

BERTRAND LE SAEC
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SATURATING RIGHT CONGRUENCES (*)

by Bertrand LE SAEC (¹)

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Abstract. – *One of the main classes of automata recognizing rational ω -languages is the family of Muller automata. Their recognition criterion is based on "state tables". We propose a criterion based on "transition tables". The transition tables automata recognize exactly the class of rational ω -languages. Their main interest is that, when they are deterministic, they are isomorph to a family of finite right congruences satisfying a "property of saturation" similar to those defined for congruences by A. Arnold, J. R. Buchi.*

Résumé. – *Une des principales classes d'automates reconnaissant les ω -langages rationnels est la famille des automates de Müller. Leur critère de reconnaissance est basé sur la notion de « tables d'états ». Nous proposons un critère basé sur des « tables de transitions ». Les automates à tables de transitions reconnaissent exactement la famille des ω -langages rationnels. Leur principal intérêt est d'être, lorsqu'ils sont déterministes, isomorphes à une famille de demi-congruences satisfaisant une propriété de « saturation » analogue à celle utilisée pour les congruences par A. Arnold, J. R. Büchi.*

INTRODUCTION

The theory of rational set of infinite words was investigated as a study of asymptotic behaviours of finite automata [BU (62), MN (66)]. One of the main classes of automata is the family of Muller automata. The criterion used for recognition is based on the notion of "states table". We propose to use a criterion based on "transitions tables". The table-transition automata and the deterministic table-transition automata also recognize exactly the class of rational ω -languages. They are smaller than classical Muller automata, but their main interest is that, when they are deterministic, they are isomorphic to a family of finite right congruences satisfying a "property of saturation"

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(¹) Université de Bordeaux-I. Unité associée au C.N.R.S. 226, 351, cours de la Libération, 33405 Talence Cedex, France.

similar to those defined for congruences [AR (85), BU (62), PE (84)]. Moreover, this “syntactic” approach of deterministic automata provides a clearer version of the property of L. H. Landweber to characterize deterministic ω -languages [LA (69)].

In part 1, we recall classical definitions and results of the theory of rational ω -languages. In part 2, we present the table-transition automata. In part 3, we study connections between right congruences and deterministic transition systems. In part 4, we introduce the “saturating right congruences” and prove that they are isomorphic to the deterministic table-transition automata. In part 5, we characterize the deterministic table-transition which recognize deterministic ω -languages. In part 6, we define the “Landweber right congruences” to characterize the deterministic ω -languages.

1. PRELIMINARIES

Let X be an finite alphabet. We denote by X^* the set of finite words on X , by ε the empty word and by X^+ the set $X^* - \{\varepsilon\}$. We denote by X^ω the set of infinite words on X . Let $B \subseteq X^+$, we denote by B^ω (resp. B^*) the set of infinite (resp. finite) words $w = w_1 w_2 \dots$ such $w_i \in B$.

Let $w \in X^* \cup X^\omega$, we denote by $LF(w)$ the set of the left factors of w defined by: $LF(w) = \{w_1 \in X^* / \exists w_2 \in X^* \cup X^\omega, w = w_1 w_2\}$. We denote by $|w|$ the length of w . We say that a language $L \subseteq X^*$ (Resp: $L \subseteq X^\omega$) is prefix iff $\forall w \in L, LF(w) \cap L = \{w\}$.

A *transition system* on X is a tuple $\mathcal{S} = (Q, I, \Delta)$ where Q is a finite set of states, $I \subseteq Q$ a set of initial states and $\Delta \subseteq Q \times X \times Q$ a set of transitions.

A transition system \mathcal{S} is *complete* iff $\forall q \in Q, \forall a \in X, \exists q' \in Q / (q, a, q') \in \Delta$.

A transition system \mathcal{S} is *deterministic* iff $\text{Card}(I) = 1$ and $\forall (q, q', q'') \in Q \times Q \times Q, \forall a \in X, (q, a, q') \in \Delta$ and $(q, a, q'') \in \Delta \Rightarrow q' = q''$.

A finite (resp. infinite) *computation* of $w \in X^*$ (Resp. X^ω) in the transition system \mathcal{S} is a finite (resp. infinite) sequence of transitions $d = \delta_0 \delta_1 \delta_2 \dots$ such that $\delta_1 = (q_1, a_1, p_1) \in \Delta, q_0 \in I$ and for $i \geq 0, p_i = q_{i+1}$ and $w = a_0 a_1 \dots$. We denote $t\text{-inf}(d) = \{\delta \in Q / \text{Card}(\{i \in \mathbb{N} / \delta_i = \delta\}) = \infty\}$ and $q\text{-inf}(d) = \{q \in Q / \text{Card}(\{i \in \mathbb{N} / q_i = q\}) = \infty\}$. If \mathcal{S} is complete deterministic, we can use $t\text{-inf}(w)$ for $t\text{-inf}(d)$ and $q\text{-inf}(w)$ for $q\text{-inf}(d)$.

A *Muller automaton* is a tuple $\mathcal{A}_M = (Q, I, \Delta, \mathcal{E})$ where $\mathcal{S} = (Q, I, \Delta)$ is a transition system and $\mathcal{E} \subseteq \mathcal{P}(Q)$ is a family of subsets of Q . The automaton \mathcal{A}_M is deterministic if \mathcal{S} is deterministic.

We say that the Muller automaton \mathcal{A}_M recognize $w \in X^\omega$ iff there exists a computation d of w in \mathcal{A}_M such that $q\text{-inf}(d) \in \mathcal{E}$.

We denote by:

Rat, the family of rational languages of X^* .

RAT, the family of finite unions of ω -languages CB^ω where C and B are rational languages of X^* .

D-RAT, the family of finite unions of ω -languages CB^ω where C and B are prefix rational languages of X^* .

M-REC, the family of ω -languages recognized by a Muller automaton.

DM-REC, the family of ω -languages recognized by a deterministic Muller automaton.

Let $K \in X^*$. We denote by \vec{K} the set $\{w \in X^\omega / \text{Card}(LF(w) \cap K) = \infty\}$.

THEOREM 1.1: [EI (74)]

- IF $K \in X^*$ is prefix then $\vec{K} = \emptyset$ and $\vec{K}^* = K^\omega$.
- $L \in D\text{-RAT} \Leftrightarrow \exists K \in \text{Rat} / \vec{K} = L$.

We have the fundamental results due to R. McNaughton and R. Buchi:

THEOREM 1.2: [BU (62), EI (74), MN (66)]

- *RAT* is a boolean algebra.
- $M\text{-REC} = DM\text{-REC} = \text{RAT} = (D\text{-RAT})^B$ where $(F)^B$ denote the boolean closure of F .

We say that $w \in X^\omega$ is an ultimately periodic word iff there exists $(x, y) \in X^* \times X^*$ such that $w = xy^\omega$. We denote $UP(L)$ the set of all the ultimately periodic words of L .

BUCHI'S LEMMA: [BU (62)]

If L and L' are rational ω -languages, we have:

$$L \neq \emptyset \Leftrightarrow UP(L) \neq \emptyset \quad \text{and} \quad L \subseteq L' \Leftrightarrow UP(L) \subseteq UP(L').$$

2. TABLE-TRANSITION AUTOMATA

Let \mathcal{S} be a transition system, we say that a set T of transitions of \mathcal{S} is coherent iff

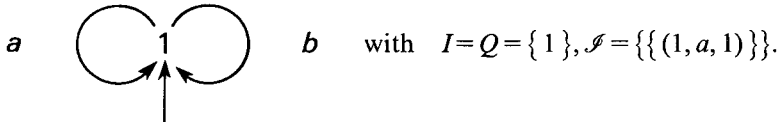
$$\begin{aligned} &\forall \delta = (q', a', p') \in T, \quad \forall \delta' = (q'', a'', p'') \in T, \\ &\exists \delta_1, \delta_2, \dots, \delta_n \in T / \forall 1 \leq i \leq n, \delta_i = (q_i, a_i, q_{i+1}), \quad q_1 = p', \quad q_{n+1} = q''. \end{aligned}$$

A *Table-transition automaton* is a tuple $\mathcal{A} = (Q, I, \Delta, \mathcal{S})$ where $\mathcal{S} = (Q, I, \Delta)$ is a transition system and $\mathcal{S} \subseteq \mathcal{P}(\Delta)$ is a family of coherent subsets of Δ .

We say that an infinite word w is recognized by the table-transition automaton \mathcal{A} iff there exists a computation $\delta_1 \delta_2 \delta_3 \dots$ of w in \mathcal{S} such that $t\text{-inf}(\delta_1 \delta_2 \delta_3 \dots) \in \mathcal{I}$.

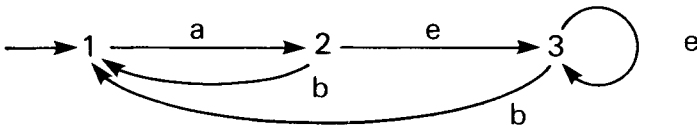
Note: If a set of transitions T is not coherent, no infinite word w can satisfy $t\text{-inf}(w, \delta_1 \delta_2 \delta_3 \dots) = T$ so we assume that all the elements of \mathcal{S} are coherent.

Examples: Let us consider the table-transition automaton \mathcal{A} :



The automaton \mathcal{A} recognizes the rational ω -language $L(\mathcal{A}) = \{a, b\}^* a^\omega$.

Let us consider the table-transition automaton \mathcal{A}' :



with $Q' = \{1, 2, 3\}, I' = \{1\}$ and $\mathcal{S} = \{(1, a, 2), (2, e, 3), (3, b, 1)\}$.

The automaton \mathcal{A}' recognizes the rational ω -language $L(\mathcal{A}') = (ae^* b)^* (aeb)^\omega$. \square

DÉFINITIONS : Let us denote:

T-REC: the family of ω -languages recognized by a table-transition automaton.

DT-REC: the family of the ω -languages recognized by a deterministic table-transition automaton.

THEOREM 2.1:

- DT-REC is a boolean algebra.
- DT-REC = T-REC = RAT.

Proof

1 a. Let $\mathcal{A} = (Q, I, \Delta, \mathcal{S})$ be a complete deterministic table-transition automaton.

It's clear that the deterministic table-transition automaton $\mathcal{A}' = (Q, I, \Delta, \mathcal{P}(\Delta) - \mathcal{S})$ recognizes the complement of $L(\mathcal{A})$.

b. Let $\mathcal{A}_1 = (Q_1, I_1, \Delta_1, \mathcal{S}_1)$ and $\mathcal{A}_2 = (Q_2, I_2, \Delta_2, \mathcal{S}_2)$ be two complete deterministic table-transition automata. Let $\mathcal{A}_x = (Q, I, \Delta, \mathcal{S})$ the deterministic

table-transition automaton with:

- $Q = Q_1 \times Q_2, I = I_1 \times I_2,$
- $\Delta = \{((q_1, q_2), a, (p_1, p_2)) / ((q_1, a, p_1) \in \Delta_1, (q_2, a, p_2) \in \Delta_2)\}$
- \mathcal{J} is defined by:

$$T \in \mathcal{J} \Leftrightarrow \begin{cases} \exists T_1 \in \mathcal{J}_1 / T_1 = \{(q_1, a, q'_1) \in \Delta_1 / ((q_1, q_2), a, (q'_1, q'_2)) \in T\} \\ \text{or} \\ \exists T_2 \in \mathcal{J}_2 / T_2 = \{(q_2, a, q'_2) \in \Delta_2 / ((q_1, q_2), a, (q'_1, q'_2)) \in T\} \end{cases}$$

It's easy to show that $L(\mathcal{A}) = L(\mathcal{A}_1) \cup L(\mathcal{A}_2).$

From 1 a and 1 b, we deduce that *DT-REC* is a Boolean algebra.

2 a. It is clear that *DT-REC* \subseteq *T-REC*.

2 b. Let $\mathcal{A} = (Q, \{q_0\}, \Delta, \mathcal{J})$ be a table-transition automaton which recognizes the rational ω -language L . Let $\mathcal{A}_M = (Q', I', \Delta', \mathcal{E})$ the Muller automaton where $Q' = \{q_0\} \cup \Delta, I' = \{q_0\}, \mathcal{E} = \mathcal{J}$ and

$$\Delta = \left\{ \begin{array}{l} \{(q_0, a, (q_0, a, q)) / (q_0, a, q) \in \Delta\} \cup \\ \{((q_1, a, q_2), b, (q_2, b, q_3)) / (q_1, a, q_2) \in \Delta, (q_2, b, q_3) \in \Delta\} \end{array} \right.$$

It is clear that $L = L(\mathcal{A}) = L(\mathcal{A}_M)$ since $\forall w \in X^\omega, w \in L \Leftrightarrow w \in L(\mathcal{A}) \Leftrightarrow t\text{-inf}(w) \in \mathcal{J} \Leftrightarrow q\text{-inf}(w) \in \mathcal{E} \Leftrightarrow w \in L(\mathcal{A}_M)$ so *T-REC* \subseteq *RAT*.

2 c. Let us consider a deterministic Muller automaton $\mathcal{A}_M = (Q, I, \Delta, \mathcal{E})$ that recognizes the rational ω -language L . We have $L(\mathcal{A}_M) = \bigcup \{L(\mathcal{A}_E) / E \in \mathcal{E}\}.$

Let $\mathcal{J} = \{T \subseteq \{\delta = (q, a, p) \in \Delta / (q, p) \in E \times E\} \text{ such that } \forall q \in E, \exists \delta \in T / \delta = (q, a, p)\}.$

Let \mathcal{A}_T be the table-transition automaton $(Q, I, \Delta, \{T\})$. It is clear that $L(\mathcal{A}) = \bigcup \{L(\mathcal{A}_T) / T \in \mathcal{J}\},$ so $L = L(\mathcal{A}_M) = \bigcup L(\mathcal{A}_E) = \bigcup L(\mathcal{A}_T)$ and *RAT* \subseteq *DT-REC*. ■

3. DETERMINISTIC TRANSITION SYSTEMS AND RIGHT CONGRUENCES

A *right congruence* \simeq over X^* is an equivalence relation which satisfies: $\forall (u, v, w) \in X^* \times X^* \times X^*, u \simeq v \Rightarrow uw \simeq vw.$ We denote u / \simeq the \simeq -class of u and we say that \simeq is finite if it has a finite number of classes.

We say that a *right congruence* \simeq preserves the successors in a ω -language L iff $\forall (v_1, v_2) \in X^* \times X^*, \forall w \in X^\omega, v_1 \simeq v_2 \Rightarrow \{v_1 w \in L \Leftrightarrow v_2 w \in L\}.$

Let $S = (Q, \{q_0\}, \Delta)$ be a deterministic transition system, we denote by \simeq_Q the relation defined by $u \simeq_Q v \Leftrightarrow \{\forall q \in Q: q_0 \xrightarrow{u} q \Leftrightarrow q_0 \xrightarrow{v} q\}.$

PROPOSITION 3.1: *The class of deterministic complete transition systems is isomorphic to the class of finite right congruences.*

– Let $S = (Q, \{q_0\}, \Delta)$ be a deterministic finite transition system, it is easy to show (from the determinism of S) that \simeq_Q is a finite right congruence. Let \simeq be a finite right congruence, we define the transition system $S \simeq = (Q \simeq, I \simeq, \Delta \simeq)$ with $Q \simeq = \{u/\simeq\}$, $I \simeq = \{\varepsilon/\simeq\}$, $\Delta \simeq = \{(u/\simeq, a, (ua)/\simeq)\}$. It is clear that $S \simeq$ is a deterministic transition system. ■

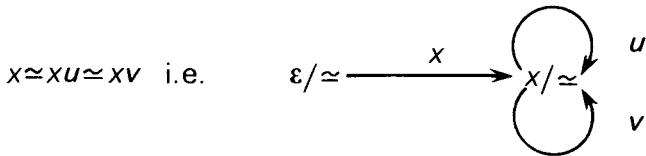
PROPOSITION 3.2: *Let \mathcal{A} be a deterministic complete Muller (resp. table-transition) automaton and \simeq_Q the associated right congruence then \simeq_Q preserves the successors in $L(\mathcal{A})$.*

– It is clear that, if two words of X^* have the same computation on \mathcal{A} , they also have the same successors in $L(\mathcal{A})$. ■

4. SATURATING RIGHT CONGRUENCES

DÉFINITION : Let L be a ω -language and \simeq a right congruence. We say that \simeq saturates L iff \simeq preserves the successors in L and satisfies:

$$\forall (x, u, v) \in X^* \times X^+ \times X^+, x \simeq xu \simeq xv \text{ and } x(u^+ v^+)^\omega \cap L \neq \emptyset \text{ implique } x(u^+ v^+)^\omega \subseteq L.$$



PROPOSITION 4.1: *Let \mathcal{A} be a deterministic complete Muller (resp. table-transition) automaton and \simeq_Q the associated right congruence then \simeq_Q saturates $L(\mathcal{A})$.*

Proof: Let us prove that if a deterministic complete Muller (resp. table-transition) automaton $\mathcal{A} = (Q, I, \Delta, \mathcal{E})$, the finite right congruence \simeq_Q saturates $L = L(\mathcal{A})$. From the proposition 3.2, \simeq_Q preserves the successors in $L(\mathcal{A})$.

Let $(x, u, v) \in X^* \times X^+ \times X^+$ such that $x \simeq_Q xu \simeq_Q xv$ and $x(u^+ v^+)^{\omega} \cap L \neq \emptyset$. Let $u = u_1 u_2 \dots u_n$ and $v = v_1 v_2 \dots v_m$ with $(u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_m) \in X^{n+m}$.

– Let us suppose that \mathcal{A} is a deterministic Muller automaton:

Since $x \simeq_Q xu \simeq_Q xv$, it is clear that $\forall w' \in x(u^+ v^+)^{\omega}$, $q\text{-inf}(w) = Q_u \cup Q_v$ where Q_u and Q_v are the sets of states of Q defined by:

$$Q_u = \{ (xu_1 \dots u_i) / \simeq_Q, \forall 1 \leq i \leq n \} \quad \text{and} \quad Q_v = \{ (xv_1 \dots v_i) / \simeq_Q, \forall 1 \leq i \leq m \},$$

If $x(u^+ v^+)^{\omega} \cap L \neq \emptyset$ then $Q_u \cup Q_v \in \mathcal{C}$ and $x(u^+ v^+)^{\omega} \subseteq L$.

– Let us suppose that \mathcal{A} is a deterministic table-transition automaton: Since $x \simeq_Q xu \simeq_Q xv$, it is clear that $\forall w' \in x(u^+ v^+)^{\omega}$, $t\text{-inf}(w) = T_u \cup T_v$ where T_u and T_v are the sets of transitions of Δ defined by:

$$- T_u = \{ ((xu_1 \dots u_i) / \simeq_Q, u_{i+1}, (xu_1 \dots u_{i+1}) / \simeq_Q) \mid \forall 1 \leq i \leq n-1 \}$$

$$- T_v = \{ ((xv_1 \dots v_i) / \simeq_Q, v_{i+1}, (xv_1 \dots v_{i+1}) / \simeq_Q) \mid \forall 1 \leq i \leq m-1 \}$$

If $x(u^+ v^+)^{\omega} \cap L \neq \emptyset$ then $T_u \cup T_v \in \mathcal{C}$ and $x(u^+ v^+)^{\omega} \subseteq L$. ■

In order to prove the converse, we need some results.

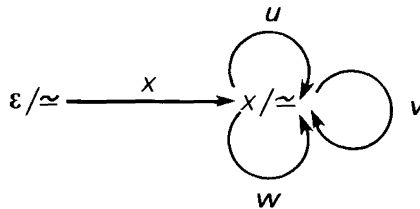
LEMMA 4.2: Let L be a ω -language and \simeq be a right congruence which saturates L . Let $(x, u, v, w) \in X^* \times X^+ \times X^+ \times X^+$ such that $x \simeq xu \simeq xv \simeq xw$. We get:

1. $x(uvw)^{\omega} \in L \Leftrightarrow x(vuuvw)^{\omega} \in L$.
2. $x(uvw)^{\omega} \in L \Leftrightarrow x(vuw)^{\omega} \in L$.
3. $x(uvw)^{\omega} \in L \Leftrightarrow x(uvuw)^{\omega} \in L$.

Proof: Let \simeq be a right congruence which saturates L .

Let $(x, u, v, w) \in X^* \times X^+ \times X^+ \times X^+$ such that $x \simeq xu \simeq xv \simeq xw$

i. e.



1. $x(uvw)^{\omega} \in L \Leftrightarrow x(uvwvw)^{\omega} \in L$ ($x \simeq xu \simeq xv$ and \simeq saturates L)
 $\Leftrightarrow x(uvwvuuvw)^{\omega} \in L$ ($x \simeq xv \simeq xu$ and \simeq saturates L)
 $\Leftrightarrow x(uvwvuuvw)^{\omega} \in L$ ($x \simeq xv \simeq xu$ and \simeq saturates L)
 $\Leftrightarrow xuvw(vuuvw)^{\omega} \in L$

$$\begin{aligned} &\Leftrightarrow x (v (uvw)^2)^\omega \in L && (\simeq \text{ preserves the successors in } L) \\ &\Leftrightarrow x (vuv)^\omega \in L && (x \simeq xv \simeq xuvw \text{ and } \simeq \text{ saturates } L). \end{aligned}$$

2. From (1), we have $x(uvw)^\omega \in L \Leftrightarrow x(vuv)^\omega \in L \Leftrightarrow xvu(vwvu)^\omega \in L \Leftrightarrow x(vwvu)^\omega \in L$. Let $U=v, V=uv, W=u$, from (1) we deduce: $x(uvw)^\omega \in L \Leftrightarrow x(vwvu)^\omega \in L \Leftrightarrow x(UVW)^\omega \in L \Leftrightarrow x(VUVW)^\omega \in L \Leftrightarrow x(wvuvu)^\omega \in L$ and thus:

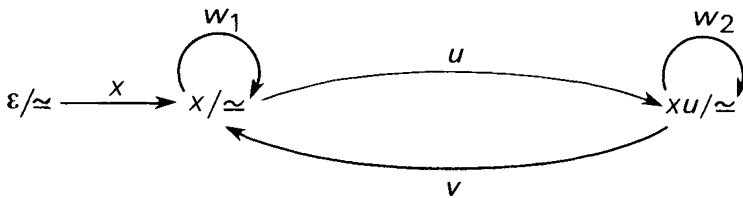
$$\begin{aligned} x(wvuvu)^\omega \in L &\Leftrightarrow x(wv^2 wvu)^\omega \in L \\ &\Leftrightarrow xw(v^2 wvu)^\omega \in L \\ &\Leftrightarrow xw(vwvu)^\omega \in L && (xw \simeq xwv \simeq xwvu, \simeq \text{ saturates } L) \\ &\Leftrightarrow x((wv)^2 u)^\omega \in L \\ &\Leftrightarrow x(wvu)^\omega \in L && (x \simeq xwv \simeq xu, \simeq \text{ saturates } L) \\ &\Leftrightarrow xw(vuv)^\omega \in L \\ &\Leftrightarrow x(vuv)^\omega \in L && (\simeq \text{ preserves the successors in } L). \end{aligned}$$

$$\begin{aligned} 3. x(uvw)^\omega \in L &\Leftrightarrow x(vuv)^\omega \in L && \text{(Lemma 4.2.2)} \\ &\Leftrightarrow x(uvuv)^\omega \in L && \text{(Lemma 4.2.1). } \blacksquare \end{aligned}$$

COROLLARY 4.3: *Let L be a ω -language and \simeq a right congruence which saturates L . Let $(x, w_1, w_2, u, v) \in X^* \times X^* \times X^* \times X^* \times X^*$ such that $x \simeq xw_1 \simeq xuv$ and $xu \simeq xuw_2$ then*

$$x(w_1 uw_2 v)^\omega \in L \Leftrightarrow x(w_1 uw_2 vuv)^\omega \in L.$$

Proof: We have $(x, u, v, w_1, w_2) \in (X^*)^5$ such that $x \simeq xw_1 \simeq xuv$ and $xu \simeq xuw_2$ and \simeq saturates L :



$$\begin{aligned} x(w_1 uw_2 uv)^\omega \in L &\Leftrightarrow x((w_1)(uw_2 v)(w_1)(uv))^\omega \in L \text{ (Lemma 4.2.3)} \\ &\Leftrightarrow xw_1 u(w_2 vw_1 uvw_1 u)^\omega \in L \\ &\Leftrightarrow xw_1 u((w_2)(vw_1 u)(w_2)(vw_1 u))^\omega \in L \text{ (Lemma 4.2.3)} \\ &\Leftrightarrow xw_1 u(w_2 vw_1 u)^\omega \in L \\ &\Leftrightarrow x(w_1 uw_2 v)^\omega \in L. \quad \blacksquare \end{aligned}$$

The following result is a generalization of the previous corollary:

COROLLARY 4.4: *Let L be a ω -language and \simeq a right congruence which saturates L and let $x, y_1, y_2, \dots, y_n, v_1, v_2, \dots, v_{n+1} \in X^*$ such that $x \simeq xv_1 \simeq xy_1 \dots y_n$ and $\forall 1 \leq i \leq n, xy_1 \dots y_i \simeq xy_1 \dots y_i v_{i+1}$ then*

$$x(v_1 y_1 v_2 \dots y_n v_{n+1})^\omega \in L \Leftrightarrow x((v_1 y_1 v_2 \dots y_n v_{n+1})(y_1 \dots y_n))^\omega \in L$$

Proof:

$$\begin{aligned} x((v_1 y_1 v_2 \dots y_n v_{n+1})(y_1 \dots y_n))^\omega \in L &\Leftrightarrow \\ x((v_1)(v_1 y_1 v_2 \dots y_n v_{n+1})(y_1 \dots y_n))^\omega \in L &\Leftrightarrow \\ x((v_1 y_1 v_2 \dots y_n v_{n+1}) v_1 (y_1 \dots y_n))^\omega \in L &\Leftrightarrow \quad (\text{Lemma 4.2.3}) \\ xv_1 y_1 ((v_2)(v_2 y_2 \dots y_n v_{n+1} v_1 y_1)(y_2 \dots y_n \dots y_n v_1 y_1))^\omega \in L &\Leftrightarrow \\ xv_1 y_1 ((v_1 y_1 \dots y_n v_{n+1} v_1 y_1) v_2 (y_2 \dots y_n v_1 y_1))^\omega \in L &\Leftrightarrow \quad (\text{Lemma 4.2.3}) \\ xv_1 y_1 v_2 (y_2 (v_3 y_3 \dots y_n v_{n+1} v_1 y_1 v_2 y_2)(y_3 \dots y_n v_1 y_1 v_2 y_2))^\omega \in L &\Leftrightarrow \\ &\vdots \\ xv_1 y_1 \dots v_n y_n ((v_{n+1} v_1 y_1 \dots v_n y_n)(v_1 y_1 \dots v_n y_n))^\omega \in L &\Leftrightarrow \\ xv_1 y_1 \dots v_n y_n ((v_{n+1})(v_{n+1} v_1 y_1 \dots v_n y_n)(v_1 y_1 \dots v_n y_n))^\omega \in L &\Leftrightarrow \\ xv_1 y_1 \dots v_n y_n ((v_{n+1} v_1 y_1 \dots v_n y_n)(v_{n+1})(v_1 y_1 \dots v_n y_n))^\omega \in L &\Leftrightarrow \\ xv_1 y_1 \dots v_n y_n v_{n+1} ((v_1 y_1 \dots v_n y_n v_{n+1})(v_1 y_1 \dots v_n y_n v_{n+1}))^\omega \in L &\Leftrightarrow \\ x(v_1 y_1 \dots v_n y_n v_{n+1})^\omega \in L. &\blacksquare \end{aligned}$$

DEFINITION: Let L be a ω -language and \simeq be a finite right congruence. We define the deterministic complete table-transition automaton $\mathcal{A} \simeq = (Q \simeq, I \simeq, \Delta \simeq, \mathcal{I} \simeq)$ by:

$$\begin{aligned} - Q \simeq &= \{u/\simeq, u \in X^*\}, \quad I \simeq = \{\varepsilon/\simeq\}, \quad \Delta \simeq = \{(u/\simeq, a, ua/\simeq), u \in X^*, a \in X\}, \\ - \mathcal{I} \simeq &= \{T \in \mathcal{P}(\Delta)/\exists w \in UP(L), T = t\text{-inf}(w)\}. \end{aligned}$$

THEOREM 4.5: *Let L be a rational ω -language and \simeq a finite right congruence which saturates L . We have: $L = L(\mathcal{A} \simeq)$.*

Proof: Since L and $L(\mathcal{A} \simeq)$ are rational ω -languages, it suffice to prove that $UP(L) = UP(L(\mathcal{A} \simeq))$.

- Let $w \in UP(L)$. Since \simeq is finite, by definition of $\mathcal{A} \simeq$, we have $t\text{-inf}(w) \in \mathcal{I} \simeq$ and then $w \in L(\mathcal{A} \simeq)$.

– Let us prove that $UP(L(\mathcal{A} \simeq)) \subseteq UP(L)$. Let $w \in UP(L(\mathcal{A} \simeq))$. By definition of $\mathcal{A} \simeq$, we have:

$$w \in UP((L(\mathcal{A} \simeq)) \Leftrightarrow \exists w' \in UP(L) / t\text{-inf}(w') = t\text{-inf}(w) \in \mathcal{I} \simeq.$$

Since $w \in UP((L(\mathcal{A} \simeq))$, there exists $x, y \in X^*$ such that $x \simeq xy, w = xy^\omega$.

Since $w' \in UP(L)$, there exists $x', y' \in X^*$ such that $x' \simeq x' y', w' = x' y'^\omega$.

Let $y = y_1 \dots y_n$ and $y' = y'_1 \dots y'_{n'}$, such that $\forall 1 \leq i \leq n, \forall 1 \leq j \leq n', y_i \in X, y'_j \in X$.

We have:

$$\begin{aligned} t\text{-inf}(xy^\omega) &= \{ (x/\simeq, y_1, xy_1/\simeq), \dots, ((xy_1, \dots, y_i)/\simeq, y_i, \\ &\quad (xy_1, \dots, y_{i+1})/\simeq), \dots \} \\ t\text{-inf}(x' y'^\omega) &= \{ (x'/\simeq, y'_1, x' y'_1/\simeq), \dots, ((x' y'_1, \dots, y'_i)/\simeq, \\ &\quad y'_i, (x' y'_1, \dots, y'_{i+1})/\simeq), \dots \}. \end{aligned}$$

Since $t\text{-inf}(xy^\omega) = t\text{-inf}(x' y'^\omega)$:

a. there exists $v_1, \dots, v_n, u_1, \dots, u_n \in X^*$ such that $\forall 1 \leq i \leq n', y = u_i y'_i v_i$ and $x u_i \simeq x y'_1 \dots y'_{i-1}$ and $x u_i y'_i v_i \simeq x$.

b. there exists $v'_1, \dots, v'_n, u'_1, \dots, u'_n \in X^*$ such that $\forall 1 \leq i \leq n, y' = u'_i y_i v'_i$ and $x' u'_i \simeq x y_1 \dots y_{i-1}$ and $x' u'_i y_i v'_i \simeq x'$.

Let $x'' = x' u'_1$ and $y'' = y_1 v'_1 u'_1 = y''_1 \dots y''_{n'}$, such that $\forall 1 \leq i \leq n'', y''_i \in X$, we have $x'' \simeq x$ and $y_1 = y''_1$. So we can suppose that x' and y' are choosen such that $y_1 = y'_1$ and $x' \simeq x$ and $u_1 = u'_1 = \varepsilon$.

We have: $w = xy^\omega = x (y^n)^\omega = x (y'_1 v_1 u_2 y'_2 v_2 u_3 y'_3 v_3 \dots u_n y'_n v_n)^\omega$

By the corollary 4.4, we have:

$$- w = xy^\omega \in L \Leftrightarrow x (y^n y')^\omega \in L \Leftrightarrow x (y y')^\omega \in L$$

By a similar construction, we have:

$$- w' = x' y'^\omega \in L \Leftrightarrow x' (y'^n y)^\omega \in L \Leftrightarrow x' (y'^n y)^\omega \in L \Leftrightarrow x (y y')^\omega \in L;$$

Hence the result. ■

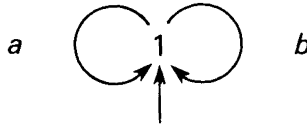
THEOREM 4.6: *The family of deterministic complete table-transition automata which recognize a rational ω -language L is isomorphic to the family of finite right congruences which saturate L .*

Proof: In the theorem 4.5, we proved that, from a finite right congruence \simeq which saturates a rational ω -language L , we can build a deterministic table-transition automaton $\mathcal{A} \simeq$ such that $L(\mathcal{A} \simeq) = L$ and $\simeq_Q = \simeq$. In the

theorem 4.1, we proved that, if \mathcal{A} is a deterministic table-transition automaton which recognizes a ω -language L then the associated finite right congruence \simeq_Q saturates L . ■

Remark: We saw, in the proof of the theorem 2.1, that any deterministic Muller automaton, is in fact a table transition automaton and so any deterministic Muller automaton which recognize $L \in \text{RAT}$, defines a finite right congruence which saturates L , but the converse is not true:

Let $L = \{a, b\}^* a^\omega \in \text{RAT}$ and \simeq the finite right congruence with one class $\varepsilon / \simeq = \{a, b\}^*$. It's obvious to see that \simeq saturates L , but the associated transition system $\mathcal{S} \simeq$ cannot leads us to build a Muller automaton which recognizes L :



5. CYCLICALLY CLOSED TABLE-TRANSITION AUTOMATA

DÉFINITION : We say that a table-transition automaton $\mathcal{A} = (Q, I, \Delta, \mathcal{S})$ is *cyclically closed* iff $\forall T \in \mathcal{S}, \forall T' \subseteq \Delta$ such that $T \cup T'$ is coherent, $T \cup T' \in \mathcal{S}$.

THEOREM 5.1: Let $\mathcal{A} = (Q, I, \Delta, \mathcal{S})$ be a deterministic complete table-transition automaton.

$$L(\mathcal{A}) \in D\text{-RAT} \quad \text{iff} \quad \mathcal{A} \text{ is cyclically closed.}$$

Proof: a. Let $\mathcal{A} = (Q, \{q_0\}, \Delta, \mathcal{S})$ be a cyclically closed deterministic table-transition automaton which recognizes the rational ω -languages L . Let us prove that $L \in D\text{-RAT}$.

Like in paragraph 2, let $\mathcal{A}_M = (Q', I', \Delta', \mathcal{E})$ the deterministic Muller automaton where $Q' = \{q_0\} \cup \Delta, I' = \{q_0\}, \mathcal{E} = \mathcal{S}$ and

$$\Delta = \left\{ \begin{array}{l} \{(q_0, a, (q_0, a, q)) / (q_0, a, q) \in \Delta\} \cup \\ \{((q_1, a, q_2), b, (q_2, b, q_3)) / (q_1, a, q_2) \in \Delta, (q_2, b, q_3) \in \Delta\} \end{array} \right\}$$

It is clear that $L = L(\mathcal{A}) = L(\mathcal{A}_M)$ since $\forall w \in X^\omega, w \in L \Leftrightarrow w \in L(\mathcal{A}) \Leftrightarrow t\text{-inf}(w) \in \mathcal{S} \Leftrightarrow q\text{-inf}(w) \in \mathcal{E} \Leftrightarrow w \in L(\mathcal{A}_M)$. Moreover \mathcal{A} is

cyclically closed so \mathcal{A}_M satisfies the property: $\forall E \in \mathcal{E}, \forall E' \in Q^*, E \cup E' \in \mathcal{F}$.
 L. H. Landweber used this property to characterize the Muller automata which recognize the deterministic ω -languages [LA (69)] so $L \in D\text{-RAT}$.

b. Let $\mathcal{A} = (Q, \{q_0\}, \Delta, \mathcal{F})$ be a deterministic table-transition automaton which recognizes a deterministic rational ω -languages L . Let us prove that \mathcal{A} is cyclically closed.

Let $T \in \mathcal{F}$ and $T' \in \Delta$ such that $T \cup T'$ is coherent. There exists $x \in X^*$, $u = u_1 u_2 \dots u_n$, $v = v_1 v_2 \dots v_m$, $\delta_1, \delta_2, \dots, \delta_n \in \Delta$, $\delta'_1, \delta'_2, \dots, \delta'_m \in \Delta$, $q \in Q$ such that:

$$- q_0 \xrightarrow{x} q;$$

$$- q \xrightarrow{u} q \text{ and}$$

$$\forall 1 \leq i \leq n \delta_i = (q_i, u_i, q_{i+1}), q = q_1 = q_{m+1} \text{ and } T = \{\delta_1, \delta_2, \dots, \delta_n\};$$

$$- q \xrightarrow{v} q \text{ and}$$

$$\forall 1 \leq i \leq m \delta'_i = (q'_i, v_i, q'_{i+1}), q = q'_1 = q'_{n+1} \text{ and } T' = \{\delta'_1, \delta'_2, \dots, \delta'_m\}.$$

We have $t\text{-inf}(xu^\omega) = T$ so $x\{u, v\}^*u^\omega \subseteq L$. Since

$$L \in D\text{-RAT}, \quad \exists K \in \text{RAT}/\bar{K} = L.$$

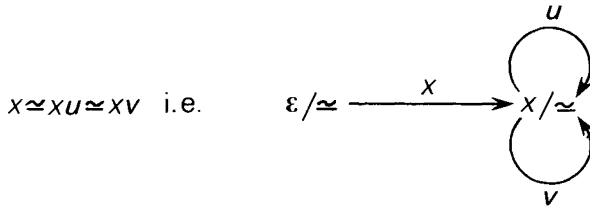
We have $x\{u, v\}^*u^\omega \subseteq \bar{K}$ so, like [AR(84) Lemma III.1], it is easy to build by induction a sequence $(w_n)_{n \in \mathbb{N}}$ such that $\forall i \in \mathbb{N}, w_i \in K, |w_i| < |w_{i+1}|$ and $\lim_{w \rightarrow \infty} (w_n) = w \in x\{u^+ v^+\}^\omega \cap L$ and since $\forall w \in x\{u^+ v^+\}^\omega, t\text{-inf}(w) = (T \cup T') \in \mathcal{F}$ and \mathcal{A} is cyclically closed. ■

6. LANDWEBER RIGHT CONGRUENCE

DEFINITION : Let L be a rational ω -language and \simeq a right congruence. We say that \simeq is *L-Landweber right congruence* iff \simeq preserves the successors in L and \simeq satisfies:

$$\forall (x, u, v) \in X^* \times X^+ \times X^+, x \simeq xu \simeq xv$$

$$\text{and } x\{u, v\}^*u^\omega \cap L \neq \emptyset \text{ implies } x(v^*u)^\omega \subseteq L^\omega$$



THEOREM 6.1: *Let L be a rational language and \simeq a finite right congruence on X^* .*

If \simeq is a L -Landweber right congruence then \simeq saturates L .

Proof: Let L be a rational language and \simeq finite L -Landweber right congruence. Let us prove that \simeq saturates L .

By definition, we have:

$$\forall (x, u, v) \in X^* \times X^+ \times X^+ / x \simeq xu \simeq xv : x \{ u, v \}^* u^\omega \cap L \neq \emptyset \Rightarrow x (v^* u)^\omega \subseteq L^\omega.$$

Let $(x, u, v) \in X^* \times X^+ \times X^+$ such that $x \simeq xu \simeq xv$ and $x (u^+ v^+)^\omega \cap L \neq \emptyset$ and let us prove that $x (u^+ v^+)^\omega \subseteq L$. Since L and $x (u^+ v^+)^\omega$ are rational languages, $x (u^+ v^+)^\omega \cap L$ is a rational language. So $UP(x (u^+ v^+)^\omega \cap L) \neq \emptyset$ and it suffice to prove that $UP(x (u^+ v^+)^\omega) \subseteq UP(L)$ (Büchi's lemma).

Let $w \in UP(x (u^+ v^+)^\omega)$, there exists $(p_1, p_2, \dots, p_n) \in \mathbb{N}^n$ such that $w = x (u^{p_1} v^{p_2} \dots u^{p_{n-1}} v^{p_n})^\omega$. Since \simeq is L -Landweber right congruence and $x \simeq xu \simeq xv \simeq xu^{p_1} v^{p_2} \dots u^{p_{n-1}} v^{p_n}$, we have:

$$\begin{aligned} x (u^{p_1} v^{p_2} \dots u^{p_{n-1}} v^{p_n})^\omega \in L &\Leftrightarrow xu (u^{p_1-1} v^{p_2} \dots u^{p_{n-1}} v^{p_n} u)^\omega \in L \\ &\Leftrightarrow xu (vu^{p_1-1} v^{p_2} \dots u^{p_{n-1}} v^{p_n} u)^\omega \in L \\ &\Leftrightarrow xuvu (u^{p_1-2} v^{p_2} \dots u^{p_{n-1}} v^{p_n} uvu)^\omega \in L \\ &\Leftrightarrow xuvu (vu^{p_1-2} v^{p_2} \dots u^{p_{n-1}} v^{p_n} uvu)^\omega \in L \\ &\Leftrightarrow x ((uv)^2 u^{p_1-2} v^{p_2} \dots u^{p_{n-1}} v^{p_n})^\omega \in L \end{aligned}$$

It now easy to show, iterating the previous calculus, that

$$\begin{aligned} w \in L &\Leftrightarrow x (u^{p_1} v^{p_2} \dots u^{p_{n-1}} v^{p_n})^\omega \in L \\ &\Leftrightarrow x ((uv)^{2^*(p_1+p_2+\dots+p_{n-1}+p_n)})^\omega \in L \\ &\Leftrightarrow x (uv)^\omega \in L. \end{aligned}$$

So we have $\forall (w, w') \in (x (u^+ v^+)^\omega) \times (x (u^+ v^+)^\omega)$, $w \in L \Leftrightarrow w' \in L$ and since $x (u^+ v^+)^\omega \cap L \neq \emptyset$, we have $x (u^+ v^+)^\omega \subseteq L$. ■

THEOREM 6.2: *The family of deterministic complete cyclically closed table-transition automata which recognize a rational ω -language L is isomorphic to the family of finite L -Landweber right congruences.*

Proof: a. Let us prove that if $\mathcal{A} = (Q, I, \Delta, \mathcal{F})$ is a deterministic cyclically closed table-transition automaton, its associated finite right congruence \simeq_Q is L -Landweber *i. e.*

$$\forall (x, u, v) \in X^* \times X^+ \times X^+ / x \simeq_Q xu \simeq_Q xv : x \{u, v\}^* u^\omega \cap L \neq \emptyset \Rightarrow x(v^* u)^\omega \in L^\omega.$$

Let $x, u, v \in X^*$ such that $x \simeq_Q xu \simeq_Q xv$ and let us suppose that $x \{u, v\}^* u^\omega \in L$.

Since \simeq_Q preserves the successors in L , $xu^\omega \in L$.

Let $T = t\text{-inf}(xu^\omega)$ and $T' = t\text{-inf}(xv^\omega)$. Since $xu^\omega \in L$, $T \in \mathcal{F}$ and since $x \simeq_Q xu \simeq_Q xv$, T and T' have a common state. Moreover, \mathcal{A} is cyclically closed then $\forall w \in x(v^* u)^\omega$, $t\text{-inf}(w) = t\text{-inf}(x(uv)^\omega) = (T \cup T') \in \mathcal{F}$ and $x(v^* u)^\omega \subseteq L$.

b. Let $L \in \text{RAT}$ and \simeq a finite right L -Landweber congruence. We denote $\mathcal{A} \simeq$ the deterministic complete table-transition automaton $(Q \simeq, I \simeq, \Delta \simeq, \mathcal{F} \simeq)$ where

$$\begin{aligned} Q \simeq &= \{x/\simeq, x \in X^*\} & \text{and} & & I \simeq &= \{\varepsilon/\simeq\} \\ \Delta \simeq &= \{(x/\simeq, a, (xa)/\simeq), x \in X^*, a \in X\} \\ \mathcal{F} \simeq &= \{T \subset P(\Delta) / \exists w \in UP(L) : t\text{-inf}(w) = T\}. \end{aligned}$$

From the theorem 6.1, \simeq saturates L from by the theorem 4.5, $L = L(\mathcal{A} \simeq)$. It remains to prove $\mathcal{A} \simeq$ is cyclically closed.

Let $(T, T') \in \mathcal{F} \simeq \times P(\Delta)$ such T and T' have a common state. Let us prove that $(T \cup T') \in \mathcal{F} \simeq$.

Let $w = xu^\omega \in UP(L)$ such $t\text{-inf}(w) = T \in \mathcal{F} \simeq$ and let $w' = x'v^\omega \in X^\omega$ such that $t\text{-inf}(w') = T'$ and $x \simeq xu$, $x' \simeq x'v$.

Since T, T' have a common state, there exists $v = v_1 v_2$ and $u = u_1 u_2$ such that $t\text{-inf}(x'(v_1 u_2 u_1 v_2))^\omega = (T \cup T')$ and $x'v_1 \simeq xu_1$. Since $x'v_1 \simeq x'v_1 v_2 v_1$ and $xu_1 \simeq xu_1 u_2 u_1$, we get $x'v_1(u_2 u_1) \simeq x'v_1(v_2 v_1)$ and, since \simeq preserves the successors in L , we have: $x'v_1(u_2 u_1)^\omega \in L$.

Since \simeq is a finite L -Landweber right congruence and $x'v_1(u_2 u_1) \simeq x'v_1(v_2 v_1)$ and $x'v_1(u_2 u_1)^\omega \in L$, we get:

$$x'v_1(u_2 u_1)^\omega \in L \quad \Rightarrow \quad x'v_1(u_2 u_1 v_2 v_1)^\omega \in L \quad \Rightarrow \quad x'(v_1 u_2 u_1 v_2)^\omega \in L.$$

Moreover, we have $t - \inf(x'(v_1 u_2 u_1 v_2)^\omega) = T \cup T'$, so $T \cup T' \in \mathcal{F} \simeq$. ■

COROLLARY 6.3: $L \in D\text{-RAT}$ iff there exists a finite L -Landweber right congruence.

Proof: Since $D\text{-RAT} \subseteq \text{RAT}$, there exists a complete deterministic table-transition automaton \mathcal{A} such $L = L(\mathcal{A})$ (theorem 2.1), so \mathcal{A} is a deterministic cyclically closed table cycle automata \mathcal{A} (theorem 5.1) and from the theorem 6.2, there is an isomorphism between the family of complete deterministic cyclically closed table cycle automata which recognize L and the family of finite right congruences which saturate L , hence the result. ■

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