

F. BLANCHET-SADRI

**On dot-depth two**

*Informatique théorique et applications*, tome 24, n° 6 (1990),  
p. 521-529

[http://www.numdam.org/item?id=ITA\\_1990\\_\\_24\\_6\\_521\\_0](http://www.numdam.org/item?id=ITA_1990__24_6_521_0)

© AFCET, 1990, tous droits réservés.

L'accès aux archives de la revue « Informatique théorique et applications » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## ON DOT-DEPTH TWO (\*)

by F. BLANCHET-SADRI

Communicated by A. ARNOLD

---

*Abstract* – For positive integers  $m_1, \dots, m_k$ , congruences  $\sim_{(m_1, \dots, m_k)}$  related to a version of the Ehrenfeucht-Fraissé game are defined which correspond to level  $k$  of the Straubing hierarchy of star-free languages. Given any finite alphabet  $A$ , a necessary and sufficient condition is given for the monoids  $A^*/\sim_{(m_1, m_2, m_3)}$  to be of dot-depth exactly 2.

*Résumé* – Étant donnés des entiers positifs  $m_1, \dots, m_k$ , on définit des congruences  $\sim_{(m_1, \dots, m_k)}$  en relation avec une version du jeu de Ehrenfeucht-Fraissé, et qui correspondent au niveau  $k$  de la hiérarchie de concaténation de Straubing. Étant donné un alphabet fini  $A$ , une condition nécessaire et suffisante est donnée pour que les monoïdes définis par ces congruences soient de dot-depth exactement 2.

### 1. INTRODUCTION

Let  $A$  be a given finite alphabet. The regular languages over  $A$  are those subsets of  $A^*$ , the free monoid generated by  $A$ , constructed from the finite languages over  $A$  by the boolean operations, the concatenation product and the star. The star-free languages are those regular languages which can be obtained from the finite languages by the boolean operations and the concatenation product only. According to Schützenberger [15],  $L \subseteq A^*$  is star-free if and only if its syntactic monoid  $M(L)$  is finite and aperiodic. General references on the star-free languages are McNaughton and Papert [10], Eilenberg [6] or Pin [12].

Natural classifications of the star-free languages are obtained based on the alternative use of the boolean operations and the concatenation product. Let  $A^+ = A^* \setminus \{1\}$ , where 1 denotes the empty word. Let

---

(\*) Received in January 1989, revised in February 1989

<sup>(1)</sup> Department of Mathematics, McGill University, 805 Sherbrooke Street West, Montreal, P. Quebec, Canada, H3A 2K6

$$A^+ \mathcal{B}_0 = \{ L \subseteq A^+ \mid L \text{ is finite or cofinite} \},$$

$$A^+ \mathcal{B}_{k+1} = \{ L \subseteq A^+ \mid L \text{ is a boolean combination of languages of the form } L_1 \dots L_n (n \geq 1) \text{ with } L_1, \dots, L_n \in A^+ \mathcal{B}_k \}.$$

Only nonempty words over  $A$  are considered to define this hierarchy; in particular, the complement operation is applied with respect to  $A^+$ . The language classes  $A^+ \mathcal{B}_0, A^+ \mathcal{B}_1, \dots$  form the so-called dot-depth hierarchy introduced by Cohen and Brzozowski in [4]. The union of the classes  $A^+ \mathcal{B}_0, A^+ \mathcal{B}_1, \dots$  is the class of star-free languages.

Our attention is directed toward a closely related and more fundamental hierarchy, this one in  $A^*$ , introduced by Straubing in [18]. Let

$$A^* \mathcal{V}_0 = \{ 0, A^* \},$$

$$A^* \mathcal{V}_{k+1} = \{ L \subseteq A^* \mid L \text{ is a boolean combination of languages of the form } L_0 a_1 L_1 a_2 \dots a_n L_n (n \geq 0) \text{ with } L_0, \dots, L_n \in A^* \mathcal{V}_k \text{ and } a_1, \dots, a_n \in A \}.$$

$L \subseteq A^*$  is star-free if and only if  $L \in A^* \mathcal{V}_k$  for some  $k \geq 0$ . The *dot-depth* of  $L$  is the smallest such  $k$ .

Using Eilenberg's correspondence, we have that for each  $k \geq 0$ , there is a variety  $V_k$  of finite monoids such that for  $L \subseteq A^*$ ,  $L \in A^* \mathcal{V}_k$  if and only if  $M(L) \in V_k$ . An outstanding open problem is whether one can decide if a language has dot-depth  $k$ , *i. e.*, can we effectively characterize the varieties  $V_k$ ? The variety  $V_0$  consists of the trivial monoid alone,  $V_1$  of all finite  $\mathcal{T}$ -trivial monoids [16]. Straubing [19] conjectured an effective characterization, based on the syntactic monoid of the language, for the case  $k=2$ . His characterization, formulated in terms of a novel use of categories in semigroup theory recently developed by Tilson [22], is shown to be necessary in general, and sufficient for an alphabet of two elements.

In the framework of semigroup theory, Brzozowski and Knast [1] showed that the dot-depth hierarchy is infinite. Thomas [21] gave a new proof of this result, which shows also that the Straubing hierarchy is infinite, based on a logical characterization of the dot-depth hierarchy that the obtained in [20] (Perrin and Pin gave one for the Straubing hierarchy [11]) and the following version of the Ehrenfeucht-Fraïssé game.

First, one regards a word  $w \in A^*$  of length  $|w|$  as a word model  $w = \langle \{1, \dots, |w|\}, <^w, (Q_a^w)_{a \in A} \rangle$  where the universe  $\{1, \dots, |w|\}$  represents the set of positions of letters in  $w$ ,  $<^w$  denotes the  $<$ -relation in  $w$ , and  $Q_a^w$  are unary relations over  $\{1, \dots, |w|\}$  containing the positions with letter  $a$ , for each  $a \in A$ . For a sequence  $\bar{m} = (m_1, \dots, m_k)$  of positive integers, where  $k \geq 0$ , the game  $\mathcal{G}_{\bar{m}}(u, v)$  is played between two players I and II on the word models  $u$  and  $v$ . A play of the game consists of  $k$  moves. In the

$i$ -th move, player I chooses, in  $u$  or in  $v$ , a sequence of  $m_i$  positions; then player II chooses, in the remaining word, also a sequence of  $m_i$  positions. After  $k$  moves, by concatenating the sequences chosen from  $u$  and  $v$ , two sequences  $p_1 \dots p_n$  from  $u$  and  $q_1 \dots q_n$  from  $v$  have been formed where  $n = m_1 + \dots + m_k$ .

Player II has won the play if

$$p_i <^u p_j \quad \text{if and only if} \quad q_i <^v q_j, \tag{1}$$

and

$$Q_a^u p_i \quad \text{if and only if} \quad Q_a^v q_i, \quad a \in A \quad \text{for } 1 \leq i, j \leq n. \tag{2}$$

If there is a winning strategy for II in the game  $\mathcal{G}_{\bar{m}}(u, v)$  to win each play we write  $u \sim_{\bar{m}} v$ .  $\sim_{\bar{m}}$  naturally defines a congruence on  $A^*$  which we denote also by  $\sim_{\bar{m}}$ . The standard Ehrenfeucht-Fraissé game [5] is the special case  $\mathcal{G}_{(1, \dots, 1)}(u, v)$ . Thomas [20], [21] and Perrin and Pin [11] imply that  $L \in A^* \mathcal{V}_k$  if and only if  $L$  is a  $\sim_{\bar{m}}$ -language for some  $\bar{m} = (m_1, \dots, m_k)$  (or  $L$  is a union of classes of the congruence  $\sim_{\bar{m}}$ ). This congruence characterization implies that the problem of deciding whether a language has dot-depth  $k$  is equivalent to the problem of effectively characterizing the monoids  $M = A^*/\sim$  with  $\sim \cong \sim_{\bar{m}}$  for some  $\bar{m} = (m_1, \dots, m_k)$ , *i. e.*,

$$V_k = \{ A^*/\sim \mid \sim \cong \sim_{\bar{m}} \text{ for some } \bar{m} = (m_1, \dots, m_k) \}.$$

This paper is concerned with an application of the above congruence characterization. We show that  $A^*/\sim_{(m_1, m_2, m_3)}$  is of dot-depth exactly 2 if and only if  $m_2 = 1$ . The proof relies on some properties of the congruences  $\sim_{\bar{m}}$  stated in the next section. [2] and [3] include other applications: among them are an answer to a conjecture of Pin [13] concerning tree hierarchies of monoids and also systems of equations satisfied in natural sublevels of level 1 of the Straubing hierarchy. The reader is referred to the books by Eilenberg [6], Lallement [9], Pin [12], Enderton [7] and Fraissé [8] for all the algebraic and logical terms not defined here.

**2. SOME PROPERTIES OF THE CHARACTERIZING CONGRUENCES**

**2 . 1. An induction lemma**

The following lemma is a basic result (similar to one in [14] regarding  $\sim_{(1, \dots, 1)}$ ) which allows to resolve games with  $k + 1$  moves into games with

$k$  moves and thereby allows to perform induction arguments. In what follows,  $u^{<p}$  ( $u^{>p}$ ) denotes the subword of  $u$  to the left (right) of position  $p$  and  $u_{>p}^{<q}$  the subword of  $u$  between positions  $p$  and  $q$ .

LEMMA 2.1.: Let  $\bar{m} = (m_1, \dots, m_k)$ .  $u \sim_{(m, m_1, \dots, m_k)} v$  if and only if

(1) for every  $p_1, \dots, p_m \in u (p_1 \leq \dots \leq p_m)$  there are  $q_1, \dots, q_m \in v (q_1 \leq \dots \leq q_m)$  such that

(i)  $Q_a^u p_i$  if and only if  $Q_a^v q_i$ ,  $a \in A$  for  $1 \leq i \leq m$ ,

(ii)  $u^{<p_1} \sim_{\bar{m}} v^{<q_1}$ ,

(iii)  $u_{>p_i}^{<p_{i+1}} \sim_{\bar{m}} v_{>q_i}^{<q_{i+1}}$  for  $1 \leq i \leq m-1$ ,

(iv)  $u_{>p_m} \sim_{\bar{m}} v_{>q_m}$  and

(2) for every  $q_1, \dots, q_m \in v (q_1 \leq \dots \leq q_m)$  there are  $p_1, \dots, p_m \in u (p_1 \leq \dots \leq p_m)$  such that (i), (ii), (iii) and (iv) hold.

## 2.2. A lemma for inclusion

Define

$$\mathcal{N}_{(m_1, \dots, m_k)} = m_1 + \dots + m_k + \sum_{1 \leq i_1 < i_2 \leq k} m_{i_1} m_{i_2} + \dots + \sum_{1 \leq i_1 < \dots < i_{k-1} \leq k} m_{i_1} \dots m_{i_{k-1}} + m_1 \dots m_k.$$

One can show that  $x^N \sim_{(m_1, \dots, m_k)} x^{N+1}$  ( $N = \mathcal{N}_{(m_1, \dots, m_k)}$ ) and that  $N$  is the smallest  $n$  such that  $x^n \sim_{(m_1, \dots, m_k)} x^{n+1}$  (the proof is similar to the one of a property of  $\sim_{(1, \dots, 1)}$  in [21]). We see that if  $u, v \in A^*$  and  $u \sim_{(m_1, \dots, m_k)} v$ , then  $|u|_a = |v|_a < \mathcal{N}_{(m_1, \dots, m_k)}$  or  $|u|_a, |v|_a \geq \mathcal{N}_{(m_1, \dots, m_k)}$  (here  $|w|_a$  denotes the number of occurrences of the letter  $a$  in  $w$ ). The following lemma follows easily from Lemma 2.1 and the above remarks.

LEMME 2.2 :

$$\sim_{(m_1, \dots, m_k)} \subseteq \sim_{(\mathcal{N}_{(m_1, \dots, m_k)})} \quad \text{and} \quad \sim_{(m_1, \dots, m_k)} \not\subseteq \sim_{(\mathcal{N}_{(m_1, \dots, m_k)} + 1)}.$$

If  $k \leq k'$  and  $\exists 0 = j_0 < \dots < j_{k-1} < j_k = k'$  such that  $m_i \leq \mathcal{N}_{(m'_{j_{i-1}+1}, \dots, m'_{j_i})}$  for  $1 \leq i \leq k$ , then  $\sim_{(m'_1, \dots, m'_k)} \subseteq \sim_{(m_1, \dots, m_k)}$ .

## 3. A SEQUENCE OF MONOIDS OF DOT-DEPTH 2

In this section, we show that for positive integers  $m_1, m_2$  and  $m_3$ ,  $A^*/\sim_{(m_1, m_2, m_3)}$  is of dot-depth exactly 2 if and only if  $m_2 = 1$ . The following lemma shows the necessity of the condition.

LEMMA 3.1: Let  $m_1$  and  $m_3$  be positive integers. Then  $A^*/\sim_{(m_1, 2, m_3)}$  is of dot-depth exactly 3.

*Proof:* Let  $m > 0$ . Consider  $u_m = ((xy)^m x (xy)^{2m} y (xy)^m)^m$ ,  $v_m = ((xy)^m y (xy)^{2m} x (xy)^m)^m$ . A result of Straubing [17] implies that monoids in  $V_2$  are 2-mutative and hence satisfy  $u_m = v_m$  for all sufficiently large  $m$ . However, for every  $N \geq \mathcal{N}_{(1, 2, 1)}$ ,  $u_N x_{(1, 2, 1)} v_N$ . To see this, we illustrate a winning strategy for player I in the game  $\mathcal{G}_{(1, 2, 1)}(u_N, v_N)$ .  $(I, i)$  denotes a position chosen by player I in the  $i$ -th move,  $i = 1, 2, 3$ . Similarly,  $(II, i)$  denotes a position chosen by player II in the  $i$ -th move. Player I, in the first move, chooses the

$$\begin{array}{c}
 \begin{array}{ccccccc}
 & & & 2N & & & N \\
 & & & \overbrace{\hspace{10em}} & & & \overbrace{\hspace{10em}} \\
 u_N = \dots (xy)^N & x & (xy)(xy) \dots (xy)(xy) & y & (xy)(xy) \dots (xy)(xy) \\
 & \uparrow & & \uparrow \uparrow & \\
 & (II, 1) & & (I, 2) & 
 \end{array} \\
 \\
 \begin{array}{ccccccc}
 & & & 2N & & & N \\
 & & & \overbrace{\hspace{10em}} & & & \overbrace{\hspace{10em}} \\
 v_N = \dots (xy)^N & x & (xy)(xy) \dots (xy)(xy) & y & (xy)(xy) \dots (xy)(xy) \\
 \\
 \begin{array}{ccccccc}
 & & & N & & & N \\
 & & & \overbrace{\hspace{10em}} & & & \overbrace{\hspace{10em}} \\
 (xy)(xy) \dots (xy)(xy) & x & (xy)(xy) \dots (xy)(xy) & \dots & (xy)(xy) \dots (xy)(xy) \\
 & \uparrow & & & \uparrow \uparrow \\
 & (I, 1) & & & (II, 2)
 \end{array}
 \end{array}
 \end{array}$$

last  $x$  followed immediately by an  $x$  in  $v_N$ . Player II, in the first move, has to choose the last  $x$  followed immediately by an  $x$  in  $u_N$  (if not, player I in the next two moves could win by choosing in the second move the last two consecutive  $x$ 's in  $u_N$ ). Player I, in the second move, chooses the last two consecutive  $y$ 's in  $u_N$ . Player II, in the second move, cannot choose two consecutive  $y$ 's in  $v_N$  to the right of the previously chosen position. Hence he is forced to choose two  $y$ 's separated by an  $x$ . Player I, in the third move, selects that  $x$ . But player II loses since he cannot choose an  $x$  between the two consecutive  $y$ 's chosen in the preceding move by I. The result follows. [ ]

Assume  $|u|_a, |v|_a > 0$ . Let  $u = u_0 a u_1 \dots a u_{|u|_a}$ ,  $v = v_0 a v_1 \dots a v_{|v|_a}$ . If  $Q_a^u p_i, Q_a^v q_j$  for  $i = 1, \dots, |u|_a, j = 1, \dots, |v|_a$ , then  $u_i = u_{>p_i+1}^{<p_i+1}$ ,  $i = 1, \dots, |u|_a - 1, v_j = v_{>q_j+1}^{<q_j+1}, j = 1, \dots, |v|_a - 1. u_0 = u^{<p_1}, v_0 = v^{<q_1}, u_{|u|_a} = u_{>p_{|u|_a}}, v_{|v|_a} = v_{>q_{|v|_a}}$ .

The next two lemmas will be used in showing that for positive integers  $m_1$  and  $m_3, A^*/\sim_{(m_1, 1, m_3)}$  is of dot-depth exactly 2.

LEMMA 3.2.: Assume  $u \sim_{(m'_1, m'_2)} v$ . Then

$$u^{<p}(s-1)m'_2+i \sim_{(m'_1-s, m'_2)} v^{<q}(s-1)m'_2+i, \tag{1}$$

$$u_{>p|u|_{a+1-(s-1)m'_2-i}} \sim_{(m'_1-s, m'_2)} v_{>q|v|_{a+1-(s-1)m'_2-i}} \tag{2}$$

for  $i=1, \dots, m'_2$  and  $s=1, \dots, m'_1-1$ .

*Proof:* (1) Let  $1 \leq i \leq m'_2$  and  $1 \leq s \leq m'_1-1$ . Let  $p'_1, \dots, p'_{m'_1-s}$  ( $p'_1 \leq \dots \leq p'_{m'_1-s}$ ) be positions in  $u^{<p}(s-1)m'_2+i$ . Consider the following play of the game  $\mathcal{G}_{(m'_1, m'_2)}(u, v)$ . Player I, in the first move, chooses  $p_{m'_2}, p_{2m'_2}, \dots, p_{(s-1)m'_2}, p_{(s-1)m'_2+i}, p'_1, \dots, p'_{m'_1-s}$ . Hence by the lemma of Induction 2.1, there exist positions  $q'_1, \dots, q'_{m'_1-s}$  ( $q'_1 \leq \dots \leq q'_{m'_1-s}$ ) in  $v^{<q}(s-1)m'_2+i$  such that player II, by choosing  $q_{m'_2}, q_{2m'_2}, \dots, q_{(s-1)m'_2}, q_{(s-1)m'_2+i}, q'_1, \dots, q'_{m'_1-s}$  for the corresponding positions, wins this play of the game. It is clear that

- (i)  $u^{<p_1} \sim_{(m'_2)} v^{<q_1}$ ,
- (ii)  $u^{<p_j+1} \sim_{(m'_2)} v^{<q_j+1}$  for  $1 \leq j \leq m'_1-s-1$ ,
- (iii)  $u^{<p_{(s-1)m'_2+i}} \sim_{(m'_2)} v^{<q_{(s-1)m'_2+i}}$ .

Note that player II has to choose  $q_{m'_2}, q_{2m'_2}, \dots, q_{(s-1)m'_2}, q_{(s-1)m'_2+i}$  because there is a number of  $a$ 's  $< m'_2$  between any two consecutive positions among  $p_{m'_2}, p_{2m'_2}, \dots, p_{(s-1)m'_2}, p_{(s-1)m'_2+i}$ .

The proof is similar, when starting with positions in  $v^{<q}(s-1)m'_2+i$ .

For (2), we consider  $p_{|u|_{a+1-m'_2}}, p_{|u|_{a+1-2m'_2}}, \dots, p_{|u|_{a+1-(s-1)m'_2}}, p_{|u|_{a+1-(s-1)m'_2-i}}, p'_1, \dots, p'_{m'_1-s}$ . [ ]

LEMMA 3.3: Assume  $u \sim_{(m'_1, m'_2)} v$ . Then

- (1)  $u_{>p_{(s-1)m'_2+i}} \sim_{(m'_1-s, m'_2)} v_{>q_{(s-1)m'_2+i}}$
- (2)  $u^{<p}|u|_{a+1-(s-1)m'_2-i} \sim_{(m'_1-s, m'_2)} v^{<q}|v|_{a+1-(s-1)m'_2-i}$  for  $i=1, \dots, m'_2$  and  $s=1, \dots, m'_1-1$ .

*Proof:* Similar to Lemma 3.2. [ ]

In the following theorem we talk about positions spelling the first and last occurrences of every subword of length  $\leq m$  of a word  $w$ . We illustrate what we mean by this with the following example. Let  $A = \{a, b, c\}$  and

$$u = abcccbaabbabbaccabababccaaaabbaa \dots$$

$$\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \quad \uparrow \quad \uparrow \uparrow \uparrow \uparrow \quad \uparrow \uparrow \uparrow \uparrow \uparrow$$

$p$

The six arrows on the left point to the positions which spell the first occurrences of every subword of length  $\leq 2$  in  $u^{<p}$  and the eight arrows on the right (before the one pointing to  $p$ ) to the positions which spell the last occurrences of every subword of length  $\leq 2$  in  $u^{<p}$ .

**THEOREM 3.4:** *Let  $m_1, m_2$  and  $m_3$  be positive integers. Then  $A^*/\sim_{(m_1, m_2, m_3)}$  is of dot-depth exactly 2 if and only if  $m_2 = 1$ .*

*Proof:* If  $A^*/\sim_{(m_1, m_2, m_3)}$  is of dot-depth exactly 2, then  $m_2 < 2$  by

**LEMMA 3.1.:** *Conversely, for  $|A| = r > 1$ , we show that for any positive integers  $m'_1, m'_2, \sim_{(m'_1+(m'_1+1)2m'_2(r+1)^{m'_2}, m'_2)} \subseteq \sim_{(m'_1, 1, m'_2)}$ .*

*To see this, suppose  $u \sim_{(m'_1+(m'_1+1)2m'_2(r+1)^{m'_2}, m'_2)} v$ . Then there is a winning strategy for player II in the game  $\mathcal{G}_{(m'_1+(m'_1+1)2m'_2(r+1)^{m'_2}, m'_2)}(u, v)$  to win each play. A winning strategy for player II in the game  $\mathcal{G}_{(m'_1, 1, m'_2)}(u, v)$  to win each play is described as follows. Let  $p'_1, \dots, p'_{m'_1}$  ( $p'_1 \leq \dots \leq p'_{m'_1}$ ) be positions in  $u$  chosen by player I in the first move. Player II chooses positions  $q'_1, \dots, q'_{m'_1}$  ( $q'_1 \leq \dots \leq q'_{m'_1}$ ) by considering the following play of the game  $\mathcal{G}_{(m'_1+(m'_1+1)2m'_2(r+1)^{m'_2}, m'_2)}(u, v)$ . In the first move, player I chooses  $p'_1, \dots, p'_{m'_1}$  and the positions which spell the first and last occurrences of every subword of length  $\leq m'_2$  in  $u^{<p'_1}, u^{<p'_2}, \dots, u^{<p'_{m'_1-1}}$  and  $u_{>p'_{m'_1}}$  for a total of no more than  $m'_1 + (m'_1 + 1)2m'_2(r+1)^{m'_2}$  positions (there are  $r^{m'_2}$  possible words of length  $m'_2$  for a total of no more than  $m'_2(r+1)^{m'_2}$  positions to spell the first (last) occurrences of every subword of length  $\leq m'_2$ ). More details follow for the special case  $u \sim_{(1+4m'_2(r+1)^{m'_2}, m'_2)} v$ . We have a winning strategy for player II in the game  $\mathcal{G}_{(1+4m'_2(r+1)^{m'_2}, m'_2)}(u, v)$  to win each play. Let us describe a winning strategy for player II in the game  $\mathcal{G}_{(1, 1, m'_2)}(u, v)$  to win each play. Let  $p$  be a position in  $u$  chosen by player I in the first move. Suppose  $Q_a^u p$  for some  $a \in A$ . If  $p$  is the  $i$ -th occurrence of  $a$  in  $u$  ( $1 \leq i \leq \mathcal{N}_{(1, m'_2)} = 2m'_2 + 1$ ), then player II chooses the same occurrence of  $a$  in  $v$ , say position  $q$ . The fact that  $u^{<p} \sim_{(1, m'_2)} v^{<q}$  and  $u_{>p} \sim_{(1, m'_2)} v_{>q}$  follows from Lemmas 3.2 and 3.3 ( $\mathcal{N}_{(1, m'_2)} \leq (4m'_2(r+1)^{m'_2})m'_2$ ). If  $p$  is the  $|u|_a + 1 - i$ -th occurrence of  $a$  in  $u$  ( $1 \leq i \leq \mathcal{N}_{(1, m'_2)}$ ), player II chooses the  $|v|_a + 1 - i$ -th occurrence of  $a$  in  $v$ . If  $p$  is among  $p_{2m'_2+2}, \dots, p_{|u|_a-2m'_2-1}$ , then player II chooses position  $q$ , an  $a$ , among  $q_{2m'_2+2}, \dots, q_{|v|_a-2m'_2-1}$  by considering the following play of the game  $\mathcal{G}_{(1+4m'_2(r+1)^{m'_2}, m'_2)}(u, v)$ . In the first move, player I chooses  $p$ , the positions which spell the first and last occurrences of every subword of length  $\leq m'_2$  in  $u^{<p}$  and in  $u_{>p}$ . Hence there exists a position  $q$  in  $v$  such that player II, by choosing  $q$ , the positions which spell the first and last occurrences of every subword of length  $\leq m'_2$  in  $v^{<q}$  and in  $v_{>q}$ , wins the play of the game. Let us show that  $u^{<p} \sim_{(1, m'_2)} v^{<q}$  (the proof that  $u_{>p} \sim_{(1, m'_2)} v_{>q}$  is similar). Let  $p'$  be*



a position in  $u^{<p}$  (the proof is similar when starting with a position in  $v^{<q}$ ). Assume  $Q_{a_i}^u p'$ .

Case 1:  $p'$  is among the first  $m'_2$  occurrences of  $a_i$  in  $u^{<p}$ .

Let  $q'$  be the same occurrence among the first  $m'_2$  occurrences of  $a_i$  in  $v^{<q}$ . It is clear that  $u_{>p'}^{<p} \sim_{(m'_2)} v_{>q'}^{<q}$ , and  $u^{<p'} \sim_{(m'_2)} v^{<q'}$ .

Case 2:  $p'$  is among the last  $m'_2$  occurrences of  $a_i$  in  $u^{<p}$ . Similar to case 1.

Case 3:  $p'$  is not among the first  $m'_2$  nor the last  $m'_2$  occurrences of  $a_i$  in  $u^{<p}$ .

Let  $p''$  and  $p'''$  ( $p'' < p'''$ ) be the closest positions to  $p'$  in  $u^{<p'}$  and  $u_{>p'}^{<p}$ , respectively among the chosen positions by player I. Let  $q''$  and  $q'''$  ( $q'' < q'''$ ) be the corresponding positions chosen by player II.

Since  $u_{>p'''}^{<p'''} \sim_{(m'_2)} v_{>q'''}^{<q'''}$ , there is  $q'$  in  $v_{>q'''}^{<q'''}$  such that  $Q_{a_i}^v q'$ .

Let us show that  $u_{>p'}^{<p} \sim_{(m'_2)} v_{>q'}^{<q}$ .  $u^{<p'} \sim_{(m'_2)} v^{<q'}$  follows similarly.

Let  $w = w_1 \dots w_{|w|}$ ,  $|w| \leq m'_2$  in  $v_{>q'}^{<q}$ . The proof is similar when starting with  $w$  in  $u_{>p'}^{<p}$ . If  $w \in v_{>q'''}^{<q'''}$ , it is clear that  $w \in u_{>p'''}^{<p''}$ , hence in  $u_{>p'}^{<p}$ . So let us assume  $w \notin v_{>q'''}^{<q'''}$ . Let  $p_{w_1}, \dots, p_{w_{|w|}}$  in  $v_{>q'}^{<q}$ , at least  $p_{w_1}$  being in  $v_{>q'''}^{<q'''}$ , which spell  $w_1 \dots w_{|w|} \cdot p_{w_1}, \dots, p_{w_{|w|}}$  are hence positions which spell an occurrence of a subword of length  $\leq m'_2$  in  $v^{<q}$ . Hence they are smaller than or equal to those positions which spell the last occurrence of  $w$  in  $v^{<q}$  which are in  $v_{\geq q'''}^{<q}$ . Hence  $w \in u_{>p'}^{<p}$ . [ ]

The following corollary gives another result for inclusion (one was Lemma 2.2).

COROLLARY 3.5: Let  $|A| = r$ . Then

$$\sim_{(m'_1 + (m'_1 + 1) 2m'_2 (r + 1)^{m'_2}, m'_2)} \subseteq \sim_{(m'_1, \mathcal{N}(1, m'_2))}$$

Proof: From Theorem 3.4 and Lemma 2.2. [ ]

ACKNOWLEDGMENTS

This research was partially supported by F.C.A.R. (Quebec). We would like to thank the referees for their comments and suggestions.

REFERENCES

1. J. A. BRZOWSKI and R. KNAST, The Dot-Depth Hierarchy of Star-Free Languages if Infinite, *J. Comp. Sys. Sci.*, 1978, 16, pp. 37-55.

2. F. BLANCHET-SADRI, Some Logical Characterizations of the Dot-Depth Hierarchy and Applications, *Technical Report No. 88-03 of the Department of Mathematics and Statistics of McGill University*, July 1988, pp. 1-44.
3. F. BLANCHET-SADRI, Games, Equations and the Dot-Depth Hierarchy, (preprint 1988), *Computers and Mathematics with applications* (à paraître).
4. R. S. COHEN and J. A. BRZOWSKI, Dot-Depth of Star-Free Events, *J. Comp. Sys. Sci.*, 1971, 5, pp. 1-16.
5. A. EHRENFUCHT, An Application of Games to the Completeness Problem for Formalized Theories, *Fund. Math.*, 1961, 49, pp. 129-141.
6. S. EILENBERG, Automata, Languages and Machines, B, *Academic Press*, New York, 1976.
7. H. B. ENDERTON, A Mathematical Introduction to Logic, *Academic Press*, New York, 1972.
8. R. FRAISSÉ, Cours de logique mathématique, tome 2, Gauthier-Villars, Paris, 1972.
9. G. LALLEMENT, Semigroups and Combinatorial Applications, *Wiley*, New York, 1979.
10. R. McNAUGHTON and S. PAPERT, Counter-Free Automata, *M.I.T. Press*, Cambridge, Mass., 1971.
11. D. PERRIN and J. E. PIN, First-Order Logic and Star-Free Sets, *J. Comp. Sys. Sci.*, 1986, 32, pp. 393-406.
12. J. E. PIN, Variétés de langages formels, *Masson*, Paris, 1984.
13. J. E. PIN, Hiérarchies de contaténation, *R.A.I.R.O. Informatique Théorique*, 1984, 18, pp. 23-46.
14. J. G. ROSENSTEIN, Linear Orderings, *Academic Press*, New York, 1982.
15. M. P. SCHÜTZENBERGER, On Finite Monoids having only Trivial Subgroups, *Information and Control*, 1965, 8, pp. 190-194.
16. I. SIMON, Piecewise Testable Events, Proc. 2nd GI Conference, *Lectures Notes in Comput. Sci.*, Springer Verlag, Berlin, 1975, 33, pp. 214-222.
17. H. STRAUBING, A Generalization of the Schützenberger Product of Finite Monoids, *Theoretical Comput. Sci.*, 1981, 13, pp. 137-150.
18. H. STRAUBING, Finite Semigroup Varieties of the Form  $V\star D$ , *J. of Pure and Applied Algebra*, 1985, 36, pp. 53-94.
19. H. STRAUBING, Semigroups and Languages of Dot-Depth Two, *Proc. 13th ICALP, Lecture Notes in Comput. Sci.*, Springer Verlag, New York, 1986, 226, pp. 416-423.
20. W. THOMAS, Classifying Regular Events in Symbolic Logic, *J. Comp. Sys. Sci.*, 1982, 25, pp. 360-376.
21. W. THOMAS, An Application of the Ehrenfeucht-Fraissé Game in Formal Language Theory, *Bull. Soc. Math. de France*, 2<sup>e</sup> série, Mémoire, 1984, No. 16, pp. 11-21.
22. B. TILSON, Categories as Algebra, *J. of Pure and Applied Algebra*, 1987, 48, pp. 83-198.