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PARTIALLY ABELIAN SQUAREFREE WORDS (*)

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Abstract. – *The notions of square-freeness and abelian squarefreeness of words are generalized by introducing the definition of θ -square free words for a commutation θ in the free monoid. Properties involving finiteness or infiniteness of the set of θ -square free words are obtained for alphabets of three and four letters.*

Résumé. – *On généralise la notion de mots sans carré et de mots sans carré abélien en introduisant celle de mot sans carré partiellement abélien pour une relation de commutation θ . Des résultats concernant le caractère fini ou infini de l'ensemble des mots sans carré partiellement abélien sont obtenus dans le cas des alphabets de trois ou quatre lettres.*

The determination of avoidable properties of words is one of the main chapters in the combinatorial theory of the free monoid [2, 10]. Among these properties, the one of containing a square has been investigated by many authors (*see* the survey of Berstel [3]). Since the work of Thue [15] it is known that there exist infinitely many square-free words in a three letter alphabet. Another avoidable property is the abelian square-freeness, an abelian square being a word fg such that f and g possess the same number of occurrences of each letter of the alphabet; Pleasants [12] has shown that the set of words which do not contain an abelian square over an alphabet of five letters is infinite. The same question for a 4-letter alphabet is still open.

The recent interest for free partially commutative monoids (introduced by Cartier and Foata [7]) motivated by the modelization of concurrency [1, 11], suggests the definition of a new notion of a square. It is that of a square

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with respect to a commutation relation \sim_θ , called a θ -square in this article. It is a word fg such that $f \sim_\theta g$. If θ is empty then the ordinary squares are obtained and if θ is the whole set $A \times A$ then the θ -squares are the abelian squares. A different definition is given by A. Carpi and A. De Luca [6]. As a consequence of the result of Pleasants, for any alphabet A containing at least five letters and for any relation θ the set of θ -square-free words is infinite. We thus restrict our investigation to the infiniteness of the set of θ -square-free words in the case of three or four letter alphabets.

For a three letter alphabet, we prove that if two or three pairs of letters commute then the set of θ -square-free words is finite. If only one pair of letters commute then it is infinite and we give a characterisation of those θ -square-free words in terms of excluded factors.

For a four letter alphabet infiniteness is proved in the case that strictly less than five pairs of letters commute; the case of five and six commutations remains an open problem.

1. PRELIMINARIES

The definitions and notation follow M. Lothaire [10] (*see* chapters 1 and 2).

A is a finite alphabet, A^* is the *free monoid* generated by A , whose elements are called *words*, $\mathbf{1}$ is the empty word. The *length* of a word w is denoted by $|w|$ and the number of occurrences of the letter a in w by $|w|_a$. The word u is a *factor* of w if $w = w_1 u w_2$. A *morphism* φ between two free monoids A^* and B^* is a mapping φ such that:

$$\forall u, v \in A^*, \quad \varphi(uv) = \varphi(u) \cdot \varphi(v).$$

Square-free words

A *square* is a word $w = uu$ with $u \neq \mathbf{1}$, and a *square-free* word is such that none of its factors is a square. If A is a 2-letter alphabet there are only six square-free words namely a, b, ab, ba, aba, bab . If the alphabet has cardinality greater than 2, Thue [15] has shown that there are infinitely many square free words; for instance the sequence $u_1 = abc$, $u_{i+1} = \varphi(u_i)$ where φ is the morphism:

$$\varphi(a) = abc, \quad \varphi(b) = ac, \quad \varphi(c) = b$$

consists of square-free words. An *infinite word* w is a mapping from the set N of natural integers into A ; such a word w is square-free if $w = w_1 u w'$ (where w, u are finite and w' infinite) implies $u = \mathbf{1}$. Clearly the existence of infinite square free words is equivalent to the infiniteness of the set of square-free finite words.

Commutation relation

A symmetrical subset θ of $A \times A$ generates a relation denoted by \sim_θ on A^* as the least congruence for with $ab \sim_\theta ba$, for all $(a, b) \in \theta$. In other words, two elements f, g of A^* are equivalent under \sim_θ if there exist h_1, h_2, \dots, h_k such that:

$$h_1 = f, h_k = g, \quad \text{and} \quad \forall i (1 \leq i < k) h_i = h'_i a_i b_i h''_i, \\ h_{i+1} = h'_i b_i a_i h''_i \quad (a_i, b_i) \in \theta.$$

Note that it is generally assumed that $(a, a) \notin \theta$ for all a but this assumption has no importance here.

DEFINITION 1.1: *A square with respect to the relation θ , or a θ -square, is a word w such that $w = uv$ and $u \sim_\theta v$. A word w is θ -square-free if none of its factors is a θ -square. The set of θ -square-free words is denoted by $L_2(\theta)$.*

Note that if θ and ρ are such that $\theta \subset \rho$, then each θ -square is also a ρ -square and then $L_2(\theta)$ contains $L_2(\rho)$. If θ is empty then θ -squares are the usual squares and if θ contains all pairs (a, b) for $a \neq b$ then θ -squares are the abelian squares.

A. Carpi and A. Deluca [6] have introduced another notion of square-freeness in the quotient monoid A^*/\sim_θ . A word is square-free in A^*/\sim_θ if all words of its \sim_θ class are square-free. It is easy to verify that if a word is square-free in A^*/\sim_θ then it is also θ -square-free, but the converse is not true. For instance in $\{a, b\}^*$ with $ba \sim_\theta ab$, the word aba is θ -square-free but not square-free in A^*/\sim_θ (it is equivalent to aab).

We end this section with a characterisation of θ -squares.

Let a, b the two letters of A and let $\pi_{a,b}$ be the morphism of A^* onto $\{a, b\}^*$ defined by:

$$\pi_{a,b}(a) = a, \quad \pi_{a,b}(b) = b, \quad \pi_{a,b}(c) = \mathbf{1}, \quad \forall c \notin \{a, b\}.$$

The following proposition is a reformulation of Proposition 1.1 of [8].

PROPOSITION 1.2: *The word $u.v$ is a θ -square if and only if conditions (i) and (ii) are satisfied:*

- (i) $|u|_a = |v|_a, \forall a \in A.$
- (ii) $\pi_{a,b}(u) = \pi_{a,b}(v), \forall (a,b) \notin \theta.$

2. PARTIALLY ABELIAN SQUARE FREE WORDS IN $\{a, b, c\}^*$

In this section A is the alphabet consisting of the three letters $\{a, b, c\}$ and θ_1 is the relation consisting of the two pairs $\{(a, c), (c, a)\}$, θ_2 consists of $\{(b, c), (c, b)\}$ and θ_3 of $\{(a, b), (b, a)\}$. We will prove that there are only finitely many $(\theta_1 \cup \theta_2)$ square-free words. We first give some necessary conditions for a word to be θ_1 -square-free. Further investigation along these lines would probably lead one to a generalization to θ_1 -square-free words of the results obtained by Shelton and Soni [14] on square-free words in $\{a, b, c\}^*$.

PROPOSITION 2.1: *Let f be a θ_1 -square-free word such that $f = f_1 bacbf_2$ or $f = f_1 bcabf_2$. Then at least one of the two words f_1 or f_2 is of length strictly less than 2.*

Proof: Because of the symmetric role played by \underline{a} and \underline{c} , we can restrict ourselves to $f = f_1 bacbf_2$. Suppose that f_1 has length at least 2; then $f_1 = f'_1 bc$, as well as any other end for f_1 , gives a square (this is the case for ab, cb, ba, ac) or a θ_1 -square (this is the case for ca). This gives :

$$f = f'_1 bcbacbf_2.$$

If f_2 begins with an \underline{a} then $cba cba$ is a square; thus f_2 begins with a \underline{c} and this occurrence of c can be followed neither by a \underline{b} (square $c b c b$) nor by an \underline{a} (θ_1 -square $b a c b c a$) thus f_2 is of length at most 1 giving the result. ■

Let us introduce the following subsets of $\{a, b, c\}^*$:

$$\begin{aligned} Y &= \{ba, baca\}, & Z &= \{bc, bcac\}, & X &= Y \cup Z \\ U &= \{1, a, c, ac, ca, aca, cac, bac, bca, abac, abca, cbac, cbca\} \\ V &= \{1, b, bac, bca, bcab, bach, bcaba, bcabc, bacba, bacbc\}. \end{aligned}$$

PROPOSITION 2.2: *The set $L_2(\theta_1)$ of θ_1 -square-free words is a subset of UX^*V . Moreover, if w is a θ_1 -square-free word such that*

$w = ux_1x_2 \dots x_k vx_i \in X$, $u \in U$, $v \in V$, then:

$$\begin{aligned} i < k, \quad x_i \in Y, \quad |x_{i+2} \dots x_k v| \neq 0 &\Rightarrow x_{i+1} \in Z \\ i < k, \quad x_i \in Z, \quad |x_{i+2} \dots x_k v| \neq 0 &\Rightarrow x_{i+1} \in Y. \end{aligned}$$

Proof: Let w be a θ_1 -square free word. If w contains one or no occurrences of b then the result is easy to obtain by inspection. If w contains more than two occurrences of b , as w is square free the words between two consecutive occurrences of b are square free over $\{a, c\}$ hence one of a, c, ac, ca, aca, cac . We rule out the possibility that they are ac or ca by Proposition 2.1. We can thus obtain:

$$w = \alpha_1 b \alpha_2 b \dots b \alpha_k b \alpha_{k+1} \quad \text{with } k \geq 2.$$

If $k \leq 3$ the result is again obtained by inspection; assume that $k > 3$. Since $|\alpha_1 b \alpha_2| \geq 2$ and $|\alpha_k b \alpha_{k+1}| \geq 2$. It follows by Proposition 2.1, that $\alpha_i \in \{a, c, aca, cac\}$ for $2 < i < k$ and:

$$w = \alpha_1 b \alpha_2 w' b \alpha_k b \alpha_{k+1}$$

with $w' \in X^*$.

If $b \alpha_2$ is an element of X then $\alpha_1 \in \{1, a, c, ac, aca, cac\}$ which is included in U ; similarly if $b \alpha_k$ is an element of X then $b \alpha_{k+1}$ belongs to X or to $\{bac, bca\}$ giving the result.

We can thus suppose $b \alpha_2, b \alpha_k \notin X$; then $\alpha_2, \alpha_k \in \{ac, ca\}$; and an easy inspection shows in this case $\alpha_1 b \alpha_2 \in U$ and $b \alpha_k b \alpha_{k+1} \in V$ as these words do not contain θ_1 -squares.

Let us now consider a decomposition of a θ_1 -square free word w in:

$$w = ux_1 \dots x_k v, \quad u \in U, \quad v \in V, \quad x_i \in X$$

then as $babaca$ contains a square, we obtain:

$$i < k; \quad x_i = ba \Rightarrow x_{i+1} \in \{bc, bcac\}.$$

If $x_i = baca$ and $x_{i+1} = ba$ then if $x_{i+2} \dots x_k v$ begins with the letter \underline{b} ; this gives the square $abab$, so that $x_{i+2} \dots x_k v$ is empty. ■

PROPOSITION 2.3 : *The length of a $(\theta_1 \cup \theta_2)$ -square free word is at most 15.*

Proof: Let w be a $(\theta_1 \cup \theta_2)$ -square free word; w being θ_1 -square free it can be written as

$$w = ux_1 \dots x_k v.$$

From Proposition 2.1 applied to θ_2 -square-free words we deduce that none of the x_i for $i=1 \dots k-2$ is $bcac$ since in that case x_{i+1} would be from the set $\{ba, baca\}$ giving the factor $acba$ for w .

The longest θ_1 -square-free word belonging to $\{ba, bc, baca\}^*$ are:

babcbba, babcbacabcbabc, bacabcbabc,
bacabcbacaba, bcbabc, bcbacabcbabc

This gives the two $(\theta_1 \cup \theta_2)$ square free words of length 15:

cabacabc bacabac
cbabcbacabcbabc. ■

Remark 2.4: Recall that $L_2(\theta)$ is the set of θ -square-free words. In next section we will prove that $L_2(\theta_1)$ (and symmetrically $L_2(\theta_2)$, and $L_2(\theta_3)$) is infinite. By easy but tedious considerations (or by using a computer) it is possible to verify that:

$$L_2(\theta_1 \cup \theta_2) = L_2(\theta_1) \cap L_2(\theta_2)$$

$$L_2(\theta_1 \cup \theta_2 \cup \theta_3) = L_2(\theta_1) \cap L_2(\theta_2) \cap L_2(\theta_3).$$

Note that these equalities do not hold for any θ, θ' since if we consider the four letter alphabet $\{a, b, c, d\}$ and the two relations $\theta_1 = \{(a, b), (b, a)\}$ and $\theta_2 = \{(c, d), (d, c)\}$ then $abcdbadc$ belongs to $L_2(\theta_1) \cap L_2(\theta_2)$ but not to $L_2(\theta_1 \cup \theta_2)$.

Remark 2.5: The number of words of length k for $(1 \leq k \leq 15)$ of $L_2(\theta_1)$, $L_2(\theta_1 \cup \theta_2)$, $L_2(\theta_1 \cup \theta_2 \cup \theta_3)$ is given by the following table:

| k | $L_2(\theta_1)$ | $L_2(\theta_1 \cup \theta_2)$ | $L_2(\theta_1 \cup \theta_2 \cup \theta_3)$ |
|---------|-----------------|-------------------------------|---|
| 1..... | 3 | 3 | 3 |
| 2..... | 6 | 6 | 6 |
| 3..... | 12 | 12 | 12 |
| 4..... | 18 | 18 | 18 |
| 5..... | 30 | 30 | 30 |
| 6..... | 38 | 34 | 30 |
| 7..... | 46 | 32 | 18 |
| 8..... | 48 | 22 | 0 |
| 9..... | 60 | 24 | 0 |
| 10..... | 68 | 24 | - |
| 11..... | 88 | 30 | - |
| 12..... | 96 | 28 | - |
| 13..... | 98 | 18 | - |
| 14..... | 100 | 6 | - |
| 15..... | 100 | 2 | - |

3. SUFFICIENT CONDITIONS FOR θ_1 -SQUARE-FREENESS

In this section we give conditions for a word w which imply that w is θ_1 -square-free and we prove that these conditions are satisfied by the sequence of Thue-Morse. We also give some conditions which have to be satisfied by a morphism in order that the image of a square-free word is a θ_1 -square-free word.

DEFINITION 3.1: A word f satisfies condition (F) if neither $bacb$ nor $bcab$ is a factor of f .

PROPOSITION 3.2: Let f be a finite square-free word satisfying (F), and containing a θ_1 -square as a factor, then f admits one of the following decompositions: (α) $f=f_1acuaauf_2$; (β) $f=f_1caucacucf_2$; (γ) $f=f_1auacaucf_2$; (δ) $f=f_1cucacucf_2$.

Moreover in such a decomposition one of f_1 or f_2 is of length at most 1.

Proof: Let f be such a word. Then:

$$f=f_1ghf_2 \quad \text{and} \quad g \sim_{\theta_1} h.$$

As f is square-free and satisfies condition (F) the only possible words between two occurrences of b are from the set $B=\{a, c, aca, cac\}$. Note that two different words in this set are not equivalent under \sim_{θ_1} . Let g and h be decomposed in the following way:

$$\begin{aligned} g &= g_1 b g_2 \dots b g_p, & \forall i=1, p : g_i \in \{a, c\}^* \\ h &= h_1 b h_2 \dots b h_q, & \forall i=1, q : h_i \in \{a, c\}. \end{aligned}$$

From Proposition 1.1 we get $p=q$ and $g_i \sim h_i$ for $i=1, \dots, p$.

From $g_i \in B$ for $i=2, \dots, p-1$, we get $g_i=h_i$ for $i=2, \dots, p-1$. As f is square-free $g_1 \neq h_1$ or $g_p \neq h_p$, by our previous remark $g_p h_1$ is an element of B and $g_p h_1 = aca$ or $g_p h_1 = cac$. As \underline{a} and \underline{c} play symmetric roles we can suppose $g_p h_1 = aca$, this gives:

$$g_p = a \quad \text{and} \quad h_1 = ca \quad \text{or} \quad g_p = ac \quad \text{and} \quad h_1 = a;$$

in the first case $h_p = a$ and $g_1 = ca$ giving decomposition (α); in the second case $h_p = ca$ and $g_1 = a$ giving decomposition (γ).

Let us consider now the decomposition:

$$f=f_1acuaauf_2$$

and let us show that at least one of f_1 or f_2 is of length at most 1; a symmetric proof will give the other ones. In such a decomposition u begins and ends with the letter \underline{b} . If u is of length more than 1, then u has one of the following decompositions:

$$u = babu', \quad u = bacabu', \quad u = bcbu', \quad u = bcacbu'.$$

The first one gives a square $abab$, the second one $bacabaca$ (with the \underline{b} at the end of the first occurrence of u). The third one cbc , as to the fourth we have

$$f = f_1 acbcacbu' ac a u a f_2.$$

Since $bacb$ is not a factor of f , f_1 doesn't end with \underline{b} ; it doesn't end with \underline{c} or \underline{a} either, since f is square-free; thus f_1 is empty. If u is of length 1, then:

$$f = f_1 ac bac a b a f_2.$$

And f_2 doesn't begin with \underline{a} (square aa) nor with \underline{b} (square $abab$); the first letter of f_2 is thus \underline{c} and one can easily prove that this \underline{c} is not followed by any other letter so that f_2 has length 1. ■

COROLLARY 3.3: *Any infinite square free word of $\{a, b, c\}^*$ beginning with a letter b and satisfying (F) is θ_1 -square-free.*

Proof: Let w be such a word and assume it has a θ_1 -square then $w = w_1 gh w_2 w'$ with $|w_2| \geq 2$. Since $w_1 gh w_2$ satisfies the hypothesis of Proposition 3.1 this gives $|w_1| \leq 1$, since w begins a letter b , we get $w_1 = b$ and among the decompositions of $w_1 gh w_2$ only the following remain because of condition (F):

$$ba u a c a u c a w_2, \quad bc u c a c u a c w_2.$$

Then u is of length greater than one and ends with $cacb$ or $acab$. This implies $|w_2| \leq 1$, a contradiction. ■

COROLLARY 3.4: *The infinite sequence of words obtained from the Thue Morse sequence by deleting the first letter consists of θ_1 -square-free words. Thus $L_2(\theta_1)$ is infinite.*

Proof: Set $u_0 = abc$, and $u_i = \varphi(u_{i-1})$ where φ is defined by $\varphi(a) = abc$, $\varphi(b) = ca$, $\varphi(c) = b$. Remark first that cbc is not a factor of u_i since $\{abc, ac, b\}^* \cap A^* cbc A^*$ is empty. We observe that, if $\varphi(u_i)$ has $bcab$ as a factor, then u_i contains aa and is not square-free. If $\varphi(u_i)$ contains the factor

$bcab$ then u_i contains necessarily cbc with is a contradiction by the previous remark. We thus obtain the result as a consequence of Corollary 3.2. ■

Note that each u_i is also θ_1 -square-free but the technical proof of this fact is of poor interest and is omitted here.

4. PARTIALLY ABELIAN SQUARE FREE WORDS IN A FOUR LETTERS ALPHABET

In this section, A is the alphabet $\{a, b, c, d\}$. We consider the two relations ρ_1 and ρ_2 which are obtained by symmetrization of:

$$\begin{aligned}\rho'_1 &= \{(a, c), (a, d), (b, d), (c, d)\} \\ \rho'_2 &= \{(a, c), (a, d), (b, c), (b, d)\}.\end{aligned}$$

We will show that there exist an infinite number of ρ_1 -square-free words and of ρ_2 -square-free words. By the symmetric role of a, b, c, d and using the fact that if $\theta \subset \rho$, any ρ -square-free word is also θ -square-free, it is easy to verify that if ρ is any relation with at most four pairs of commutations then the set of ρ -square free words is infinite. The cases where ρ has five or six pairs of commutations remain an open question, the last one is a reformulation of the problem of the existence of an infinite word without an abelian square, in a four letters alphabet.

To prove these results we use the Thue Morse sequence t defined by the iteration of morphism $\varphi : \varphi(a) = abc, \varphi(b) = ac, \varphi(c) = b$, or any infinite sequence with no θ_1 -square.

Let ψ be the morphism defined by

$$\psi(a) = a; \quad \psi(b) = bd; \quad \psi(c) = c;$$

then we have

THEOREM: $\psi(t)$ is a ρ_1 and a ρ_2 -square free infinite word.

1. It is not difficult to prove that $\psi(t)$ is ρ_1 -square free. Assume $\psi(t)$ contains a ρ_1 -square uv , then by Proposition 1.1.:

$$\pi_{a,b}(u) = \pi_{a,b}(v) \quad \text{and} \quad \pi_{b,c}(u) = \pi_{b,c}(v).$$

Let u' and v' be obtained from u and v by deleting all the occurrences of d . Let:

$$t = t_1 u' v' t_2$$

and

$$\pi_{a,b}(u') = \pi_{a,b}(v'), \quad \pi_{b,c}(u') = \pi_{b,c}(v')$$

giving a θ_1 -square for t which is in contradiction with Corollary 3.3.

2. Suppose that $w = \psi(t)$ contains a ρ_2 -square uv , let t_1 (resp. $t_1 x$) be the longest factor of t such that $\psi(t_1)$ is a left factor of w_1 (resp. $\psi(t_1 x)$ is a left factor of $w_1 u$), and let $t_1 xy$ be the smallest such that $\psi(t_1 xy)$ has $w_1 uv$ as a left factor.

Then we have:

$$t = t_1 xy t_2, \quad w = w_1 u v w_2$$

and one of the following pair (i), (j)' of conditions holds:

| | |
|----------------------|------------------------|
| (1) $u = \psi(x)$ | (1') $v = \psi(y)$ |
| (2) $u = \psi(x) b$ | (2') $bv = \psi(y)$ |
| (3) $u = d\psi(x)$ | (3') $vd = \psi(y)$ |
| (4) $u = d\psi(x) b$ | (4') $bvd = \psi(y)$. |

Note that as uv is a ρ_2 -square we have

$$|u|_b = |v|_b \quad \text{and} \quad |u|_d = |v|_d.$$

This gives that the only possible combinations are:

- (1) or (4) with (1)' or (4)',
- (2) with (3)',
- (3) with (2)'.

As x and y are to be consecutive in t and u and v are in w then (1) with (4)', (4) with (1)', (2) with (3)' and (3) with (2)' are to be discarded:

- (1) with (4)' gives $ubvd = \psi(x)\psi(y)$,
- (4) with (1)' gives $uv = d\psi(x)b\psi(y)$,
- (2) with (3)' gives $uvd = \psi(x)b\psi(y)$,
- (3) with (2)' gives $ubv = d\psi(x)\psi(y)$.

We have only to consider (1), (1)' and (4), (4)'.

If (1) and (1)' hold then:

$$uv = \psi(x)\psi(y);$$

uv being a ρ_2 -square this gives:

$$\pi_{a,b}(u) = \pi_{a,b}(v) \quad \text{and} \quad \pi_{c,d}(u) = \pi_{c,d}(v).$$

But $\pi_{a,b}(u) = \pi_{a,b}(x)$ and $\pi_{c,d}(u)$ is obtained from $\pi_{b,c}(x)$ replacing the occurrences of b by d . Thus:

$$\pi_{a,b}(x) = \pi_{a,b}(y) \quad \text{and} \quad \pi_{b,c}(x) = \pi_{b,c}(y)$$

and again by Proposition 1.1, xy is a ρ_1 -square in t , a contradiction.

If (4) and (4)' hold then:

$$uvd = d\psi(x)\psi(y)$$

and as uv is a ρ_2 -square, $\pi_{a,b}(u) = \pi_{a,b}(v)$ and $\pi_{c,d}(u) = \pi_{c,d}(v)$. We thus get

$$\pi_{a,b}(bud) = \pi_{a,b}(bvd), \quad \pi_{c,d}(bud) = \pi_{c,d}(bvd).$$

From (4), and (4)' we obtain:

$$\pi_{a,b}(b\psi(x)b) = \pi_{a,b}(\psi(y)), \quad \pi_{c,d}(d\psi(x)d) = \pi_{c,d}(\psi(y))$$

$\pi_{c,d}(\psi(x))$ is obtained from $\pi_{b,c}(x)$ by replacing the occurrences of b by d ; we obtain

$$\pi_{a,b}(bxb) = \pi_{a,b}(y) \quad \text{and} \quad \pi_{b,c}(bxb) = \pi_{b,c}(y).$$

Thus bxb and y are equivalent under \sim_{θ_1} , giving $y = by'b$ and $x \sim_{\theta_1} y'$ (b commutes with no letter under θ_1) since t contains the factor xy , we have $xy = xby'b$ which is a θ_1 -square, and we also obtain a contradiction. ■

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