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SATURATING RIGHT CONGRUENCES (*)

by Bertrand Le Saec ([1])

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Abstract. – *One of the main classes of automata recognizing rational ω-langages is the family of Muller automata. Their recognition criterion is based on "state tables". We propose a criterion based on "transition tables". The transition tables automata recognize exactly the class of rational ω-langages. Their main interest is that, when they are deterministic, they are isomorph to a family of finite right congruences satisfying a "property of saturation" similar to those defined for congruences by A. Arnold, J. R. Buchi.*

Résumé. – *Une des principales classes d'automates reconnaissant les ω-langages rationnels est la famille des automates de Müller. Leur critère de reconnaissance est basé sur la notion de « tables d'états ». Nous proposons un critère basé sur des « tables de transitions ». Les automates à tables de transitions reconnaissent exactement la famille des ω-langages rationnels. Leur principal intérêt est d'être, lorsqu'ils sont déterministes, isomorphes à une famille de demi-congruences satisfaisant une propriété de « saturation » analogue à celle utilisée pour les congruences par A. Arnold, J. R. Büchi.*

INTRODUCTION

The theory of rational set of infinite words was investigated as a study of asymptotic behaviours of finite automata [BU (62), MN (66)]. One of the main classes of automata is the family of Muller automata. The criterion used for recognition is based on the notion of "states table". We propose to use a criterion based on "transitions tables". The table-transition automata and the deterministic table-transition automata also recognize exactly the class of rational ω-languages. They are smaller than classical Muller automata, but their main interest is that, when they are deterministic, they are isomorphic to a family of finite right congruences satisfying a "property of saturation"

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similar to those defined for congruences [AR (85), BU (62), PE (84)]. Moreover, this "syntactic" approach of deterministic automata provides a clearer version of the property of L. H. Landweber to characterize deterministic ω-languages [LA (69)].

In part 1, we recall classical definitions and results of the theory of rational ω-languages. In part 2, we present the table-transition automata. In part 3, we study connections between right congruences and deterministic transition systems. In part 4, we introduce the "saturating right congruences" and prove that they are isomorphic to the deterministic table-transition automata. In part 5, we characterize the deterministic table-transition which recognize deterministic ω-languages. In part 6, we define the "Landweber right congruences" to characterize the deterministic ω-languages.

1. PRELIMINARIES

Let X be an finite alphabet. We denote by X^* the set of finite words on X, by ε the empty word and by X^+ the set $X^* - \{\varepsilon\}$. We denote by X^ω the set of infinite words on X. Let $B \subseteq X^+$, we denote by B^ω (resp. B^*) the set of infinite (resp. finite) words $w = w_1 w_2 \ldots$ such $w_i \in B$.

Let $w \in X^* \cup X^\omega$, we denote by $LF(w)$ the set of the left factors of w defined by: $LF(w) = \{ w_1 \in X^* / \exists\, w_2 \in X^* \cup X^\omega,\ w = w_1 w_2 \}$. We denote by $/w/$ the length of w. We say that a language $L \subseteq X^*$(Resp: $L \subseteq X^\omega$) is prefix iff $\forall\, w \in L,\ LF(w) \cap L = \{ w \}$.

A *transition system* on X is a tuple $\mathscr{S} = (Q, I, \Delta)$ where Q is a finite set of states, $I \subseteq Q$ a set of initial states and $\Delta \subseteq Q \times X \times Q$ a set of transitions.

A transition system $\mathscr{S}$ is *complete* iff $\forall\, q \in Q,\ \forall\, a \in X,\ \exists\, q' \in Q / (q, a, q') \in \Delta$.

A transition system $\mathscr{S}$ is *deterministic* iff $\mathrm{Card}(I) = 1$ and $\forall\, (q, q', q'') \in Q \times Q \times Q,\ \forall\, a \in X,\ (q, a, q') \in \Delta$ and $(q, a, q'') \in \Delta \Rightarrow q' = q''$.

A finite (resp. infinite) *computation* of $w \in X^*$ (Resp. X^ω) in the transition system $\mathscr{S}$ is a finite (resp. infinite) sequence of transitions $d = \delta_0 \delta_1 \delta_2 \ldots$ such that $\delta_1 = (q_1, a_1, p_1) \in \Delta$, $q_0 \in I$ and for $i \geqq 0$, $p_i = q_{i+1}$ and $w = a_0 a_1 \ldots$ We denote $t-\inf(d) = \{ \delta \in Q / \mathrm{Card}(\{ i \in \mathbb{N} / \delta_i = \delta \}) = \infty \}$ and $q\text{-}\inf(d) = \{ q \in Q / \mathrm{Card}(\{ i \in \mathbb{N} / q_i = q \}) = \infty \}$. If $\mathscr{S}$ is complete deterministic, we can use $t\text{-}\inf(w)$ for $t\text{-}\inf(d)$ and $q-\inf(w)$ for $q\text{-}\inf(d)$.

A *Muller automaton* is a tuple $\mathscr{A}_M = (Q, I, \Delta, \mathscr{E})$ where $\mathscr{S} = (Q, I, \Delta)$ is a transition system and $\mathscr{E} \subseteq \mathscr{P}(Q)$ is a family of subsets of Q. The automaton $\mathscr{A}_M$ is deterministic if $\mathscr{S}$ is deterministic.

We say that the Muller automaton $\mathscr{A}_M$ recognize $w \in X^\omega$ iff there exists a computation d of w in $\mathscr{A}_M$ such that $q\text{-inf}(d) \in \mathscr{E}$.

We denote by:

Rat, the family of rational languages of X^*.

RAT, the family of finite unions of ω-languages CB^ω where C and B are rational languages of X^*.

D-RAT, the family of finite unions of ω-languages CB^ω where C and B are prefix rational languages of X^*.

M-REC, the family of ω-langages recognized by a Muller automaton.

DM-REC, the family of ω-languages recognized by a deterministic Muller automaton.

Let $K \in X^*$. We denote by $\vec{K}$ the set $\{ w \in X^\omega / \text{Card}(LF(w) \cap K) = \infty \}$.

THEOREM 1.1: [EI (74)]

– *IF* $K \in X^*$ *is prefix then* $\vec{K} = \varnothing$ *and* $\overrightarrow{K^*} = K^\omega$.

– $L \in D\text{-RAT} \Leftrightarrow \exists K \in \text{Rat} / \vec{K} = L$.

We have the fundamental results due to R. McNaughton and R. Buchi:

THEOREM 1.2: [BU (62), EI (74), MN (66)]

– RAT is a boolean algebra.

– $M\text{-REC} = DM\text{-REC} = \text{RAT} = (D\text{-RAT})^B$ where $(F)^B$ denote the boolean closrue of F.

We say that $w \in X^\omega$ is an ultimately periodic word iff there exists $(x, y) \in X^* \times X^*$ such that $w = xy^\omega$. We denote $UP(L)$ the set of all the ultimately periodic words of L.

BUCHI'S LEMMA: [BU (62)]

If *L and L′ are rational* ω-*languages, we have:*

$$L \neq \varnothing \Leftrightarrow UP(L) \neq \varnothing \quad \text{and} \quad L \subseteq L' \Leftrightarrow UP(L) \subseteq UP(L').$$

2. TABLE-TRANSITION AUTOMATA

Let $\mathscr{S}$ be a transition system, we say that a set T of transitions of $\mathscr{S}$ is coherent iff

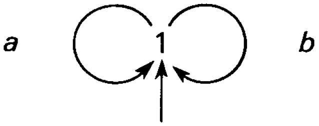
$$\forall \delta = (q', a', p') \in T, \quad \forall \delta' = (q'', a'', p'') \in T,$$
$$\exists \delta_1, \delta_2, \ldots, \delta_n \in T / \forall 1 \leqq i \leqq n,\ \delta_i = (q_i, a_i, q_{i+1}),\quad q_1 = p',\quad q_{n+1} = q''.$$

A *Table-transition automaton* is a tuple $\mathscr{A} = (Q, I, \Delta, \mathscr{I})$ where $\mathscr{S} = (Q, I, \Delta)$ is a transition system and $\mathscr{I} \subseteq \mathscr{P}(\Delta)$ is a family of coherent subsets of Δ.

We say that an infinite word w is recognized by the table-transition automaton $\mathscr{A}$ iff there exists a computation $\delta_1 \delta_2 \delta_3 \ldots$ of w in $\mathscr{S}$ such that $t\text{-inf}(\delta_1 \delta_2 \delta_3 \ldots) \in \mathscr{I}$.

Note: If a set of transitions T is not coherent, no infinite word w can satisfy $t\text{-inf}(w, \delta_1 \delta_2 \delta_3 \ldots) = T$ so we assume that all the elements of $\mathscr{I}$ are coherent.

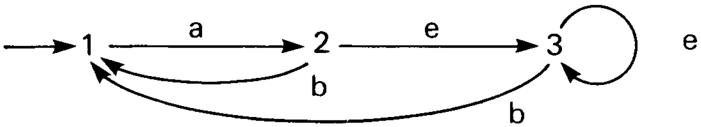
Examples: Let us consider the table-transition automaton $\mathscr{A}$:



with $I = Q = \{1\}, \mathscr{I} = \{\{(1, a, 1)\}\}$.

The automaton $\mathscr{A}$ recognizes the rational ω-language $L(\mathscr{A}) = \{a, b\}^* a^\omega$.

Let us consider the table-transition automaton $\mathscr{A}'$:



with $Q' = \{1, 2, 3\}$, $I' = \{1\}$ and $\mathscr{I} = \{\{(1, a, 2), (2, e, 3), (3, b, 1)\}\}$.

The automaton $\mathscr{A}'$ recognizes the rational ω-language $L(\mathscr{A}') = (ae^* b)^* (aeb)^\omega$. □

DÉFINITIONS : Let us denote:

T-REC: the family of ω-languages recognized by a table-transition automaton.

DT-REC: the family of the ω-languages recognized by a deterministic table-transition automaton.

THEOREM 2.1:
- *DT*-REC *is a boolean algebra.*
- *DT*-REC = *T*-REC = RAT.

Proof

1 *a*. Let $\mathscr{A} = (Q, I, \Delta, \mathscr{I})$ be a complete deterministic table-transition automaton.

It's clear that the deterministic table-transition automaton $\mathscr{A}' = (Q, I, \Delta, \mathscr{P}(\Delta)\text{-}\mathscr{I})$ recognizes the complement of $L(\mathscr{A})$.

b. Let $\mathscr{A}_1 = (Q_1, I_1, \Delta_1, \mathscr{I}_1)$ and $\mathscr{A}_2 = (Q_2, I_2, \Delta_2, \mathscr{I}_2)$ be two complete deterministic table-transition automata. Let $\mathscr{A}_x = (Q, I, \Delta, \mathscr{I})$ the deterministic table-transition automaton with:

- $Q=Q_1 \times Q_2$, $I=I_1 \times I_2$,
- $\Delta=\{((q_1,q_2), a, (p_1,p_2))/(q_1,a,p_1)\in\Delta_1, (q_2,a,p_2)\in\Delta_2\}$
- $\mathscr{I}$ is defined by:

$$T\in\mathscr{I}\Leftrightarrow\begin{cases}\exists T_1\in\mathscr{I}_1/T_1=\{(q_1,a,q'_1)\in\Delta_1/((q_1,q_2),a,(q'_1,q'_2))\in T\}\\ \text{or}\\ \exists T'_2\in\mathscr{I}_2/T_2=\{(q_2,a,q'_2)\in\Delta_2/((q_1,q_2),a,(q'_1,q'_2))\in T\}\end{cases}$$

It's easy to show that $L(\mathscr{A})=L(\mathscr{A}_1)\cup L(\mathscr{A}_2)$.
From 1 *a* and 1 *b*, we deduce that DT-REC is a Boolean algebra.

2 *a*. It is clear that DT-REC $\subseteq$ T-REC.

2 *b*. Let $\mathscr{A}=(Q,\{q_0\}, \Delta,\mathscr{I})$ be a table-transition automaton which recognizes the rational ω-language L. Let $\mathscr{A}_M=(Q',I',\Delta',\mathscr{E})$ the Muller automaton where $Q'=\{q_0\}\cup\Delta$, $I'=\{q_0\}$, $\mathscr{E}=\mathscr{I}$ and

$$\Delta=\begin{cases}\{(q_0,a,(q_0,a,q))/(q_0,a,q)\in\Delta\}\cup\\ \{((q_1,a,q_2),b,(q_2,b,q_3))/(q_1,a,q_2)\in\Delta\,(q_2,b,q_3)\in\Delta\}\end{cases}$$

It is clear that $L=L(\mathscr{A})=L(\mathscr{A}_M)$ since $\forall w\in X^\omega$, $w\in L\Leftrightarrow w\in L(\mathscr{A})\Leftrightarrow t\text{-}\inf(w)\in\mathscr{I}\Leftrightarrow q\text{-}\inf(w)\in\mathscr{E}\Leftrightarrow w\in L(\mathscr{A}_M)$ so T-REC $\subseteq$ RAT.

2 *c*. Let us consider a deterministic Muller automaton $\mathscr{A}_M=(Q,I,\Delta,\mathscr{E})$ that recognizes the rational ω-language L. We have $L(\mathscr{A}_M)=\bigcup\{L(\mathscr{A}_E)/E\in\mathscr{E}\}$.

Let $\mathscr{I}=\{T\subseteq\{\delta=(q,a,p)\in\Delta/(q,p)\in E\times E\}$ such that $\forall q\in E$, $\exists\delta\in T/\delta=(q,a,p)\}$.

Let $\mathscr{A}_T$ be the table-transition automaton $(Q,I,\Delta,\{T\})$. It is clear that $L(\mathscr{A})=\bigcup\{L(\mathscr{A}_T)/T\in\mathscr{I}\}$, so $L=L(\mathscr{A}_M)=\bigcup L(\mathscr{A}_E)=\bigcup L(\mathscr{A}_T)$ and RAT $\subseteq$ DT-REC. ■

3. DETERMINISTIC TRANSITION SYSTEMS AND RIGHT CONGRUENCES

A *right congruence* $\simeq$ over X^* is an equivalence relation which satisfies: $\forall(u,v,w)\in X^*\times X^*\times X^*$, $u\simeq v\Rightarrow uw\simeq vw$. We denote $u/\simeq$ the $\simeq$-class of u and we say that $\simeq$ is finite if it has a finite number of classes.

We say that a *right congruence* $\simeq$ *preserves the successors* in a ω-language L iff $\forall(v_1,v_2)\in X^*\times X^*$, $\forall w\in X^\omega$, $v_1\simeq v_2\Rightarrow\{v_1 w\in L\Leftrightarrow v_2 w\in L\}$.

Let $S=(Q,\{q_0\},\Delta)$ be a deterministic transition system, we denote by $\simeq_Q$ the relation defined by $u\simeq_Q V\Leftrightarrow\{\forall q\in Q: q_0\xrightarrow{u}q\Leftrightarrow q_0\xrightarrow{v}q\}$.

Proposition 3.1: *The class of deterministic complete transition systems is isomorphic to the class of finite right congruences.*

– Let $S=(Q, \{q_0\}, \Delta)$ be a deterministic finite transition system, it is easy to show (from the determinism of S) that $\simeq_Q$ is a finite right congruence. Let $\simeq$ be a finite right congruence, we define the transition system $S\simeq=(Q\simeq, I\simeq, \Delta\simeq)$ with $Q\simeq=\{u/\simeq\}$, $I\simeq=\{\varepsilon/\simeq\}$, $\Delta\simeq=\{(u/\simeq, a, (ua)/\simeq)\}$. It is clear that $S\simeq$ is a deterministic transition system. ■

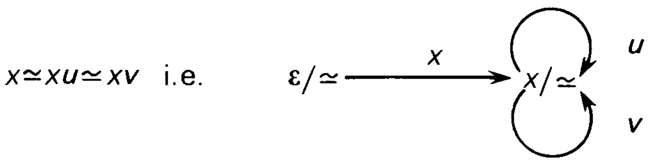
Proposition 3.2: *Let $\mathscr{A}$ be a deterministic complete Muller (resp. table-transition) automaton and $\simeq_Q$ the associated right congruence then $\simeq_Q$ preserves the successors in $L(\mathscr{A})$.*

– It is clear that, if two words of X^* have the same computation on $\mathscr{A}$, they also have the same successors in $L(\mathscr{A})$. ■

4. SATURATING RIGHT CONGRUENCES

Définition : Let L be a ω-language and $\simeq$ a right congruence. We say that $\simeq$ *saturates* L iff $\simeq$ preserves the successors in L and satisfies:

$$\forall (x, u, v) \in X^* \times X^+ \times X^+, x \simeq xu \simeq xv \text{ and } x(u^+ v^+)^\omega \cap L \neq \varnothing \text{ implie } x(u^+ v^+)^\omega \subseteq L.$$



Proposition 4.1: *Let $\mathscr{A}$ be a deterministic complete Muller (resp. table-transition) automaton and $\simeq_Q$ the associated right congruence then $\simeq_Q$ saturates $L(\mathscr{A})$.*

Proof: Let us prove that if a deterministic complete Muller (resp. table-transition) automaton $\mathscr{A}=(Q, I, \Delta, \mathscr{E})$, the finite right congruence $\simeq_Q$ saturates $L=L(\mathscr{A})$. From the proposition 3.2, $\simeq_Q$ preserves the successors in $L(\mathscr{A})$.

Let $(x, u, v) \in X^* \times X^+ \times X^+$ such that $x \simeq_Q xu \simeq_Q xv$ and $x(u^+ v^+)^\omega \cap L \neq \varnothing$. Let $u = u_1 u_2 \ldots u_n$ and $v = v_1 v_2 \ldots v_m$ with $(u_1, u_2, \ldots, u_n, v_1, v_{21} \ldots, v_m) \in X^{n+m}$.

– Let us suppose that $\mathscr{A}$ is a deterministic Muller automaton:

Since $x \simeq_Q xu \simeq_Q xv$, it is clear that $\forall w' \in x(u^+ v^+)^\omega$, $q\text{-inf}(w) = Q_u \cup Q_v$ where Q_u and Q_v are the sets of states of Q defined by:

$$Q_u = \{(xu_i \ldots u_i)/\simeq_Q, \forall 1 \leqq i \leqq n\} \quad \text{and} \quad Q_v = \{(xv_1 \ldots v_i)/\simeq_Q, \forall_1 \leqq i \leqq m\},$$

If $x(u^+ v^+)^\omega \cap L \neq \varnothing$ then $Q_u \cup Q_v \in \mathscr{E}$ and $x(u^+ v^+)^\omega \subseteq L$.

– Let us suppose that $\mathscr{A}$ is a deterministic table-transition automaton: Since $x \simeq_Q xu \simeq_Q xv$, it is clear that $\forall w' \in x(u^+ v^+)^\omega$, $t\text{-inf}(w) = T_u \cup T_v$ where T_u and T_v are the sets of transitions of Δ defined by:

– $T_u = \{((xu_1 \ldots u_i)/\simeq_Q, u_{i+1}, (xu_1 \ldots u_{i+1})/\simeq_Q) \ \forall 1 \leqq i \leqq n-1\}$

– $T_v = \{((xv_1 \ldots v_i)/\simeq_Q, v_{i+1}, (xv_1 \ldots v_{i+1})/\simeq_Q) \ \forall 1 \leqq i \leqq m-1\}$

If $x(u^+ v^+)^\omega \cap L \neq \varnothing$ then $T_u \cup T_v \in \mathscr{C}$ and $x(u^+ v^+)^\omega \subseteq L$. ■

In order to prove the converse, we need some results.

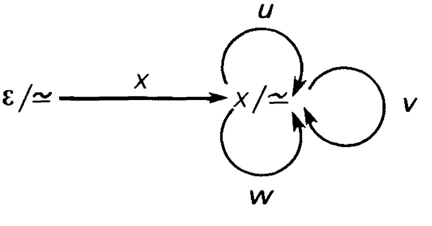
LEMMA 4.2: *Let L be a ω-language and $\simeq$ be a right congruence which saturates L. Let $(x, u, v, w) \in X^* \times X^+ \times X^+$ such that $x \simeq xu \simeq xv \simeq xw$. We get:*

1. $x(uvw)^\omega \in L \Leftrightarrow x(vuvw)^\omega \in L$.
2. $x(uvw)^\omega \in L \Leftrightarrow x(vuw)^\omega \in L$.
3. $x(uvw)^\omega \in L \Leftrightarrow x(uvuw)^\omega \in L$.

Proof: Let $\simeq$ be a right congruence which saturates L.

Let $(x, u, v, w) \in X^* \times X^+ \times X^+$ such that $x \simeq xu \simeq xv \simeq xw$

i.e.



1. $x\,(uvw)^\omega \in L \Leftrightarrow x\,(uvwvw)^\omega \in L$ $\quad$ ($x \simeq xu \simeq xvw$ and $\simeq$ saturates L)

$\Leftrightarrow x\,(uvwvuvwvw)^\omega \in L$ $\quad$ ($x \simeq xw \simeq xuvwv$ and $\simeq$ saturates L)

$\Leftrightarrow x\,(uvwvuvw)^\omega \in L$ $\quad$ ($x \simeq xvw \simeq xuvwvu$ and $\simeq$ saturates L)

$\Leftrightarrow xuvw\,(vuvwuvw)^\omega \in L$

$$\Leftrightarrow x\,(v\,(uvw)^2)^\omega \in L \qquad (\simeq \text{ preserves the successors in } L)$$
$$\Leftrightarrow x\,(vuvw)^\omega \in L \qquad (x \simeq xv \simeq xuvw \text{ and } \simeq \text{saturates } L).$$

2. From (1), we have $x\,(uvw)^\omega \in L \Leftrightarrow x\,(vuvw)^\omega \in L \Leftrightarrow xvu\,(vwvu)^\omega \in L \Leftrightarrow x\,(vwvu)^\omega \in L$. Let $U=v$, $V=wv$, $W=u$, from (1) we deduce: $x\,(uvw)^\omega \in L \Leftrightarrow x\,(vwvu)^\omega \in L \Leftrightarrow x\,(UVW)^\omega \in L \Leftrightarrow x\,(VUVW)^\omega \in L \Leftrightarrow x\,(wvvwvu)^\omega \in L$ and thus:

$$x\,(wvvwvu)^\omega \in L \Leftrightarrow x\,(wv^2\,wvu)^\omega \in L$$
$$\Leftrightarrow xw\,(v^2\,wvuw)^\omega \in L$$
$$\Leftrightarrow xw\,(vwvuw)^\omega \in L \qquad (xw \simeq xwv \simeq xwvuw,\ \simeq \text{saturates } L)$$
$$\Leftrightarrow x\,((wv)^2\,u)^\omega \in L$$
$$\Leftrightarrow x\,(wvu)^\omega \in L \qquad (x \simeq xwv \simeq xu,\ \simeq \text{saturates } L)$$
$$\Leftrightarrow xw\,(vuw)^\omega \in L$$
$$\Leftrightarrow x\,(vuw)^\omega \in L \qquad (\simeq \text{ preserves the successors in } L).$$

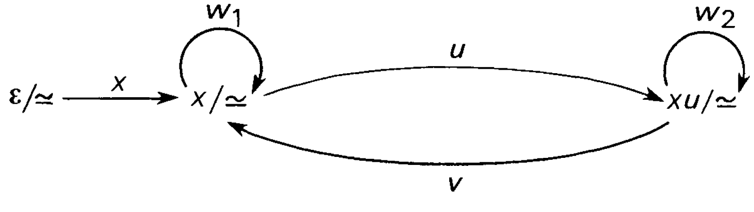
3. $x\,(uvw)^\omega \in L \Leftrightarrow x\,(vuw)^\omega \in L$ (Lemma 4.2.2)

$$\Leftrightarrow x\,(uvuw)^\omega \in L \qquad \text{(Lemma 4.2.1).} \ \blacksquare$$

COROLLARY 4.3: *Let L be a ω-language and $\simeq$ a right congruence which saturates L. Let $(x, w_1, w_2, u, v) \in X^* \times X^* \times X^* \times X^* \times X^*$ such that $x \simeq xw_1 \simeq xuv$ and $xu \simeq xuw_2$ then*

$$x\,(w_1\,uw_2\,v)^\omega \in L \quad \Leftrightarrow \quad x\,(w_1\,uw_2\,vuv)^\omega \in L.$$

Proof: We have $(x, u, v, w_1, w_2) \in (X^*)^5$ such that $x \simeq xw_1 \simeq xuv$ and $xu \simeq xuw_2$ and $\simeq$ saturates L:



$$x\,(w_1\,uw_2\,uv)^\omega \in L \quad \Leftrightarrow x\,((w_1)\,(uw_2\,v)\,(w_1)\,(uv))^\omega \in L \ \text{(Lemma 4.2.3)}$$
$$\Leftrightarrow xw_1\,u\,(w_2\,vw_1\,uvw_1\,u)^\omega \in L$$
$$\Leftrightarrow xw_1\,u\,((w_2)\,(vw_1\,u)\,(w_2)\,(vw_1\,u))^\omega \in L \ \text{(Lemma 4.2.3)}$$
$$\Leftrightarrow xw_1\,u\,(w_2\,vw_1\,u)^\omega \in L$$
$$\Leftrightarrow x(w_1\,uw_2\,v)^\omega \in L. \ \blacksquare$$

The following result is a generalization of the previous corollary:

COROLLARY 4.4: *Let L be a* ω*-language and* $\simeq$ *a right congruence which saturates* L *and let* $x, y_1, y_2, \ldots, y_n, v_1, v_2, \ldots, v_{n+1} \in X^*$ *such that* $x \simeq xv_1 \simeq xy_1 \ldots y_n$ *and* $\forall\, 1 \leqq i \leqq n$, $xy_1 \ldots y_i \simeq xy_1 \ldots y_1 v_{i+1}$ *then*

$$x(v_1 y_1 v_2 \ldots y_n v_{n+1})^\omega \in L \quad \Leftrightarrow \quad x((v_1 y_1 v_2 \ldots y_n v_{n+1})(y_1 \ldots y_n))^\omega \in L$$

Proof:

$x((v_1 y_1 v_2 \ldots y_n v_{n+1})(y_1 \ldots y_n))^\omega \in L \Leftrightarrow$

$x((v_1)(v_1 y_1 v_2 \ldots y_n v_{n+1})(y_1 \ldots y_n))^\omega \in L \Leftrightarrow$

$x((v_1 y_1 v_2 \ldots y_n v_{n+1}) v_1 (y_1 \ldots y_n))^\omega \in L \Leftrightarrow$ (Lemma 4.2.3)

$xv_1 y_1 ((v_2)(v_2 y_2 \ldots y_n v_{n+1} v_1 y_1)(y_2 \ldots y_n \ldots y_n v_1 y_1))^\omega \in L \Leftrightarrow$

$xv_1 y_1 ((v_1 y_1 \ldots y_n v_{n+1} v_1 y_1) v_2 (y_2 \ldots y_n v_1 y_1))^\omega \in L \Leftrightarrow$ (Lemma 4.2.3)

$xv_1 y_1 v_2 (y_2 (v_3 y_3 \ldots y_n v_{n+1} v_1 y_1 v_2 y_2)(y_3 \ldots y_n v_1 y_1 v_2 y_2))^\omega \in L \Leftrightarrow$

$\vdots$

$xv_1 y_1 \ldots v_n y_n ((v_{n+1} v_1 y_1 \ldots v_n y_n)(v_1 y_1 \ldots v_n y_n))^\omega \in L \Leftrightarrow$

$xv_1 y_1 \ldots v_n y_n ((v_{n+1})(v_{n+1} v_1 y_1 \ldots v_n y_n)(v_1 y_1 \ldots v_n y_n))^\omega \in L \Leftrightarrow$

$xv_1 y_1 \ldots v_n y_n ((v_{n+1} v_1 y_1 \ldots v_n y_n)(v_{n+1})(v_1 y_1 \ldots v_n y_n))^\omega \in L \Leftrightarrow$

$xv_1 y_1 \ldots v_n y_n v_{n+1} ((v_1 y_1 \ldots v_n y_n v_{n+1})(v_1 y_1 \ldots v_n y_n v_{n+1}))^\omega \in L \Leftrightarrow$

$x(v_1 y_1 \ldots v_n y_n v_{n+1})^\omega \in L.$ ■

DEFINITION: Let L be a ω-language and $\simeq$ be a finite right congruence. We define the deterministic complete table-transition automaton $\mathscr{A}\simeq = (Q\simeq, I\simeq, \Delta\simeq, \mathscr{I}\simeq)$ by:

– $Q\simeq = \{u/\simeq,\ u \in X^*\}$, $I\simeq = \{\varepsilon/\simeq\}$, $\Delta\simeq = \{(u/\simeq, a, ua/\simeq),\ u \in X^*, a \in X\}$,

– $\mathscr{I}\simeq = \{T \in \mathscr{P}(\Delta) / \exists\, w \in UP(L), T = t\text{-inf}(w)\}$.

THEOREM 4.5: *Let L be a rational* ω*-language and* $\simeq$ *a finite right congruence which saturates L. We have:* $L = L(\mathscr{A}\simeq)$.

Proof: Since L and $L(\mathscr{A}\simeq)$ are rational ω-languages, it suffice to prove that $UP(L) = UP(L(\mathscr{A}\simeq))$.

– Let $w \in UP(L)$. Since $\simeq$ is finite, by definition of $\mathscr{A}\simeq$, we have $t\text{-inf}(w) \in \mathscr{I}\simeq$ and then $w \in L(\mathscr{A}\simeq)$.

– Let us prove that $UP(L(\mathscr{A}\simeq)) \subseteq UP(L)$. Let $w \in UP(L(\mathscr{A}\simeq))$. By definition of $\mathscr{A}\simeq$, we have:

$$w \in UP((L(\mathscr{A}\simeq)) \quad \Leftrightarrow \quad \exists\, w' \in UP(L)/t-\inf(w') = t\text{-}\inf(w) \in \mathscr{I}\simeq.$$

Since $w \in UP((L(\mathscr{A}\simeq))$, there exists $x,\ y \in X^*$ such that $x \simeq xy$, $w = xy^\omega$.

Since $w' \in UP(L)$, there exists $x',\ y' \in X^*$ such that $x' \simeq x'\,y'$, $w' = x'\,y'^\omega$.

Let $y = y_1 \ldots y_n$ and $y' = y'_1 \ldots y'_{n'}$ such that $\forall\, 1 \leqq i \leqq n$, $\forall\, 1 \leqq j \leqq n'$, $y_i \in X$, $y'_j \in X$.

We have:

$$t\text{-}\inf(xy^\omega) = \{(x/\simeq, y_1, xy_1/\simeq), \ldots, ((xy_1, \ldots, y_i)/\simeq, y_i, (xy_1, \ldots, y_{i+1})/\simeq), \ldots\}$$

$$t\text{-}\inf(x'\,y'^\omega) = \{(x'/\simeq, y'_1, x'\,y'_1/\simeq), \ldots, ((x'\,y'_1, \ldots, y'_i)/\simeq, y'_i, (x'y'_1, \ldots, y'_{i+1})/\simeq), \ldots\}.$$

Since $t\text{-}\inf(xy^\omega) = t\text{-}\inf(x'\,y'^\omega)$:

a. there exists $v_1, \ldots, v_n,\ u_1, \ldots, u_n \in X^*$ such that $\forall\, 1 \leqq i \leqq n'$, $y = u_i\, y'_i\, v_i$ and $xu_i \simeq xy'_1 \ldots y'_{i-1}$ and $x\, u_i\, y'_i\, v_i \simeq x$.

b. there exists $v'_1, \ldots, v'_{n'},\ u'_1, \ldots, u'_{n'} \in X^*$ such that $\forall\, 1 \leqq i \leqq n$, $y' = u'_i\, y_i\, v'_i$ and $x'\, u'_i \simeq xy_1 \ldots y_{i-1}$ and $x'\, u'{}_i\, y_i\, v'_i \simeq x'$.

Let $x'' = x'\,u'_1$ and $y'' = y_1\, v'_1\, u'_1 = y''_1 \ldots y''_{n''}$ such that $\forall\, 1 \leqq i \leqq n''$, $y''_i \in X$, we have $x'' \simeq x$ and $y_1 = y''_1$. So we can suppose that x' and y' are choosen such that $y_1 = y'_1$ and $x' \simeq x$ and $u_1 = u'_1 = \varepsilon$.

We have: $w = xy^\omega = x\,(y^n)^\omega = x\,(y'_1\, v_1\, u_2\, y'_2\, v_2\, u_3\, y'_3\, v_3 \ldots u_n\, y'_n\, v_n)^\omega$

By the corollary 4.4, we have:

– $w = xy^\omega \in L \Leftrightarrow x\,(y^n\, y')^\omega \in L \Leftrightarrow x\,(yy')^\omega \in L$

By a similar construction, we have:

– $w' = x'\,y'^\omega \in L \Leftrightarrow x'\,(y'^n)^\omega \in L \Leftrightarrow x'\,(y'^{n'}\, y)^\omega \in L \Leftrightarrow x\,(yy')^\omega \in L$;

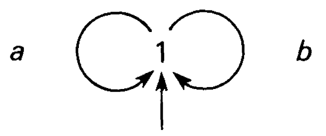
Hence the result. ■

THEOREM 4.6: *The family of deterministic complete table-transition automata which recognize a rational ω-language L is isomorphic to the family of finite right congruences which saturate L.*

Proof: In the theorem 4.5, we proved that, from a finite right congruence $\simeq$ which saturates a rational ω-language L, we can build a deterministic table-transition automaton $\mathscr{A}\simeq$ such that $\mathrm{L}(\mathscr{A}\simeq) = L$ and $\simeq_Q = \simeq$. In the theorem 4.1, we proved that, if $\mathscr{A}$ is a deterministic table-transition automaton which recognizes a ω-language L then the associated finite right congruence $\simeq_Q$ saturates L. ■

Remark: We saw, in the proof of the theorem 2.1, that any deterministic Muller automaton, is in fact a table transition automaton and so any deterministic Muller automaton which recognize $L \in \text{RAT}$, defines a finite right congruence which saturates L, but the converse is not true:

Let $L = \{a, b\}^* a^\omega \in \text{RAT}$ and $\simeq$ the finite right congruence with one class $\varepsilon/\simeq\, = \{a, b\}^*$. It's obvious to see that $\simeq$ saturates L, but the associated transition system $\mathscr{S}\simeq$ cannot leads us to build a Muller automaton which recognizes L:



5. CYCLICALLY CLOSED TABLE-TRANSITION AUTOMATA

DÉFINITION : We say that a table-transition automaton $\mathscr{A} = (Q, I, \Delta, \mathscr{I})$ is *cyclically closed* iff $\forall\, T \in \mathscr{I}$, $\forall\, T' \subseteq \Delta$ such that $T \cup T'$ is coherent, $T \cup T' \in \mathscr{I}$.

THEOREM 5.1: *Let $\mathscr{A} = (Q, I, \Delta, \mathscr{I})$ be a deterministic complete table-transition automaton.*

$$L(\mathscr{A}) \in D\text{-RAT} \quad \text{iff} \quad \mathscr{A} \text{ is cyclically closed.}$$

Proof: a. Let $\mathscr{A} = (Q, \{q_0\}, \Delta, \mathscr{I})$ be a cyclically closed deterministic table-transition automaton which recognizes the rational ω-languages L. Let us prove that $L \in D$-RAT.

Like in paragraph 2, let $\mathscr{A}_M = (Q', I', \Delta', \mathscr{E})$ the deterministic Muller automaton where $Q' = \{q_0\} \cup \Delta$, $I' = \{q_0\}$, $\mathscr{E} = \mathscr{I}$ and

$$\Delta = \begin{cases} \{(q_0, a, (q_0, a, q))/(q_0, a, q) \in \Delta\} \cup \\ \{((q_1, a, q_2), b, (q_2, b, q_3))/(q_1, a, q_2) \in \Delta\, (q_2, b, q_3) \in \Delta\} \end{cases}$$

It is clear that $L = L(\mathscr{A}) = L(\mathscr{A}_M)$ since $\forall\, w \in X^\omega$, $w \in L \Leftrightarrow w \in L(\mathscr{A}) \Leftrightarrow t\text{-inf}(w) \in \mathscr{I} \Leftrightarrow q\text{-inf}(w) \in \mathscr{E} \Leftrightarrow w \in L(\mathscr{A}_M)$. Moreover $\mathscr{A}$ is cyclically closed so $\mathscr{A}_M$ satisfies the property: $\forall E \in \mathscr{E}$, $\forall E'\ Q^*$, $E \cup E' \in \mathscr{I}$. L. H. Landweber used this property to characterize the Muller automata which recognize the deterministic ω-languages [LA (69)] so $L \in D$-RAT.

b. Let $\mathscr{A} = (Q, \{q_0\}, \Delta, \mathscr{I})$ be a deterministic table-transition automaton which recognizes a deterministic rational ω-languages L. Let us prove that $\mathscr{A}$ is cyclically closed.

Let $T \in \mathscr{I}$ and $T' \in \Delta$ such that $T \cup T'$ is coherent. There exists $x \in X^*$, $u = u_1 u_2 \ldots u_n$, $v = v_1 v_2 \ldots v_m$, $\delta_1, \delta_2, \ldots, \delta_n \in \Delta$, $\delta_{1'}, \delta_{2'}, \ldots, \delta_{m'} \in \Delta$, $q \in Q$ such that:

– $q_0 \overset{x}{\rightarrow} q$;

– $q \overset{u}{\rightarrow} q$ and
$\forall\, 1 \leqq i \leqq n$ $\delta_i = (q_i, u_i, q_{i+1})$, $q = q_{1'} = q_{m+1}$ and $T = \{\delta_1, \delta_2, \ldots, \delta_n\}$;

– $q \overset{v}{\rightarrow} q$ and
$\forall\, 1 \leqq i \leqq m$ $\delta'_1 = (q'_1, v_i, q'_{i+1})$, $q = q'_1 = q_{n+1}$ and $T' = \{\delta'_1, \delta'_2, \ldots; \delta'_m\}$.

We have $t\text{-inf}(xu^\omega) = T$ so $x\{u, v\}^* u^\omega \subseteq L$. Since

$$L \in D\text{-RAT}, \qquad \exists K \in \mathrm{RAT}/\vec{K} = L.$$

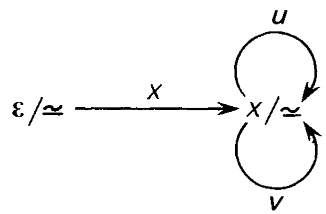
We have $x\{u, v\}^* u^\omega \subseteq \vec{K}$ so, like [AR (84) Lemma III.1], it is easy to build by induction a sequence $(w_n)_{n \in \mathbb{N}'}$ such that $\forall i \in \mathbb{N}$, $w_i \in K$, $/w_i/ < /w_{i+1}/$ and $\lim_{w \to \infty} (w_n) = w \in x\{u^+ v^+_,\}^\omega \cap L$ and since $\forall w \in x\{u^+ v^+\}^\omega$, t-inf$(w) = (T \cup T') \in \mathscr{I}$ and $\mathscr{A}$ is cyclically closed. ■

6. LANDWEBER RIGHT CONGRUENCE

DÉFINITION : Let L be a rational ω-language and $\simeq$ a right congruence. We say that $\simeq$ is *L-Landweber right congruence* iff $\simeq$ preserves the successors in L and $\simeq$ satisfies:

$$\forall (x, u, v) \in X^* \times X^+ \times X^+,\ x \simeq xu \simeq xv$$
$$\text{and} \quad x\{u, v\}^* u^\omega \cap L \neq \varnothing \quad \text{implies}\ x(v^* u)^\omega \subseteq L^\omega$$

$x \simeq xu \simeq xv$ i.e.



THEOREM 6.1: *Let L be a rational language and* $\simeq$ *a finite right congruence on* X^*.

If $\simeq$ *is a L-Landweber right congruence then* $\simeq$ *saturates L.*

Proof: Let L be a rational language and $\simeq$ finite L-Landweber right congruence. Let us prove that $\simeq$ saturates L.

By definition, we have:

$$\forall (x, u, v) \in X^* \times X^+ \times X^+ / x \simeq xu \simeq xv : x\{u, v\}^* u^\omega \cap L \neq \emptyset \Rightarrow x(v^* u)^\omega \subseteq L^\omega.$$

Let $(x, u, v) \in X^* \times X^+ \times X^+$ such that $x \simeq xu \simeq xv$ and $x(u^+ v^+)^\omega \cap L \neq \emptyset$ and let us prove that $x(u^+ v^+)^\omega \subseteq L$. Since L and $x(u^+ v^+)^\omega$ are rational languages, $x(u^+ v^+)^\omega \cap L$ is a rational language. So $UP(x(u^+ v^+)^\omega \cap L) \neq \emptyset$ and it suffice to prove that $UP(x(u^+ v^+)^\omega) \subseteq UP(L)$ (Büchi's lemma).

Let $w \in UP(x(u^+ v^+)^\omega)$, there exists $(p_1, p_2, \dots, p_n) \in \mathbb{N}^n$ such that $w = x(u^{p_1} v^{p_2} \dots u^{p_{n-1}} v^{p_n})^\omega$. Since $\simeq$ is L-Landweber right congruence and $x \simeq xu \simeq xv \simeq xu^{p_1} v^{p_2} \dots u^{p_{n-1}} v^{p_n}$, we have:

$$\begin{aligned} x(u^{p_1} v^{p_2} \dots u^{p_{n-1}} v^{p_n})^\omega \in L &\Leftrightarrow xu(u^{p_1-1} v^{p_2} \dots u^{p_{n-1}} v^{p_n} u)^\omega \in L \\ &\Leftrightarrow xu(vu^{p_1-1} v^{p_2} \dots u^{p_{n-1}} v^{p_n} u)^\omega \in L \\ &\Leftrightarrow xuvu(u^{p_1-2} v^{p_2} \dots u^{p_{n-1}} v^{p_n} uvu)^\omega \in L \\ &\Leftrightarrow xuvu(vu^{p_1-2} v^{p_2} \dots u^{p_{n-1}} v^{p_n} uvu)^\omega \in L \\ &\Leftrightarrow x((uv)^2 u^{p_1-2} v^{p_2} \dots u^{p_{n-1}} v^{p_n})^\omega \in L \end{aligned}$$

It now easy to show, iterating the previous calculus, that

$$\begin{aligned} w \in L &\Leftrightarrow x(u^{p_1} v^{p_2} \dots u^{p_{n-1}} v^{p_n})^\omega \in L \\ &\Leftrightarrow x((uv)^{2*(P_1+P_2+\dots+P_{-1}+P_{n'})})^\omega \in L \\ &\Leftrightarrow x(uv)^\omega \in L. \end{aligned}$$

So we have $\forall (w, w') \in (x(u^+ v^+)^\omega) \times (x(u^+ v^+)^\omega)$, $w \in L \Leftrightarrow w' \in L$ and since $x(u^+ v^+)^\omega \cap L \neq \emptyset$, we have $x(u^+ v^+)^\omega \subseteq L$. ■

THEOREM 6.2: *The family of deterministic complete cyclically closed table-transition automata which recognize a rational ω-language L is isomorphic to the family of finite L-Landweber right congruences.*

Proof: a. Let us prove that if $\mathcal{A}=(Q, I, \Delta, \mathcal{I})$ is a deterministic cyclically closed table-transition automaton, its associated finite right congruence $\simeq_Q$ is L-Landweber *i.e.*

$$\forall (x, u, v)\in X^* \times X^+ \times X^+/x\simeq_Q xu\simeq_Q xv: x\{u, v\}^* u^\omega \cap L\neq\varnothing \Rightarrow x(v^* u)^\omega\in L^\omega.$$

Let x, u, $v\in X^*$ such that $x\simeq_Q xu\simeq_Q xv$ and let us suppose that $x\{u, v\}^* u^\omega\in L$.

Since $\simeq_q$ preserves the successors in L, $xu^\omega\in L$.

Let $T=t-\inf(xu^\omega)$ and $T'=t\text{-}\inf(xv^\omega)$. Since $xu^\omega\in L$, $T\in\mathcal{I}$ and since $x\simeq xu\simeq xv$, T and T' have a common state. Moreover, $\mathcal{A}$ is cyclically closed then $\forall w\in x(v^* u)^\omega$, $t\text{-}\inf(w)=t\text{-}\inf(x(uv)^\omega)=(T\cup T')\in\mathcal{I}$ and $x(v^* u)^\omega\subseteq L$.

b. Let $L\in\mathrm{RAT}$ and $\simeq$ a finite right L-Landweber congruence. We denote $\mathcal{A}\simeq$ the deterministic complete table-transition automaton $(Q\simeq, I\simeq, \Delta\simeq, \mathcal{I}\simeq)$ where

$$Q\simeq=\{x/\simeq, x\in X^*\} \quad \text{and} \quad I\simeq=\{\varepsilon/\simeq\}$$

$$\Delta\simeq=\{(x/\simeq, a, (xa)/\simeq), x\in X^*, a\in X\}$$

$$\mathcal{I}\simeq=\{T\subset P(\Delta)/\exists w\in UP(L): t\text{-}\inf(w)=T\}.$$

From the theorem 6.1, $\simeq$ saturates L from by the theorem 4.5, $L=L(\mathcal{A}\simeq)$. It remains to prove $\mathcal{A}\simeq$ is cyclically closed.

Let $(T, T')\in\mathcal{I}\simeq\times P(\Delta)$ such T and T' have a common state. Let us prove that $(T\cup T')\in\mathcal{I}\simeq$.

Let $w=xu^\omega\in UP(L)$ such t-inf $(w)=T\in\mathcal{I}\simeq$ and let $w'=x'v^\omega\in X^\omega$ such that $t\text{-}\inf(w')=T'$ and $x\simeq xu$, $x'\simeq x'v$.

Since T, T' have a common state, there exists $v=v_1 v_2$ and $u=u_1 u_2$ such that $t\text{-}\inf(x'(v_1 u_2 u_1 v_2))^\omega=(T\cup T')$ and $x' v_1\simeq xu_1$. Since $x' v_1\simeq x' v_1 v_2 v_1$ and $xu_1\simeq xu_1 u_2 u_1$, we get $x' v_1 (u_2 u_1)\simeq x' v_1 (v_2 v_1)$ and, since $\simeq$ preserves the successors in L, we have: $x' v_1 (u_2 u_1)^\omega\in L$.

Since $\simeq$ is a finite L-Landweber right congruence and $x' v_1 (u_2 u_1)\simeq x' v_1 (v_2 v_1)$ and $x' v_1 (u_2 u_1)^\omega\in L$, we get:

$$x' v_1 (u_2 u_1)^\omega\in L \quad \Rightarrow \quad x' v_1 (u_2 u_1 v_2 v_1)^\omega\in L \quad \Rightarrow \quad x'(v_1 u_2 u_1 v_2)^\omega\in L.$$

Moreover, we have $t-\inf(x'(v_1 u_2 u_1 v_2)^{\omega}) = T \cup T'$, so $T \cup T' \in \mathcal{I}_{\simeq}$. ■

COROLLARY 6.3: *$L \in D$-RAT iff there exists a finite L-Landweber right congruence.*

Proof: Since D-RAT $\subseteq$ RAT, there exists a complete deterministic table-transition automaton $\mathcal{A}$ such $L = L(\mathcal{A})$ (theorem 2.1), so $\mathcal{A}$ is a deterministic cyclically closed table cycle automata $\mathcal{A}$ (theorem 5.1) and from the theorem 6.2, there is an isomorphism between the family of complete deterministic cyclically closed table cycle automata which recognize L and the family of finite right congruences which saturate L, hence the result. ■

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