INFORMATIQUE THÉORIQUE ET APPLICATIONS

F. BLANCHET-SADRI

On dot-depth two

Informatique théorique et applications, tome 24, nº 6 (1990), p. 521-529

<http://www.numdam.org/item?id=ITA_1990__24_6_521_0>

Informatique théorique et Applications/Theoretical Informatics and Applications
(vol 24, n° 6, 1990, p 521 à 530)

ON DOT-DEPTH TWO (*)

by F. BLANCHET-SADRI

Communicated by A ARNOLD

Abstract – *For positive integers* $m_1, \ldots, m_k$, *congruences* $\sim_{(m_1, \ldots, m_k)}$ *related to a version of the Ehrenfeucht-Fraissé game are defined which correspond to level k of the Sraubing hierarchy of star-free languages Given any finite alphabet A, a necessary and sufficient condition is given for the monoids* $A^*/\sim_{(m_1\ m_2\ m_3)}$ *to be of dot-deàth exactly* 2

Résumé – *Étant donnés des entiers positifs* $m_1, \ldots, m_k$, *on définit des congruences* $\sim_{(m_1, \ldots, m_k)}$ *en relation avec une versuin du jeu de Ehrenfeucht-Fraissé, et qui correspondent au niveau k de la hiérarchie de concaténation de Straubing Étant donné un alphabet fini A, une condition nécessaire et suffisante est donnée pour que les monoides définis par ces congruences soient de dot-depth exactement* 2

1. INTRODUCTION

Let A be a given finite alphabet. The regular languages over A are those subsets of A^*, the free monoid generated by A, constructed from the finite languages over A by the boolean operations, the concatenation product and the star. The star-free languages are those regular languages which can be obtained from the finite languages by the boolean operations and the concatenation product only. According to Schützenberger [15], $L \subseteq A^*$ is star-free if and only if its syntactic monoid $M(L)$ is finite and aperiodic. General references on the star-free languages are McNaughton and Papert [10], Eilenberg [6] or Pin [12].

Natural classifications of the star-free languages are obtained based on the alternative use of the boolean operations and the concatenation product. Let $A^+ = A^* \setminus \{1\}$, where 1 denotes the empty word. Let

(*) Received in January 1989, revised in February 1989

([1]) Department of Mathematics, Mcgill University, 805 Sherbrooke Street West, Montreal, P Quebec, Canada, H3A 2K6

$A^+\mathscr{B}_0=\{L\subseteq A^+ \mid L \text{ is finite or cofinite}\}$,

$A^+\mathscr{B}_{k+1}=\{L\subseteq A^+ \mid L$ is a boolean combination of languages of the form $L_1 \ldots L_n (n\geqq 1)$ with $L_1, \ldots, L_n\in A^+\mathscr{B}_k\}$.

Only nonempty words over A are considered to define this hierarchy; in particular, the complement operation is applied with respect to A^+. The language classes $A^+\mathscr{B}_0$, $A^+\mathscr{B}_1, \ldots$ form the so-called dot-depth hierarchy introduced by Cohen and Brzozowski in [4]. The union of the classes $A^+\mathscr{B}_0$, $A^+\mathscr{B}_1, \ldots$ is the class of star-free languages.

Our attention is directed toward a closely related and more fundamental hierarchy, this one in A^*, introduced by Straubing in [18]. Let

$A^*\mathscr{V}_0=\{0, A^*\}$,

$A^*\mathscr{V}_{k+1}=\{L\subseteq A^* \mid L$ is a boolean combination of languages of the form $L_0 a_1 L_1 a_2 \ldots a_n L_n (n\geqq 0)$ with $L_0, \ldots, L_n\in A^*\mathscr{V}_k$ and $a_1, \ldots, a_n\in A\}$.

$L\subseteq A^*$ is star-free if and only if $L\in A^*\mathscr{V}_k$ for some $k\geqq 0$. The *dot-depth* of L is the smallest such k.

Using Eilenberg's correspondence, we have that for each $k\geqq 0$, there is a variety V_k of finite monoids such that for $L\subseteq A^*$, $L\in A^*\mathscr{V}_k$ if and only if $M(L)\in V_k$. An outstanding open problem is whether one can decide if a language has dot-depth k, *i.e.*, can we effectively characterize the varieties V_k? The variety V_0 consists of the trivial monoid alone, V_1 of all finite $\mathscr{T}$-trivial monoids [16]. Straubing [19] conjectured an effective characterization, based on the syntactic monoid of the language, for the case $k=2$. His characterization, formulated in terms of a novel use of categories in semigroup theory recently developed by Tilson [22], is shown to be necessary in general, and sufficient for an alphabet of two elements.

In the framework of semigroup theory, Brzozowski and Knast [1] showed that the dot-depth hierarchy is infinite. Thomas [21] gave a new proof of this result, which shows also that the Straubing hierarchy is infinite, based on a logical characterization of the dot-depth hierarchy that the obtained in [20] (Perrin and Pin gave one for the Straubing hierarchy [11]) and the following version of the Ehrenfeucht-Fraissé game.

First, one regards a word $w\in A^*$ of length $|w|$ as a word model $w=\langle\{1, \ldots, |w|\}, <^w, (Q_a^w)_{a\in A}\rangle$ where the universe $\{1, \ldots, |w|\}$ represents the set of positions of letters in w, $<^w$ denotes the $<$-relation in w, and Q_a^w are unary relations over $\{1, \ldots, |w|\}$ containing the positions with letter a, for each $a\in A$. For a sequence $\bar{m}=(m_1, \ldots, m_k)$ of positive integers, where $k\geqq 0$, the game $\mathscr{G}_{\bar{m}}(u, v)$ is played between two players I and II on the word models u and v. A play of the game consists of k moves. In the

i-th move, player I chooses, in u or in v, a sequence of m_i positions; then player II chooses, in the remaining word, also a sequence of m_i positions. After k moves, by concatenating the sequences chosen from u and v, two sequences $p_1 \ldots p_n$ from u and $q_1 \ldots q_n$ from v have been formed where $n=m_1+\ldots+m_k$.

Player II has won the play if

$$p_i <^u p_j \quad \text{if and only if} \quad q_i <^v q_j, \tag{1}$$

and

$$Q_a^u p_i \quad \text{if and only if} \quad Q_a^v q_i, \qquad a \in A \quad \text{for } 1 \leqq i, j \leqq n. \tag{2}$$

If there is a winning strategy for II in the game $\mathscr{G}_{\bar{m}}(u, v)$ to win each play we write $u \sim_{\bar{m}} v$. $\sim_{\bar{m}}$ naturally defines a congruence on A^* which we denote also by $\sim_{\bar{m}}$. The standard Ehrenfeucht-Fraissé game [5] is the special case $\mathscr{G}_{(1, \ldots, 1)}(u, v)$. Thomas [20], [21] and Perrin and Pin [11] imply that $L \in A^* \mathscr{V}_k$ if and only if L is a $\sim_{\bar{m}}$-language for some $\bar{m}=(m_1, \ldots, m_k)$ (or L is a union of classes of the congruence $\sim_{\bar{m}}$). This congruence characterization implies that the problem of deciding whether a language has dot-depth k is equivalent to the problem of effectively characterizing the monoids $M=A^*/\sim$ with $\sim \supseteqq \sim_{\bar{m}}$ for some $\bar{m}=(m_1, \ldots, m_k)$, *i.e.*,

$$V_k=\{A^*/\sim \mid \sim \supseteqq \sim_{\bar{m}} \text{ for some } \bar{m}=(m_1, \ldots, m_k)\}.$$

This paper is concerned with an application of the above congruence characterization. We show that $A^*/\sim_{(m_1, m_2, m_3)}$ is of dot-depth exactly 2 if and only if $m_2=1$. The proof relies on some properties of the congruences $\sim_{\bar{m}}$ stated in the next section. [2] and [3] include other applications: among them are an answer to a conjecture of Pin [13] concerning tree hierarchies of monoids and also systems of equations satisfied in natural sublevels of level 1 of the Straubing hierarchy. The reader is referred to the books by Eilenberg [6], Lallement [9], Pin [12], Enderton [7] and Fraissé [8] for all the algebraic and logical terms not defined here.

2. SOME PROPERTIES OF THE CHARACTERIZING CONGRUENCES

2.1. An induction lemma

The following lemma is a basic result (similar to one in [14] regarding $\sim_{(1, \ldots, 1)}$) which allows to resolve games with $k+1$ moves into games with

k moves and thereby allows to perform induction arguments. In what follows, $u^{<p}(u_{>p})$ denotes the subword of u to the left (right) of position p and $u_{>p}^{<q}$ the subword of u between positions p and q.

LEMMA 2.1.: *Let* $\bar{m}=(m_1, \ldots, m_k)$. $u \sim_{(m, m_1, \ldots, m_k)} v$ *if and only if*

(1) *for every* $p_1, \ldots, p_m \in u (p_1 \leqq \ldots \leqq p_m)$ *there are* $q_1, \ldots, q_m \in v$ $(q_1 \leqq \ldots \leqq q_m)$ *such that*

(i) $Q_a^u p_i$ *if and only if* $Q_a^v q_i$, $a \in A$ *for* $1 \leqq i \leqq m$,

(ii) $u^{<p_1} \sim_{\bar{m}} v^{<q_1}$,

(iii) $u_{>p_i}^{\leq p_{i+1}} \sim_{\bar{m}} v_{>q_i}^{\leq q_{i+1}}$ *for* $1 \leqq i \leqq m-1$,

(iv) $u_{>p_m} \sim_{\bar{m}} v_{>q_m}$ and

(2) *for every* $q_1, \ldots, q_m \in v (q_1 \leqq \ldots \leqq q_m)$ *there are* $p_1, \ldots, p_m \in u$ $(p_1 \leqq \ldots \leqq p_m)$ *such that* (i), (ii), (iii) *and* (iv) *hold.*

2.2. A lemma for inclusion

Define

$$\mathcal{N}_{(m_1, \ldots, m_k)} = m_1 + \ldots + m_k + \Sigma_{1 \leqq i_1 < i_2 \leqq k} m_{i_1} m_{i_2} + \ldots + \Sigma_{1 \leqq i_1 < \ldots < i_{k-1} \leqq k} m_{i_1} \ldots m_{i_{k-1}} + m_1 \ldots m_k.$$

One can show that $x^N \sim_{(m_1, \ldots, m_k)} x^{N+1}$ $(N = \mathcal{N}_{(m_1, \ldots, m_k)})$ and that N is the smallest n such that $x^n \sim_{(m_1, \ldots, m_k)} x^{n+1}$ (the proof is similar to the one of a property of $\sim_{(1, \ldots, 1)}$ in [21]). We see that if $u, v \in A^*$ and $u \sim_{(m_1, \ldots, m_k)} v$, then $|u|_a = |v|_a < \mathcal{N}_{(m_1, \ldots, m_k)}$ or $|u|_a, |v|_a \geqq \mathcal{N}_{(m_1, \ldots, m_k)}$ (here $|w|_a$ denotes the number of occurences of the letter a in w). The following lemma follows easily from Lemma 2.1 and the above remarks.

LEMME 2.2 :

$$\sim_{(m_1, \ldots, m_k)} \subseteqq \sim_{(\mathcal{N}_{(m_1, \ldots, m_k)})}, \quad \text{and} \quad \sim_{(m_1, \ldots, m_k)} \nsubseteq \sim_{(\mathcal{N}_{(m_1, \ldots, m_k)}+1)}.$$

If $k \leqq k'$ *and* $\exists\, 0 = j_0 < \ldots < j_{k-1} < j_k = k'$ *such that* $m_i \leqq \mathcal{N}_{(m'_{j_{i-1}+1}, \ldots, m'_{j_i})}$ *for* $1 \leqq i \leqq k$, *then* $\sim_{(m'_1, \ldots, m'_{k'})} \subseteqq \sim_{(m_1, \ldots, m_k)}$.

3. A SEQUENCE OF MONOIDS OF DOT-DEPTH 2

In this section, we show that for positive integers m_1, m_2 and m_3, $A^*/\sim_{(m_1, m_2, m_3)}$ is of dot-depth exactly 2 if and only if $m_2 = 1$. The following lemma shows the necessity of the condition.

LEMMA 3.1: *Let* m_1 *and* m_3 *be positive integers. Then* $A^*/\sim_{(m_1, 2, m_3)}$ *is of dot-depth exactly* 3.

Proof: Let $m>0$. Consider $u_m=((xy)^m x (xy)^{2m} y (xy)^m)^m$, $v_m=((xy)^m y (xy)^{2m} x (xy)^m)^m$. A result of Straubing [17] implies that monoids in V_2 are 2-mutative and hence satisfy $u_m=v_m$ for all sufficiently large m. However, for every $N \geqq \mathcal{N}_{(1,\,2,\,1)}$, $u_N \not\sim_{(1,\,2,\,1)} v_N$. To see this, we illustrate a winning strategy for player I in the game $\mathcal{G}_{(1,\,2,\,1)}(u_N, v_N)$. (I, i) denotes a position chosen by player I in the i-th move, $i=1, 2, 3$. Similarly, (II, i) denotes a position chosen by player II in the i-th move. Player I, in the first move, chooses the

$$u_N=\ldots(xy)^N \underset{\substack{\uparrow\\(\mathrm{II},\,1)}}{x} \overbrace{(xy)(xy)\ldots(xy)(x\underset{\uparrow}{y})}^{2N}\underset{\substack{\uparrow\\(\mathrm{I},\,2)}}{y}\overbrace{(xy)(xy)\ldots(xy)(xy)}^{N}$$

$$v_N=\ldots(xy)^N x \overbrace{(xy)(xy)\ldots(xy)(xy)}^{2N} y \overbrace{(xy)(xy)\ldots(xy)(xy)}^{N}$$

$$\overbrace{(xy)(xy)\ldots(xy)(xy)}^{N} \underset{\substack{\uparrow\\(\mathrm{I},\,1)}}{x} \overbrace{(xy)(xy)\ldots(x\underset{\uparrow}{y})(x\underset{\substack{\uparrow\\(\mathrm{II},\,2)}}{y})\ldots(xy)(xy)}^{N}$$

last x followed immediately by an x in v_N. Player II, in the first move, has to choose the last x followed immediately by an x in u_N (if not, player I in the next two moves could win by choosing in the second move the last two consecutive x's in u_N). Player I, in the second move, chooses the last two consecutive y's in u_N. Player II, in the second move, cannot choose two consecutive y's in v_N to the right of the previously chosen position. Hence he is forced to choose two y's separated by an x. Player I, in the third move, selects that x. But player II looses since he cannot choose an x between the two consecutive y's chosen in the preceding move by I. The result follows. [ ]

Assume $|u|_a$, $|v|_a>0$. Let $u=u_0 a u_1 \ldots a u_{|u|_a}$, $v=v_0 a v_1 \ldots a v_{|v|_a}$. If $Q^u_a p_i$, $Q^v_a q_j$ for $i=1, \ldots, |u|_a$, $j=1, \ldots, |v|_a$, then $u_i=u^{<p_{i+1}}_{>p_i}$, $i=1, \ldots, |u|_{a}-1$, $v_j=v^{<q_{j+1}}_{>q_j}$, $j=1, \ldots, |v|_{a}-1$. $u_0=u^{<p_1}$, $v_0=v^{<q_1}$, $u_{|u|_a}=u_{>p_{|u|_a}}$, $v_{|v|_a}=v_{>q_{|v|_a}}$.

The next two lemmas will be used in showing that for positive integers m_1 and m_3, $A^*/\sim_{(m_1,\,1,\,m_3)}$ is of dot-depth exactly 2.

LEMMA 3.2.: *Assume* $u \sim_{(m_1', m_2')} v$. *Then*

$$u^{<p}(s-1)m_2'+i \sim_{(m_1'-s,\ m_2')} v^{<q}(s-1)m_2'+i, \tag{1}$$

$$u_{>p|u|_a+1-(s-1)m_2'-i} \sim_{(m_1'-s,\ m_2')} v_{>q|v|_a+1-(s-1)m_2'-i}$$

$$\textit{for} \quad i=1, \ldots, m_2' \textit{ and } s=1, \ldots, m_1'-1. \tag{2}$$

Proof: (1) Let $1 \leqq i \leqq m_2'$ and $1 \leqq s \leqq m_1'-1$. Let $p_1', \ldots, p_{m_1'-s}'$ $(p_1' \leqq \ldots \leqq p_{m_1'-s}')$ be positions in $u^{<p}(s-1)m_2'+i$. Consider the following play of the game $\mathcal{G}_{(m_1', m_2')}(u, v)$. Player I, in the first move, chooses $p_{m_2'}, p_{2m_2'}, \ldots, p_{(s-1)m_2'}, p_{(s-1)m_2'+i}, p'_1, \ldots, p_{m_1'-s}'$. Hence by the lemma of Induction 2.1, there exist positions $q_1', \ldots, q_{m_1'-s}'(q_1' \leqq \ldots \leqq q_{m_1'-s}')$ in $v^{<q}(s-1)m_2'+i$ such that player II, by choosing $q_{m_2'}, q_{2m_2'}, \ldots, q_{(s-1)m_2'}$, $q_{(s-1)m_2'+i}, q_1', \ldots, q_{m_1'-s}'$ for the corresponding positions, wins this play of the game. It is clear that

(i) $u^{<p_1'} \sim_{(m_2')} v^{<q_1'}$,

(ii) $u_{>p_j'}^{<p_{j+1}'} \sim_{(m_2')} v_{>q_j'}^{<q_{j+1}'}$ for $1 \leqq j \leqq m_1'-s-1$,

(iii) $u_{>p_{m_1'-s}'}^{<p_{(s-1)m_2'+i}} \sim_{(m_2')} v_{>q_{m_1'-s}'}^{<q_{(s-1)m_2'+i}}$.

Note that player II has to choose $q_{m_2'}, q_{2m_2'}, \ldots, q_{(s-1)m_2'}, q_{(s-1)m_2'+i}$ because there is a number of $a's < m_2'$ between any two consecutive positions among $p_{m_2'}, p_{2m_2'}, \ldots, p_{(s-1)m_2'}, p_{(s-1)m_2'+i}$.

The proof is similar, when starting with positions in $v^{<q}(s-1)m_2'+i$.

For (2), we consider $p_{|u|_a+1-m_2'}$, $p_{|u|_a+1-2m_2'}, \ldots, p_{|u|_a+1-(s-1)m_2'}$, $p_{|u|_a+1-(s-1)m_2'-i}, p_1', \ldots, p_{m_1'-s}'$. [ ]

LEMMA 3.3: *Assume* $u \sim_{(m_1', m_2')} v$. *Then*

(1) $u_{>p(s-1)m_2'+i} \sim_{(m_1'-s,\ m_2')} v_{>q(s-1)m_2'+i}$,

(2) $u^{<p}|u|_a+1-(s-1)m_2'-i \sim_{(m_1'-s,\ m_2')} v^{<q}|v|_a+1-(s-1)m_2'-i$ *for* $i=1, \ldots, m_2'$ *and* $s=1, \ldots, m_1'-1$.

Proof: Similar to Lemma 3.2. [ ]

In the following theorem we talk about positions spelling the first and last occurences of every subword of length $\leqq m$ of a word w. We illustrate what we mean by this with the following example. Let $A = \{a, b, c\}$ and

$$u = abccccaabbabbacccabababccaaaabbaa \ldots.$$

↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

p

The six arrows on the left point to the positions which spell the first occurences of every subword of length $\leqq 2$ in $u^{<p}$ and the eight arrows on the right (before the one pointing to p) to the positions which spell the last occurences of every subword of length $\leqq 2$ in $u^{<p}$.

THEOREM 3.4: *Let m_1, m_2 and m_3 be positive integers. Then $A^*/\sim_{(m_1, m_2, m_3)}$ is of dot-depth exactly 2 if and only if $m_2 = 1$.*

Proof: If $A^*/\sim_{(m_1, m_2, m_3)}$ is of dot-depth exactly 2, then $m_2 < 2$ by

LEMMA 3.1.: *Conversely, for $|A| = r > 1$, we show that for any positive integers m'_1, m'_2, $\sim_{(m'_1 + (m'_1 + 1)\, 2m'_2\, (r+1)^{m'_2},\, m'_2)} \subseteqq \sim_{(m'_1,\, 1,\, m'_2)}$.*
To see this, suppose $u \sim_{(m'_1 + (m'_1 + 1)\, 2m'_2\, (r+1)^{m'_2},\, m'_2)} v$. Then there is a winning strategy for player II in the game $\mathscr{G}_{(m'_1 + (m'_1 + 1)\, 2m'_2\, (r+1)^{m'_2},\, m'_2)}(u, v)$ to win each play. A winning strategy for player II in the game $\mathscr{G}_{(m'_1,\, 1,\, m'_2)}(u, v)$ to win each play is described as follows. Let $p'_1, \ldots, p'_{m'_1}$ ($p'_1 \leqq \ldots \leqq p'_{m'_1}$) be positions in u chosen by player I in the first move. Player II chooses positions $q'_1, \ldots, q'_{m'_1}$ ($q'_1 \leqq \ldots \leqq q'_{m'_1}$) by considering the following play of the game $\mathscr{G}_{(m'_1 + (m'_1 + 1)\, 2m'_2\, (r+1)^{m'_2},\, m'_2)}(u, v)$. In the first move, player I chooses $p'_1, \ldots, p'_{m'_1}$ and the positions which spell the first and last occurences of every subword of length $\leqq m'_2$ in $u^{<p'_1}$, $u^{<p'_2}_{>p'_1}, \ldots, u^{<p'_{m'_1}}_{>p'_{m'_1 - 1}}$ and $u_{>p'_{m'_1}}$ for a total of no more than $m'_1 + (m'_1 + 1)\, 2m'_2\, (r+1)^{m'_2}$ positions (there are $r^{m'_2}$ possible words of length m'_2 for a total of no more than $m'_2\, (r+1)^{m'_2}$ positions to spell the first (last) occurences of every subword of length $\leqq m'_2$). More details follow for the special case $u \sim_{(1 + 4m'_2\, (r+1)^{m'_2},\, m'_2)} v$. We have a winning strategy for player II in the game $\mathscr{G}_{(1 + 4m'_2\, (r+1)^{m'_2},\, m'_2)}(u, v)$ to win each play. Let us describe a winning strategy for player II in the game $\mathscr{G}_{(1,\, 1,\, m'_2)}(u, v)$ to win each play. Let p be a position in u chosen by player I in the first move. Suppose $Q^u_a p$ for some $a \in A$. If p is the i-th *occurence of a in u ($1 \leqq i \leqq \mathscr{N}_{(1,\, m'_2)} = 2m'_2 + 1$), then player II chooses the same occurence of a in v, say position q. The fact that $u^{<p} \sim_{(1,\, m'_2)} v^{<q}$ and $u_{>p} \sim_{(1,\, m'_2)} v_{>q}$ follows from Lemmas 3.2 and 3.3 ($\mathscr{N}_{(1,\, m'_2)} \leqq (4m'_2\, (r+1)^{m'_2})\, m'_2$). If p is the $|u|_a + 1 - i$*-th occurence of a in u ($1 \leqq i \leqq \mathscr{N}_{(1,\, m'_2)}$), player II chooses the $|v|_a + 1 - i$-th occurence of a in v. If p is among $p_{2m'_2 + 2}, \ldots, p_{|u|_a - 2m'_2 - 1}$, then player II chooses position q, an a, among $q_{2m'_2 + 2}, \ldots, q_{|v|_a - 2m'_2 - 1}$ by considering the following play of the game $\mathscr{G}_{(1 + 4m'_2\, (r+1)^{m'_2},\, m'_2)}(u, v)$. In the first move, player I chooses p, the positions which spell the first and last occurences of every subword of length $\leqq m'_2$ in $u^{<p}$ and in $u_{>p}$. Hence there exists a position q in v such that player II, by choosing q, the positions which spell the first and last occurences of every subword of length $\leqq m'_2$* in *$v^{<q}$ and in $v_{>q}$, wins the play of the game. Let us show that $u^{<p} \sim_{(1,\, m'_2)} v^{<q}$ (the proof that $u_{>p} \sim_{(1,\, m'_2)} v_{>q}$ is similar). Let p' be*

a position in $u^{<p}$ *(the proof is similar when starting with a position in* $v^{<q}$*). Assume* $Q^u_{a_i} p'$.

Case 1: p' is among the first m_2' occurences of a_i in $u^{<p}$.

Let q' be the same occurence among the first m_2' occurences of a_i in $v^{<q}$. It is clear that $u^{<p}_{>p'} \sim_{(m_2')} v^{<q}_{>q'}$ and $u^{<p'} \sim_{(m'_2)} v^{<q'}$.

Case 2: p' is among the last m_2' occurences of a_i in $u^{<p}$. Similar to case 1.

Case 3: p' is not among the first m_2' nor the last m_2' occurences of a_i in $u^{<p}$.

Let p'' and p''' ($p'' < p'''$) be the closest positions to p' in $u^{<p'}$ and $u^{<p}_{>p'}$ respectively among the chosen positions by player I. Let q'' and q''' ($q'' < q'''$) be the corresponding positions chosen by player II.

Since $u^{<p'''}_{>p''} \sim_{(m_2')} v^{<q'''}_{>q''}$, there is q' in $v^{<q'''}_{>q''}$ such that $Q^v_{a_i} q'$.

Let us show that $u^{<p}_{>p'} \sim_{(m_2')} v^{<q}_{>q'}$. $u^{<p'} \sim_{(m_2')} v^{<q'}$ follows similarly.

Let $w = w_1 \ldots w_{|w|}$, $|w| \leqq m_2'$ in $v^{<q}_{>q'}$. The proof is similar when starting with w in $u^{<p}_{>p'}$. If $w \in v^{<q}_{>q'''}$, it is clear that $w \in u^{<p}_{>p'''}$, hence in $u^{<p}_{>p'}$. So let us assume $w \notin v^{<q}_{>q'''}$. Let $p_{w_1}, \ldots, p_{w_{|w|}}$ in $v^{<q}_{>q'}$, at least p_{w_1} being in $v^{<q'''}_{>q'}$, which spell $w_1 \ldots w_{|w|}$. $p_{w_1}, \ldots, p_{w_{|w|}}$ are hence positions which spell an occurence of a subword of length $\leqq m_2'$ in $v^{<q}$. Hence they are smaller than or equal to those positions which spell the last occurence of w in $v^{<q}$ which are in $v^{<q}_{\geqq q'''}$. Hence $w \in u^{<p}_{>p'}$. [ ]

The following corollary gives another result for inclusion (one was Lemma 2.2).

COROLLARY 3.5: *Let* $|A| = r$. *Then*

$$\sim_{(m_1' + (m_1'+1)\, 2 m_2' (r+1)^{m_2'},\ m_2')} \subseteqq \sim_{(m_1',\ \mathcal{N}(1,\ m_2'))}$$

Proof: From Theorem 3.4 and Lemma 2.2. [ ]

ACKNOWLEDGMENTS

This research was partially supported by F.C.A.R. (Quebec). We would like to thank the referees for their comments and suggestions.

REFERENCES

1. J. A. BRZOZOWSKI and R. KNAST, The Dot-Depth Hierarchy of Star-Free Languages if Infinite, *J. Comp. Sys. Sci.*, 1978, *16*, pp. 37-55.

2. F. Blanchet-Sadri, Some Logical Characterizations of the Dot-Depth Hierarchy and Applications, *Technical Report No.* 88-03 *of the Department of Mathematics and Statistics of McGill University*, July 1988, pp. 1-44.
3. F. Blanchet-Sadri, Games, Equations and the Dot-Depth Hierarchy, (preprint 1988), *Computers and Mathematics with applications* (à paraître).
4. R. S. Cohen and J. A. Brzozowski, Dot-Depth of Star-Free Events, *J. Comp. Sys. Sci.*, 1971, *5*, pp. 1-16.
5. A. Ehrenfeucht, An Application of Games to the Completeness Problem for Formalized Theories, *Fund. Math.*, 1961, *49*, pp. 129-141.
6. S. Eilenberg, Automata, Languages and Machines, B, *Academic Press*, New York, 1976.
7. H. B. Enderton, A Mathematical Introduction to Logic, *Academic Press*, New York, 1972.
8. R. Fraissé, Cours de logique mathématique, tome 2, Gauthier-Villars, Paris, 1972.
9. G. Lallement, Semigroups and Combinatorial Applications, *Wiley*, New York, 1979.
10. R. McNaughton and S. Papert, Counter-Free Automata, *M.I.T. Press*, Cambridge, Mass., 1971.
11. D. Perrin and J. E. Pin, First-Order Logic and Star-Free Sets, *J. Comp. Sys. Sci.*, 1986, *32*, pp. 393-406.
12. J. E. Pin, Variétés de langages formels, *Masson*, Paris, 1984.
13. J. E. Pin, Hiérarchies de contaténation, *R.A.I.R.O. Informatique Théorique*, 1984, *18*, pp. 23-46.
14. J. G. Rosenstein, Linear Orderings, *Academic Press*, New York, 1982.
15. M. P. Schützenberger, On Finite Monoids having only Trivial Subgroups, *Information and Control*, 1965, *8*, pp. 190-194.
16. I. Simon, Piecewise Testable Events, Proc. 2nd GI Conference, *Lectures Notes in Comput Sci.*, *Springer Verlag*, Berlin, 1975, *33*, pp. 214-222.
17. H. Straubing, A Generalization of the Schützenberger Product of Finite Monoids, *Theoretical Comput Sci.*, 1981, *13*, pp. 137-150.
18. H. Straubing, Finite Semigroup Varieties of the Form $V*D$, *J. of Pure and Applied Algebra*, 1985, *36*, pp. 53-94.
19. H. Straubing, Semigroups and Languages of Dot-Depth Two, *Proc.* 13*th ICALP*, *Lecture Notes in Comput. Sci.*, *Springer Verlag*, New York, 1986, *226*, pp. 416-423.
20. W. Thomas, Classifying Regular Events in Symbolic Logic, *J. Comp. Sys. Sci.*, 1982, *25*, pp. 360-376.
21. W. Thomas, An Application of the Ehrenfeucht-Fraissé Game in Formal Language Theory, *Bull. Soc. Math. de France*, 2e série, Mémoire, 1984, No. 16, pp. 11-21.
22. B. Tilson, Categories as Algebra, *J. of Pure and Applied Algebra*, 1987, *48*, pp. 83-198.