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ON EQUALITY SETS OF MORPHISMS IN TOPOLOGICAL FREE MONOIDS (*)

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Abstract. – *In a free monoid endowed with the finite group topology we consider an equality set of morphisms and prove that it is nowhere dense.*

Résumé. – *Dans un monoide libre muni de la topologie de groupes finis nous considérons l'ensemble d'égalité de morphismes et nous démontrons qu'il est dense nulle part.*

1. INTRODUCTION

Morphisms of free monoids and their equality sets constitute a classical topic within formal language theory. Being directly connected with Post correspondence problem and some decision problems in the theory of L systems they have been of great interest to language theoretists. Also from the mathematical point of view equality sets, as sets of solutions of morphic equations in free monoids, are well motivated objects for research.

We consider a free monoid A^* endowed with the finite group topology. This topology was introduced by M. Hall Jr in 1950 [2] and then Ch. Reutenauer extended this notion to the case of free monoids [4]. The aim of this paper is to establish a topological property of an equality set of morphisms defined on free monoids endowed with the finite group topology. The result is that it is a nowhere dense subset of A^*. This implies the nowhere density of a set of fixed points of a morphism.

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2. PRELIMINARIES

Now we briefly recall definitions, notations and the main properties of introduced notions that are used throughout this paper. For the more detailed description of these notions the reader is referred to [1, 2, 3].

Let A be any set, finite/or not, and let A^* denotes a free monoid generated by A. Let X be a set and let $\mathcal{F}=(f_i)_{i \in I}$ be a family of mappings $f_i: X \to (Y_i, \tau_i)$ where (Y_i, τ_i) is a topological space for any $i \in I$. The initial topology on X defined by the family $\mathcal{F}$ is the coarsests topology such that each f_i is a continuous mapping. The base for this topology is formed by finite intersections of subsets $f_i^{-1}(P_i)$ where for any $i \in I P_i$ is an open set of τ_i.

Definition 1: The finite group topology on A^* is the initial topology defined by the class of all monoid morphisms $f: A^* \to G$ where G is a finite discrete group.

Similarly, we define a p-adic topology on A^*.

Definition 2: Let p be a prime number. The p-adic topology on A^* is the initial topology defined by the class of all monoid morphisms $f: A^* \to G$ where G is a finite discrete p-group.

In what follows we shall denote by τ the finite group topology and by τ_p the p-adic topology of a free monoid A^*.

The following facts are true for the defined above topologies [2, 3].

Fact 1: A free monoid A^* endowed with the topology τ (τ_p) is a Hausdorff space and a topological monoid (multiplication continuous).

Fact 2: Every morphism $f: A^* \to B^*$ between two free monoids is continuous with respect to $\tau(\tau_p)$.

Fact 3: For any $w \in A^*$

$$\lim_{n \to \infty} w^{n!} = 1 \quad \text{in } \tau,$$

$$\lim_{n \to \infty} w^{p^n} = 1 \quad \text{in } \tau_p.$$

Now we recall the notion of an equality set.

Definition 3: Let $h: A^* \to B^*$ and $g: A^* \to B^*$ be any two morphisms. A set

$$\mathrm{Eq}(g, h) = \{w \in A^* : g(w) = h(w)\}$$

is called an equality set of morphisms g and h.

For the sake of completness we recall.

DEFINITION 4: A subset X of any topological space is nowhere dense if $\operatorname{int}\overline{X}=\varnothing$ where the bar denotes a topological closure of a set X.

3. RESULT

In this section we prove that any equality set which is a proper subset of A^* is nowhere dense. First we state some properties of an equality set.

LEMMA 1: *Let A^* be a free monoid endowed with the finite group topology τ (p-adic topology τ_p). Any equality set $Eq(g, h)$ is closed.*

The standard proof using the fact that A^* is a Hausdorff space and the equality $\operatorname{Eq}(g, h)=(g, h)^{-1}(\Delta)$ where $\Delta=\{(w, w): w\in A^*\}$ is omitted.

Notice that taking into account the result of Lemma 1 to obtain nowhere density of $\operatorname{Eq}(g, h)$ it is enough to prove that $\operatorname{int}\operatorname{Eq}(g, h)=\varnothing$.

LEMMA 2: *For any equality set* $\operatorname{Eq}(g, h)$ *it holds:*

1° *if* $u^p\in\operatorname{Eq}(g, h)$ *then* $u\in\operatorname{Eq}(g, h)$ *for* $u\in A^*$.

2° *if* u, $uv\in\operatorname{Eq}(g, h)$ *then* $v\in\operatorname{Eq}(g, h)$ *for* $u, v\in A^*$.

Proof: For the proof of 1° notice that $u^p\in\operatorname{Eq}(g, h)$ implies that $h(u)^p=g(u)^p$. Because the word $w=h(u)^p=g(u)^p$ admits a unique factorization as a product of letters of A it follows that $h(u)$ and $g(u)$ have the same length and finally $h(u)=g(u)$.

The second implication is justified by the observation that directly from the assumptions we have $h(u)h(v)=g(u)g(v)$ what in view of the equality $h(u)=g(u)$ and from the unique factorization property of A^* gives as the result $v\in\operatorname{Eq}(g, h)$.

Now we are ready to formulate the following.

THEOREM: *Let A^* be a free monoid endowed with the finite group topology τ (p-adic topology τ_p) and $g: A^*\to B^*$ and $h: A^*\to B^*$ be any, but different, morphisms. An equality set $Eq(g, h)$ is a nowhere dense subset of A^*.*

Proof: The proof is given for τ, but for τ_p is similar. We have to show that for an equality set $\operatorname{Eq}(g, h)$ the condition $\operatorname{int}\overline{\operatorname{Eq}(g, h)}=\varnothing$ is true. But in view of Lemma 1 it is enough to prove that $\operatorname{int}\operatorname{Eq}(g, h)=\varnothing$. Let us assume for the contrary that $\operatorname{int}\operatorname{Eq}(g, h)=A^*\setminus\overline{(A^*\setminus\operatorname{Eq}(g, h))}$ is non-empty. Thus there exists $w\in A^*$ such that $w\in L$ and there is no sequence of elements from $A^*\setminus\operatorname{Eq}(g, h)$ which convergences to w. Let $u\in A^*$ be such a word that $u\notin\operatorname{Eq}(g, h)$. It is clear by Lemma 2 that $u^{n!}\notin\operatorname{Eq}(g, h)$ for any $n\in\mathbb{N}$.

Now $u^{n!}w$ converges (in τ) to w if $n \to \infty$. Thus $w \in \overline{(A^* \setminus \mathrm{Eq}(g, h))}$ what creates a contradiction. Finally we obtain the equality $\mathrm{int}\,\mathrm{Eq}(g, h) = \varnothing$ what finishes the proof.

Remark 1: The reciprocal statement to the Theorem is not true. Let us consider for example a set $P = \{a^P : P \text{ is a prime number}\}$. Of course $P = a^* \setminus \bigcup_{p, q > 2} (a^{pq})^*$. Each set $(a^{pq})^*$ is open as an inverse image of an open set $\{O\}$ under a continuous morphism $f : a^* \to \mathbb{Z}/(pq)\mathbb{Z}$ defined by $h(a) = 1$. Hence P is closed [4]. In the same manner as above one can show that P is nowhere dense. It is clear that there are no morphisms g and h such that $\mathrm{Eq}(g, h) = P$.

Remark 2: Actually even nowhere dense free submonoid of A^* not necessarily have to be an equality set of some morphisms g and h. Take as an example $(A^n)^*$ for any fixed even n. It is closed (and also open) subset of A^*. Let us fix a word $u \notin (A^n)^*$ of an odd length. Now let us consider a sequence $u^{k!}$ for $k \in \mathbb{N}$. Because the number 2 is a prime element in a ring $\mathbb{Z}$ it follows that $u^{k!} \notin (A^n)^*$. Using once again the idea of the proof of Theorem we come to the conclusion that $(A^n)^*$ is nowhere dense. If g and h are morphisms such that $\mathrm{Eq}(g, h) = (A^n)^*$ then for any $a \in A$ $g(a^n) = h(a^n)$. This implies $g(a) = h(a)$ and consequently $g = h$, a contradiction.

Finishing our considerations let us formulate the conclusion from Theorem regarding fixed points of a morphism defined on free monoid.

Let $h : A^* \to A^*$ be any morphism and $e : A^* \to A^*$ denotes an identity on A^*. Of course

$$\mathrm{Eq}(h, e) = Fp\,h = \{w \in A^* : h(w) = w\}.$$

Hence from Theorem we can derive the following.

COROLLARY: *Let A^* be a free monoid endowed with the finite group topology τ (p-adic topology τ_p) and $h : A^* \to A^*$ any morphism, but no an identity e. Then the set $Fp\,h$ of fixed points h is nowhere dense.*

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