF(G)$ for each T_i.

Therefore, let $T_i \subseteq \mathbb{Z} g$, $T_i \in \mathrm{Alg}(G)$. It follows from Lemma 4.1 that $T_i g^{-1} \in \mathrm{Alg}(G)$. Moreover, $T_i g^{-1} \subseteq \mathbb{Z}$. We conclude from Proposition 5.4 that $T_i g^{-1} \in \mathrm{Alg}(\mathbb{Z})$. Parikh's theorem (*cf.* [14]) gives $T_i g^{-1} \in \mathrm{Rat}(\mathbb{Z})$, since $\mathbb{Z}$ is a commutative monoid. Thus, $T_i g^{-1} \in \mathrm{Rat}(G)$. It follows from Lemma 4.1 that $T_i g^{-1} g = T_i \in \mathrm{Rat}(G)$. Thus $T_i \in CF(G)$ by Lemma 4.2.

$(d) \Rightarrow (a)$: $\{1\} \in CF(G)$. Therefore, G is a context-free group and the results from [19, 11] imply that there exists a finitely generated free normal subgroup N of G with finite index.

Assume that N has rank more or equal 2. It follows from Proposition 4.9 that there is a $T \subseteq N$, $T \in \mathrm{Alg}(N)$, $T \notin CF(N)$. Certainly, $T \in \mathrm{Alg}(G)$. We conclude from Corollary 4.4 that $T \notin CF(G)$ which contradicts our assumption.

$(a) \Rightarrow (f)$: same as $(a) \Rightarrow (d)$.

$(f) \Rightarrow (a)$: analogously to $(d) \Rightarrow (a)$.

$(e) \Rightarrow (a)$: trivial.

$(a) \Rightarrow (e)$: So far we have $\mathrm{Rat}(G)=\mathrm{Alg}(G)$. Therefore it is sufficient to show $\mathrm{Rat}(G) \subseteq \mathrm{Ocl}(G)$. But that is an easy consequence of Lemma 4.1 (b).

$(a) \Rightarrow (g)$: trivial.

$(g) \Rightarrow (a)$: Let $G = u_1 v_1^* w_1 \cup \ldots \cup u_n v_n^* w_n$, $1 \leqq i \leqq n$.

It follows

$$G=(u_1 v_1 u_1^{-1})^* u_1 w_1 \cup \ldots \cup (u_n v_n u_n^{-1})^* u_n w_n$$
$$=\langle u_1 v_1 u_1^{-1} \rangle u_1 w_1 \cup \ldots \cup \langle u_n v_n u_n^{-1} \rangle u_n w_n.$$

Therefore we can assume that there are a_i, $g_i \in G$ such that

$$G=\langle a_1 \rangle g_1 \cup \ldots \cup \langle a_n \rangle g_n, \qquad 1 \leqq i \leqq n.$$

Each $\langle a_i \rangle$ is a cyclic subgroup of G. We can gather all finite $\langle a_i \rangle g_i$ to get a finite set E. Hence

$$G=E \cup \langle b_1 \rangle g'_1 \cup \ldots \cup \langle b_m \rangle g'_m$$

and

$$\langle b_i \rangle \cong \mathbb{Z}, \qquad 1 \leqq i \leqq m.$$

$\langle b_1 \rangle \cap \langle b_i \rangle$ is either trivial or isomorphic to $\mathbb{Z}$ and in the latter case a subgroup of $\langle b_1 \rangle$ and of $\langle b_i \rangle$ with finite index.

Therefore there exists c, c_i, g''_i, $h_j \in G$, $1 \leqq i \leqq p$, $1 \leqq j \leqq k$, such that

$$G=E \cup \langle c \rangle \{ h_1, \ldots, h_k \} \cup \langle c_1 \rangle g''_1 \cup \ldots \cup \langle c_p \rangle g''_p$$

and $\langle c \rangle \cap \langle c_i \rangle = \{ 1 \}$ for every $1 \leqq i \leqq p$.

Now assume that G is no one counter group.

Then $G \neq E \cup \langle c \rangle \{ h_1, \ldots, h_k \}$ and there exists $h \in G$, $h \notin \{ h_1, \ldots, h_k \}$. $\langle c \rangle h$ contains infinitely many elements not in $E \cup \langle c \rangle \{ h_1, \ldots, h_k \}$. Hence there exists $1 \leqq i \leqq p$ such that $\langle c \rangle h \cap \langle c_i \rangle g''_i$ is infinite. Especially there are two different elements in both cosets.

Thus, $c^q h = c_i^r g''_i$ and $c^s h = c_i^t g''_i$, where $q \neq s$, $r \neq t \in \mathbb{Z}$. It follows $c^{s-q} = c_i^{t-r}$, which contradicts the assumption $\langle c \rangle \cap \langle c_i \rangle = \{ 1 \}$.

6. PROPERTIES OF ONE COUNTER GROUPS

Let Z_i, F_i, and φ defined as in chapter 4.

$D_i^* := \varphi^{-1}(1)$ where $i=1, 2$.

We start with a remark concerning the significance of one counter groups. From a certain formal language point of view they prove to be the most important subclass between finite and context-free groups. A precise formulation is given in Proposition 6.2. To prove this we need the following statement.

LEMMA 6.1: *Let $\mathfrak{C}$ be a family of languages closed under inverse homomorphism, $\mathfrak{C} \subseteqq \mathfrak{L}_2$.*

(*a*) *If there is an infinite group G such that $\{1\} \in \mathfrak{C}^{\forall}(G)$, then $D_1^* \in \mathfrak{C}$.*

(*b*) *If there is a group G, G no one counter group, such that $\{1\} \in \mathfrak{C}^{\forall}(G)$, then $D_2^* \in \mathfrak{C}$.*

Proof: (*a*) G is context-free and infinite. Thus F_1 is subgroup of G. The claim follows from Lemma 4.3 if we consider the natural embedding.

(*b*) is similar to (*a*).

PROPOSITION 6.2: *Let $\mathfrak{C}$ be a cone, $\mathfrak{C} \subset \mathfrak{L}_2$. Let $\mathfrak{G}$ be the class of all groups whose word problem is in $\mathfrak{C}$. Then $\mathfrak{G}$ is the class of finite or one counter groups. Moreover, the following two statements are equivalent:*

(*a*) $D_1^* \in \mathfrak{C}$.

(*b*) *$\mathfrak{G}$ is the class of one counter groups.*

Proof: Every group, whose word problem is in $\mathfrak{C}$, is a context-free group and therefore has a free subgroup of finite index. Let m be the maximum of the ranks of all those subgroups.

If $m=0$, then every group in $\mathfrak{G}$ is finite. On the other hand, every finite group is in $\mathfrak{G}$, since every cone contains $\mathfrak{L}_3$.

If $m=1$, then, by Lemma 6.1, $D_1^* \in \mathfrak{C}$. We derive from Proposition 5.5 (*a*) that $\mathfrak{G}$ is the class of one counter groups.

The case $m>1$ is impossible: assume $m>1$.

It follows from Lemma 6.1 (*b*) that $D_2^* \in \mathfrak{C}$. But D_2^* is a cone generator of $\mathfrak{L}_2$ (*cf.* [6]).

The property of a context-free group G to be a one counter group is decidable if the word problem of G is given. Observe that the word problem of G is always deterministic context-free (*cf.* [19, 11]). We give an algorithm which also can be seen as a partial solution of the more general problem to decide whether a deterministic context-free language is one counter which is to our knowledge still an open problem (*cf.* [25]).

PROPOSITION 6.3: *Let G be a finitely generated group, $\varphi : X^* \to G$ a surjective homomorphism, and $L := \varphi^{-1}(1)$ a deterministic context-free language which may be given by a deterministic pushdown automaton. Then it is decidable whether G is a one counter group.*

Proof: In a first step we test whether L is regular (*cf.* [24]). Note that L is regular iff G is finite.

If L is not regular, we determine $w \in X^*$ such that $\varphi(w)$ has infinite order in G by testing whether $w^* w \cap L$ is empty. Such w does exist, since every periodic context-free group is finite.

Next we search for $w' \in X^*$ such that $ww' \in L$. This can be done by enumerating the words in X^* and testing successively the membership. At last we test whether $L(\{w, w'\}^*)^{-1}$ is regular which is equivalent to testing whether $\langle \varphi(w) \rangle$ has finite index.

Now we come back to the conjecture of Sakarovitch. If the monoid is a context-free group we are able to prove it. Observe that every finitely generated group is syntactic.

PROPOSITION 6.4: *Let G be a context-free group.*

G is deterministic $\Leftrightarrow$ G is thin.

Proof: "$\Leftarrow$" By Theorem 3.1 we deduce that G is one counter and that the image of every context-free language in G is a rational set. The remainder of the proof can be done analogously to the proof of Lemma 4.1 (*b*), since deterministic context-free languages are closed under right quotient with regular languages.

"$\Rightarrow$" Assume that G is not thin and therefore not one counter. Then there exists a free normal subgroup $F = F(x_1, x_2, \ldots, x_n)$ of finite index in G and $n \geqq 2$.

$T := \{ x_1^n x_2^n \mid n \in \mathbb{N}_0 \}$.

Let $\varphi : X^* \to G$ be a surjective homomorphism and $L := \varphi^{-1}(T)$.

It follows from [26] that the syntactic monoid of L is isomorphic to G iff T is a disjunctive subset. (Recall that a subset T is disjunctive in G if for all $a \neq b$ in G there are $u, v \in G$ such that $uav \in T \Leftrightarrow ubv \notin T$).

It is an easy exercise to show that T is disjunctive.

Using Lemma 4.6 and Proposition 5.5 (*a*) we obtain $T \in CF(G)$. Assume that L is already deterministic context-free. It follows from Lemma 4.3 that the inverse image of T in F is deterministic context-free. But that is a contradiction with an example given in [21, p. 141].

Proposition 6.4 could be extended to all kinds of groups if it is possible to show that a group which is syntactic monoid of a deterministic context-free language, is always a context-free group. But we failed in proving this.

The last proposition we state in this paper hasn't that much to do with one counter groups but is more a property of context-free groups which are not one counter. Nevertheless we find it worth mentioning.

PROPOSITION 6.5: *Let G be a context-free group, G not one counter group. Let $U \leqq G$ be a finitely generated subgroup of G, $U \notin \mathrm{Rec}(G)$ (or equivalent, U has infinite index in G). Let $\emptyset \neq T \subseteqq U$ be any nonempty subset of U and $\mathfrak{C}$ the cone generated by the inverse image of T.*

Then $\mathfrak{L}_2 \subseteqq \mathfrak{C}$.

Proof: To prove the claim it is sufficient to show that $D_2^* \in \mathfrak{C}$.

G is context-free and not one counter. Therefore there exists a free normal subgroup F of finite index and finite rank n in G with $n \geqq 2$.

Let $H := U \cap F$. H is free and of finite rank, since every subgroup of a free group is free [18] and the intersection of a finitely generated subgroup and a subgroup of finite index is finitely generated [13].

$U \notin \mathrm{Rec}(G)$ and therefore $H \notin \mathrm{Rec}(F)$.

By Lemma 4.1 (*b*) we may assume without loss of generality $\{1\} \in T$ and therefore $T \cap H \neq \emptyset$.

Let $\{h_1, \ldots, h_k\}$ be a free generating system of H.

First, assume $k \geqq 2$.

According to [10] there is a free generating system $\{h_1, \ldots, h_k, g_1, \ldots, g_m\}$ of a subgroup N of finite index in F and $m \geqq 1$, since $H \notin \mathrm{Rec}(F)$.

Let $F_2' := \langle h_1, h_2 \rangle$ and let $\tau : F_2' \to G$ be the natural embedding. We can deduce from Lemma 4.1 (*b*) and (*c*) that $T' := g_1^{-1} T g_1 \in \mathfrak{C}^{\forall}(G)$. But $\tau^{-1}(T') = \{1\}$. Hence, by Lemma 4.3, $D_2^* \in \mathfrak{C}$.

Now assume $k = 1$. This implies $\emptyset \neq T \cap F \subseteqq \langle h_1 \rangle$.

Let F_2 be the free group generated by x_1 and x_2, and $h \in F$ such that $h_1 h \neq h h_1$. Such h exists, since $\mathrm{rank}(F) \geqq 2$.

$\rho : F_2 \to G$ defined by $\rho(x_1) = h_1 h$, $\rho(x_2) = h h_1$.

From this definition we conclude $\rho^{-1}(T) = \{1\}$ and furthermore, by Lemma 4.3, $D_2^* \in \mathfrak{C}$.

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