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*RAIRO. Informatique théorique et applications*, tome 24, n° 1 (1990),  
p. 67-87

[http://www.numdam.org/item?id=ITA\\_1990\\_\\_24\\_1\\_67\\_0](http://www.numdam.org/item?id=ITA_1990__24_1_67_0)

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## BI-INFINITARY CODES (\*)

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Communicated by A. ARNOLD

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*Abstract.* – The notion of bi-infinity codes is introduced. For this purpose, the monoid  ${}^{\infty}A^{\infty}$  of finite, infinite and bi-infinite words over an alphabet  $A$  is defined. A necessary and sufficient condition for a set of words to be a bi-infinity code is formulated. Conditions for a submonoid of  ${}^{\infty}A^{\infty}$  to have a minimal generator set are established. Using a specific kind of Thue system, the notion of bi-quasi free sub-monoids is introduced. An "algebraic" characterization of the submonoids generated by bi-infinity codes is obtained. Finally, a "combinatorial" characterization of bi-quasi free submonoids is studied.

*Résumé.* – On introduit la notion de code biinfini. On définit d'abord le monoïde  ${}^{\infty}A^{\infty}$  des mots finis, infinis ou biinfinis sur un alphabet  $A$ . On énonce une condition nécessaire et suffisante pour qu'un ensemble de mots soit un code biinfini. On donne également des conditions pour qu'un sous-monoïde de  ${}^{\infty}A^{\infty}$  ait un ensemble minimal de générateurs. En utilisant un système de Thue spécifique, on introduit la notion de sous-monoïde bi-quasi libre. Une caractérisation « algébrique » des sous-monoïdes engendrés par des codes bi-infinis est alors obtenue. Finalement, on étudie une caractérisation « combinatoire » des sous-monoïdes bi-quasi libres.

### INTRODUCTION

There has been a systematic study of codes consisting of finite words, initiated by M. P. Schützenberger [16] and developed by many others taking motivation from information theory (see [11-13]).

Recently, infinity languages consisting of finite and infinite words have served as an adequate tool for studying behaviours of processes. This is the approach of M. Nivat and A. Arnold [14] in some problems of synchronization which stimulated the study of infinite words including bi-infinite words [15].

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(\*) Received June 1987, revised in February 1988.

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Motivated by the theory of codes [1] and the theory of infinitary languages, the notion of infinitary codes has been introduced and examined in [3-10].

This paper is devoted to a study of bi-infinitary codes which are a natural generalization of infinitary codes to bi-infinitary languages *i. e.*, languages of finite, left-infinite, right-infinite and bi-infinite words.

## SECTION 1

### MONOID ${}^{\infty}A^{\infty}$ AND BI-INFINITARY CODES

Let  $A$  be an alphabet. We denote by  $A^*$ , the free monoid generated by  $A$ . Elements of  $A^*$  are called finite words. The length of a word  $x$  in  $A^*$  is denoted by  $|x|$ , the empty word by  $\varepsilon$  and  $A^+ = A^* - \{\varepsilon\}$ .

We denote by  $A^N$ , the set of all right-infinite words, by  $A^{-N}$ , the set of all left-infinite words and by  $A^Z$ , the set of all bi-infinite words over  $A$ . Every (bi) infinite word  $u$  has a countable length  $|u| = \omega$ . For any  $X \subseteq A^*$ , we denote by  $X^{\omega}$  ( ${}^{\omega}X$ ,  ${}^{\omega}X^{\omega}$ ), the set of all right-infinite (left-infinite, bi-infinite) words of the form  $x_1 x_2 \dots (\dots x_2 x_1, \dots x_1 x_2 x_3 \dots)$  for  $x_i \in X$ . In particular, if  $x \in A^*$ , then  $x^{\omega} = xxx \dots$ ,  ${}^{\omega}x = \dots xxx$  and  ${}^{\omega}x^{\omega} = \dots xxx \dots$ . We write  $A^{\infty} = A^* \cup A^N$ ,  ${}^{\infty}A = A^* \cup A^{-N}$  and  ${}^{\infty}A^{\infty} = A^* \cup A^N \cup A^{-N} \cup A^Z$ .

We define a product on elements of  ${}^{\infty}A^{\infty}$  as follows:

$$\alpha \cdot \beta = \begin{cases} \alpha, & \text{if } \alpha \in A^N \cup A^Z \\ \alpha\beta, & \text{if } \alpha \in A^* \cup A^{-N}, \beta \in A^* \cup A^N \\ \beta, & \text{if } \alpha \in A^* \cup A^{-N}, \beta \in A^{-N} \cup A^Z. \end{cases}$$

It is not difficult to verify that the product is associative and therefore  ${}^{\infty}A^{\infty}$  is a monoid. This monoid has  $A^*$ ,  $A^{\infty}$  and  ${}^{\infty}A$  as its submonoids. For simplicity, instead of  $\alpha \cdot \beta$ , we write  $\alpha\beta$ . For any  $X \subseteq {}^{\infty}A^{\infty}$ , we denote by  $X^*$ , the submonoid of  ${}^{\infty}A^{\infty}$  generated by  $X$  and write  $X^+ = X^* - \{\varepsilon\}$ . If  $\alpha$  is a word, instead of  $\{\alpha\}^*$ , we write  $\alpha^*$ .

For any  $X \subseteq {}^{\infty}A^{\infty}$ , we write  $X_{\text{fin}} = X \cap A^*$ ,

$$X_{\text{inf}} = X \cap A^N, \quad X_{-\text{inf}} = X \cap A^{-N}, \quad X_{\text{binf}} = X \cap A^Z,$$

$$X^{\infty} = X_{\text{fin}} \cup X_{\text{inf}}, \quad {}^{\infty}X = X_{\text{fin}} \cup X_{-\text{inf}},$$

$$\bar{X}^{(0)} = X^{(0)} = \{\varepsilon\},$$

$$\bar{X}^{(1)} = X^{(1)} = X,$$

$$X^{(\vec{n})} = \{(x_1, x_2, \dots, x_n) / x_1, x_2, \dots, x_{n-1} \in X_{\text{fin}}, x_n \in X^{\infty}\} \quad \text{for } n \geq 2,$$

$$X^{(\vec{n})} = \{ (x_1, x_2, \dots, x_n) / x_1 \in {}^\omega X, x_2, x_3, \dots, x_n \in X_{\text{fin}} \} \quad \text{for } n \geq 2,$$

$$X^{(\vec{n})} = \{ (x_1, x_2, \dots, x_n) / x_1 \in X_{-\text{inf}}, x_n \in X_{\text{inf}}, x_2, x_3, \dots, x_{n-1} \in X_{\text{fin}} \} \quad \text{for } n \geq 2$$

$$X^{(n)} = X^{(\vec{n})} \cup X^{(\vec{n})} \cup X^{(\vec{n})} \quad \text{for } n \geq 2,$$

$$X^{(*)} = \bigcup_{n \geq 0} X^{(n)}$$

$$\bar{X}^{(\vec{n})} = \{ x_1 x_2 \dots x_n / (x_1, x_2, \dots, x_n) \in X^{(\vec{n})} \} \quad \text{for } n \geq 2$$

$$\bar{X}^{(\vec{n})} = \{ x_1 x_2 \dots x_n / (x_1, x_2, \dots, x_n) \in X^{(\vec{n})} \} \quad \text{for } n \geq 2,$$

$$\bar{X}^{(\vec{n})} = \{ x_1 x_2 \dots x_n / (x_1, x_2, \dots, x_n) \in X^{(\vec{n})} \} \quad \text{for } n \geq 2$$

and

$$\bar{X}^{(n)} = \bar{X}^{(\vec{n})} \cup \bar{X}^{(\vec{n})} \cup \bar{X}^{(\vec{n})} \quad \text{for } n \geq 2.$$

We say that a word  $\alpha \in {}^\omega A^\omega$  has a factorization on elements of  $X$  if  $\alpha = x_1 x_2 \dots x_n$  for some  $(x_1, x_2, \dots, x_n) \in X^{(*)}$ .

DEFINITION 1.1: A subset  $X$  of  ${}^\omega A^\omega$  is called a bi-infinitary code if every word  $\alpha \in {}^\omega A^\omega$  has atmost one factorization on elements of  $X$ . More precisely,  $X$  is a bi-infinitary code if for any  $n, m \geq 1$  and for any  $(x_1, x_2, \dots, x_n) \in X^{(n)}$ ,  $(x'_1, x'_2, \dots, x'_m) \in X^{(m)}$ , the equality  $x_1 x_2 \dots x_n = x'_1 x'_2 \dots x'_m$  implies  $n = m$  and  $x_i = x'_i (i = 1, 2, \dots, n)$ .

Unless otherwise stated, from now on code means bi-infinitary code.

Example 1.1 : If  $A = \{ a, b \}$ , the subset

$$X = \{ {}^\omega(ab)^\omega, {}^\omega a, b^\omega, ba \}$$

is a code whereas the subset

$$Y = \{ {}^\omega(ab)^\omega, {}^\omega a, b^\omega, ab \}$$

is not a code, since we have,

$$\begin{aligned} {}^\omega ab^\omega &= {}^\omega a . ab . b^\omega \\ &= {}^\omega a . b^\omega. \end{aligned}$$

## SECTION 2

## A CHARACTERIZATION OF BI-INFINITARY CODES

In this section, we establish a characterization of codes. We first introduce certain concepts and formulate a fundamental formula.

Let  $X$  and  $Y$  be two subsets of  ${}^\infty A^\infty$ . Define the sets

$$\begin{aligned} Y^{-1}X &= \{ \alpha \in {}^\infty A^\infty \mid \exists \beta \in Y: \beta \alpha \in X, \\ &\quad (\beta \in Y_{\text{inf}} \cup Y_{\text{biinf}} \Rightarrow \alpha = \varepsilon), \\ &\quad (\beta \in {}^\infty Y \text{ and } \alpha \in A^{-N} \cup A^Z \Rightarrow \beta = \varepsilon) \}, \\ XY^{-1} &= \{ \alpha \in {}^\infty A^\infty \mid \exists \beta \in Y: \alpha \beta \in X, (\alpha \in A^N \cup A^Z \Rightarrow \beta = \varepsilon), \\ &\quad (\alpha \in {}^\infty A \text{ and } \beta \in Y_{-\text{inf}} \cup Y_{\text{biinf}} \Rightarrow \alpha = \varepsilon) \}. \end{aligned}$$

We note that if  $u, v \in A^{-N}$  and  $u \leq v$ , then  $u^{-1}v$  is a subset of  $A^*$ . For example, if  $u = {}^\omega a$  and  $v = {}^\omega a = {}^\omega a \cdot a^*$ , then  $u^{-1}v = a^*$ .

We associate with every subset  $X \subseteq {}^\infty A^\infty$ , a sequence of subsets, denoted by  $U_n(X)$  or simply by  $U_n$ , defined recursively by

$$\begin{aligned} U_1 &= X^{-1}X - \{ \varepsilon \} \\ U_{n+1} &= X^{-1}U_n \cup U_n^{-1}X, \quad n \geq 1. \end{aligned}$$

LEMMA 2.1: For any subset  $X$  of  ${}^\infty A^\infty - \{ \varepsilon \}$ , (i) if  $n$  is the smallest natural number such that  $\varepsilon \in U_n$ , then  $\forall k \in \{ 1, 2, \dots, n \}$ ,  $\exists u \in U_k$ ,  $\exists i, j \geq 0$ :

$$\begin{aligned} u(\bar{X}^{(i)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j)} &\neq \Phi, \quad i+j+k=n, \\ u \in A^N \cup A^Z &\Rightarrow i=0 \end{aligned} \tag{2.1}$$

(ii)  $\forall n \geq 1$ ,  $\forall k \in \{ 1, 2, \dots, n \}$ :

$$\begin{aligned} (\exists u \in U_k, \exists i, j \geq 0: u(\bar{X}^{(i)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j)} &\neq \Phi, \\ i+j+k &= n, \\ u \in A^N \cup A^Z &\Rightarrow i=0) \Rightarrow \varepsilon \in U_n. \end{aligned}$$

*Proof:* We prove by recurrence on  $k$ .

(i) Let  $n$  be the smallest natural number such that  $\varepsilon \in U_n$ . If  $k=n$ , then (2.1) holds obviously with  $u=\varepsilon$ ,  $i=j=0$ . Let  $n > k \geq 1$  and suppose the statement is true for  $n, n-1, \dots, k+1$ . We prove for  $k$ . Since the statement

is true for  $k + 1$ , there exist  $v \in U_{k+1}$  and integers  $i', j'$  such that

$$v(\bar{X}^{(i')} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j')} \neq \Phi, \quad i' + j' + k + 1 = n,$$

$v \in A^N \cup A^Z \Rightarrow i' = 0$ . Thus we have  $x \in \bar{X}^{(i')} - (A^{-N} \cup A^Z)$  and  $y \in \bar{X}^{(j')}$  such that  $vx = y$ . The fact that  $v \in U_{k+1}$  gives rise to two cases.

Case (a):  $v \in X^{-1} U_k$ . Then, there exists  $z \in X, u \in U_k$  such that

$$zv = u, (z \in X_{\text{inf}} \cup X_{\text{biinf}} \Rightarrow v = \varepsilon)$$

and

$$(z \in {}^\infty X \text{ and } v \in A^{-N} \cup A^Z \Rightarrow z = \varepsilon).$$

If  $v \in A^N$ , then  $i' = 0, x = \varepsilon, z \in {}^\infty X$  and  $u = zy$ . Hence  $u(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j'+1)} \neq \Phi$ . Thus (2.1) holds with  $i = 0, j = j' + 1$ .

If  $v \in A^{-N} \cup A^Z$ , then  $z \in {}^\infty X$  and so  $z = \varepsilon$ . Thus  $\varepsilon \in X$  which contradicts the hypothesis that  $X \subseteq {}^\infty A^\infty - \{\varepsilon\}$ .

If  $v \in A^*$  and  $z \in X_{\text{inf}} \cup X_{\text{biinf}}$ , then  $v = \varepsilon$  and  $u = z$ . Hence  $u(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(1)} \neq \Phi$  and therefore (2.1) holds with  $i = 0, j = 1$ .

If  $v \in A^*$  and  $z \in {}^\infty X$ , then  $ux = zy$  and so

$$u(\bar{X}^{(i')} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j'+1)} \neq \Phi.$$

Thus (2.1) holds with  $i = i', j = j' + 1$ .

Case (b):  $v \in U_k^{-1} X$ . Then, there exist  $u \in U_k$  and  $z \in X$  such that  $uv = z, (u \in A^N \cup A^Z \Rightarrow v = \varepsilon)$  and  $(u \in {}^\infty A, v \in A^{-N} \cup A^Z \Rightarrow u = \varepsilon)$ .

If  $v \in A^N$ , then  $i' = 0, x = \varepsilon, v = y, u \in {}^\infty A$  and  $uy = z$ . Hence  $u(\bar{X}^{(j')} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(1)} \neq \Phi$ . So, (2.1) holds with  $i = j', j = 1$ .

If  $v \in A^{-N} \cup A^Z$ , then  $u \in {}^\infty A$  and therefore  $u = \varepsilon$ . Thus  $\varepsilon = u \in U_k$  with  $k < n$ , which is contrary to the hypothesis that  $n$  is the smallest natural number such that  $\varepsilon \in U_n$ .

If  $v \in A^*$  and  $z \in X_{\text{inf}} \cup X_{\text{biinf}}$ , then  $v = \varepsilon, u = z$  and  $y = x$ . If  $i' = j' = 0$ , then  $k + 1 = n$  and the equality  $u = z$  implies  $u(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(1)} \neq \Phi$ . That is, (2.1) holds with  $i = 0, j = 1$ . Otherwise we have  $k + 1 < n$  and  $v = \varepsilon \in U_{k+1}$  which gives a contradiction.

If  $v \in A^*$  and  $z \in {}^\infty X$  then  $u \in {}^\infty A$ . The equation  $uy = zx$  gives  $u(\bar{X}^{(j')} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(i'+1)} \neq \Phi$ . Thus (2.1) holds with  $i = j', j = i' + 1$ .

(ii) Suppose there exist  $u \in U_k$  and two integers  $i, j \geq 0$  such that  $u(\bar{X}^{(i)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j)} \neq \Phi, i + j + k = n, u \in A^N \cup A^Z \Rightarrow i = 0$ . We have to prove that  $\varepsilon \in U_n$ . If  $k = n$ , then  $i = j = 0$  and so  $u = \varepsilon$ . Hence  $\varepsilon \in U_n$ . Let now

$n > k \geq 1$  and suppose the statement is true for  $n, n-1, \dots, k+1$ . We prove for  $k$ . Suppose  $x_1 x_2 \dots x_i \in \bar{X}^{(i)} - (A^{-N} \cup A^Z)$  and  $x'_1 x'_2 \dots x'_j \in \bar{X}^{(j)}$  such that  $u x_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j$ . We discuss the following cases:

Case (a): Suppose  $u \in A^N \cup A^Z$ . Then  $i=0, j+k=n, j \geq 1$  and  $u = x'_1 x'_2 \dots x'_j$ . Let  $u' = x'_2 x'_3 \dots x'_j$ . Clearly  $u' \in U_{k+1}$  and  $u'(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi, 0+j-1+k+1=n$ . By recurrence hypothesis  $\varepsilon \in U_n$ .

Case (b): Suppose  $u \in A^*$ . If  $j=0$ , then  $i=0, u=\varepsilon$  and  $k=n$ . Thus we have  $\varepsilon \in U_n$ . Let  $j \geq 1$ . If  $|u| \geq |x'_1|$ , that is,  $u = x'_1 u'$  for some  $u'$ , then  $u' \in U_{k+1}$  and

$$u' x_1 x_2 \dots x_i = x'_2 x'_3 \dots x'_j.$$

So  $u'(\bar{X}^{(i)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi, i+j-1+k+1=n$ . By recurrence hypothesis,  $\varepsilon \in U_n$ . If  $|u| < |x'_1|$ , that is,  $x'_1 = uu''$  for some  $u''$ , then  $u'' \in U_{k+1}$  and  $u'' x'_2 x'_3 \dots x'_j = x_1 x_2 \dots x_i$ . Hence

$$u''(\bar{X}^{(j-1)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(i)} \neq \Phi, \quad j-1+i+k+1=n.$$

This implies  $\varepsilon \in U_n$ .

Case (c): Suppose  $u \in A^{-N}$ . Then  $j \geq 1$ . If  $j=1$ , then  $u x_1 x_2 \dots x_i = x'_1$  which implies  $x_1 x_2 \dots x_i \in u^{-1} x'_1$ . Let  $u' = x_1 x_2 \dots x_i$ . We have  $u' \in U_{k+1}$  and

$$u'(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(i)} \neq \Phi, \quad 0+i+k+1=n.$$

By recurrence hypothesis  $\varepsilon \in U_n$ . If  $j > 1$ , there are two subcases.

If  $u$  is a left factor of  $x'_1$ , we have

$$x_1 x_2 \dots x_i = u' x'_2 x'_3 \dots x'_j$$

with  $u' \in u^{-1} x'_1$ . So, we have  $u' \in U_{k+1}$  and

$$u'(\bar{X}^{(j-1)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(i)} \neq \Phi, \quad j-1+i+k+1=n.$$

By recurrence hypothesis  $\varepsilon \in U_n$ .

If  $x'_1$  is a left factor of  $u$ , we have

$$x'_2 x'_3 \dots x'_j = u'' x_1 x_2 \dots x_i$$

with  $u'' \in (x'_1)^{-1} u$ . Then  $u'' \in U_{k+1}$  and

$$u''(\bar{X}^{(i)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi, \quad i+j-1+k+1=n.$$

By recurrence hypothesis  $\varepsilon \in U_n$ . This proves lemma 2.1.

We are now in a position to formulate the main result of this section which is a generalization of the result proved by Do Long Van in [5, 10]. The latter is a generalization of Sardinas-Patterson theorem. This in many cases gives us a procedure to check whether or not a given set is a bi-infinity code.

**THEOREM 2.1:** *A subset  $X$  of  ${}^\infty A^\infty - \{\varepsilon\}$  is a code iff for all  $n \geq 1$ ,  $U_n(X)$  does not contain the empty word  $\varepsilon$ .*

*Proof:* Suppose  $\alpha \notin U_n(X)$ ,  $n \geq 1$ . Assume that  $X$  is not a code. Then there exists a word  $\alpha \in {}^\infty A^\infty$  having two different factorizations on elements of  $X$ :

$$\alpha = x_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j \quad \text{where } (x_1, x_2, \dots, x_i) \in X^{(i)}$$

and  $(x'_1, x'_2, \dots, x'_j) \in X^{(j)}$ .

Case (a): Suppose  $\alpha \in A^* \cup A^N$ . We may assume that  $x_1 \neq x'_1$  and  $|x_1| > |x'_1|$ . Let  $x_1 = x'_1 u$  for some  $u \neq \varepsilon$ . Clearly  $u \in U_1$ .

If  $x_1 \in X_{\text{fin}}$ , then  $x'_1 \in X_{\text{fin}}$  and  $u \in A^+$ . So we have

$$u x_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j, \quad j \geq 2.$$

Hence

$$u(\bar{X}^{(i-1)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi.$$

By lemma 2.1 (ii),  $\varepsilon \in U_{i+j-1}$  which is a contradiction.

If  $x_1 \in X_{\text{inf}}$ , then  $i = 1$ ,  $x'_1 \in X_{\text{fin}}$  and  $u \in A^N$ . Therefore we have  $u = x'_2 x'_3 \dots x'_j$ ,  $j \geq 2$ . This implies

$$u(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi.$$

Again by lemma 2.1 (ii),  $\varepsilon \in U_j$  which is a contradiction.

Case (b): Suppose  $\alpha \in A^{-N}$ . Clearly  $x_1, x'_1 \in X_{\text{inf}}$ . Since the case  $i = j = 1$  is impossible, we may assume that  $i \geq 2$ . There are two possibilities.

(i) If  $x_1 \neq x'_1$  we can assume that  $x_1 = x'_1 u$  with  $u \in A^+$  such that  $u x_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j$ ,  $j \geq 2$ . Then clearly  $u \in U_1$  and

$$u(\bar{X}^{(i-1)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi.$$

Again by lemma 2.1 (ii),  $\varepsilon \in U_{i+j-1}$ . This is a contradiction.

(ii) Suppose  $x_1 = x'_1$ . Here, if  $j = 1$ , then  $x_1 x_2 \dots x_i = x'_1$  and so  $x_1 = {}^o(x_2 x_3 \dots x_i)$ . Let  $x_2 x_3 \dots x_i = u$ . Clearly  $u \in x_1^{-1} x'_1 \subseteq U_1$ . Hence  $u(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(i-1)} \neq \Phi$ . This implies  $\varepsilon \in U_i$  which is a contradiction.

If  $j \geq 2$ , then  $x_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j$  with

$$x_2 x_3 \dots x_i, x'_2 x'_3 \dots x'_j \in A^*.$$

If  $x_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j$ , we may assume that  $x_2 \neq x'_2$ , and as in case (a), get a contradiction. If  $x_2 x_3 \dots x_i \neq x'_2 x'_3 \dots x'_j$ , then we have either  $x_1 = x'_1 u$  and  $ux_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j$  or  $x'_1 = x_1 u$  and  $x_2 x_3 \dots x_i = ux'_2 x'_3 \dots x'_j$  for some  $u \in A^+$ . By symmetry, we shall discuss one of the two possibilities.

Consider  $x_1 = x'_1 u$  and  $ux_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j$ . Now  $x_1 = x'_1$  and  $x_1 = x'_1 u$  imply  $x_1 = x'_1 = {}^m u$  and  $u \in (x'_1)^{-1} x_1 \subseteq U_1$ . Thus  $ux_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j$  gives  $u(\bar{X}^{(i-1)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi$  and so  $\varepsilon \in U_{i+j-1}$  which is a contradiction.

Case (c): Suppose  $\alpha \in A^Z$ . The case  $i=j=1$  is impossible. We assume  $j \geq 2$ . If  $i=1$  then  $x_1 = x'_1 x'_2 \dots x'_j$  and so we have  $u = (x'_1)^{-1} x_1 \in U_1$  with  $u = x'_2 x'_3 \dots x'_j$ . Hence  $u(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi$  which gives a contradiction.

If  $i \geq 2$ , then  $x_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j$ . Now  $x_1, x'_1 \in X_{-\text{inf}}$ . There are two possibilities.

- (i) If  $x_1 \neq x'_1$ , as in case (a), we obtain a contradiction.
- (ii) If  $x_1 = x'_1$ , then we have either

$$x_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j \quad \text{or} \quad x_2 x_3 \dots x_i \neq x'_2 x'_3 \dots x'_j.$$

If  $x_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j$ , then we can assume  $x_2 \neq x'_2$  and as in case (a), get a contradiction since  $x_2 x_3 \dots x_i, x'_2 x'_3 \dots x'_j \in A^N$ . If  $x_2 x_3 \dots x_i \neq x'_2 x'_3 \dots x'_j$ , we can obtain a contradiction as in the last part of Case b (ii). Thus  $X$  is a code.

We shall prove the converse. Suppose  $X$  is a code. Assume that there are some sets  $U_i(X)$  containing  $\varepsilon$ . Let  $U_n(X)$  be one among these, with the smallest index. By lemma 2.1 (i), there exists a word  $u \in U_1$  with two integers  $i, j \geq 0$  such that

$$u(\bar{X}^{(i)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j)} \neq \Phi, \quad i+j+1=n,$$

$u \in A^N \cup A^Z \Rightarrow i=0$ . So, we have  $ux_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j$  for some  $x_1 x_2 \dots x_i \in \bar{X}^{(i)} - (A^{-N} \cup A^Z)$  and  $x'_1 x'_2 \dots x'_j \in \bar{X}^{(j)}$ . Since  $u \in U_1$ , there exist words  $x, x' \in X$  with either  $x \neq x'$  and  $x = x' u$  or  $x = x'$  and  $x = x' u$ .

If  $u \in A^+$ , then both  $x, x'$  are either in  $X_{\text{fin}}$  or in  $X_{-\text{inf}}$ . Let  $x, x' \in X_{\text{fin}}$ . Then we have  $x \neq x'$  and  $x = x' u$ . So  $xx_1 x_2 \dots x_i = x' x'_1 \dots x'_j$  and therefore

$X$  is not a code, a contradiction. Let  $x, x' \in X_{-\text{inf}}$ . If  $x \neq x'$  and  $x = x'u$ , then as before, we get a contradiction. If  $x = x'$  and  $x = x'u$ , then  $x = x' = {}^\omega u$  and either

$$x_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j \quad \text{or} \quad x_1 x_2 \dots x_i \neq x'_1 x'_2 \dots x'_j.$$

If  $x_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j$ , then  $i = j$  and  $x_k = x'_k$  ( $k = 1, 2, \dots, i$ ) since  $X$  is a code. Then the equation  $ux_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j$  implies  $u = \varepsilon$ , a contradiction. If  $x_1 x_2 \dots x_i \neq x'_1 x'_2 \dots x'_j$  then the equation  $x'x_1 x_2 \dots x_i = x'x'_1 x'_2 \dots x'_j$  shows that  $X$  is not a code, a contradiction.

If  $u \in A^N$ , then  $i = 0$  and either  $x \in X_{\text{inf}}$ ,  $x' \in X_{\text{fin}}$  or  $x \in X_{\text{biinf}}$ ,  $x' \in X_{-\text{inf}}$ . In both cases, we have  $x = x'x'_1 \dots x'_j$  which shows  $X$  is not a code, a contradiction.

If  $u \in A^Z$ , then  $i = 0$ ,  $x \in X_{\text{biinf}}$  and  $x' = \varepsilon$ . Since  $x' \in X \subseteq {}^\infty A^\infty - \{\varepsilon\}$ , this case is not possible.

If  $u \in A^{-N}$ , then  $x = u$  and  $x' = \varepsilon$ . As before, this case is also not possible. Thus  $\varepsilon \notin U_n(X), \forall n \geq 1$ .

*Example 2.1:* (i) Let  $X = \{ {}^\omega(ab)^\omega, {}^\omega a, b^\omega, ab \}$ .  $U_1(X) = \{ a^+ \}$ ,  $U_2(X) = \{ b \}$ ,  $U_3(X) = \{ b^\omega \}$  and  $U_4(X) = \{ \varepsilon \}$ . So  $X$  is not a code.

(ii) Let  $X = \{ {}^\omega(ab)^\omega, {}^\omega a, b^\omega, ba \}$ .  $U_1(X) = \{ a^+ \}$ ,  $U_2(X) = \Phi$ . So,  $X$  is a code.

**SECTION 3**

**MINIMAL GENERATOR SET OF A SUBMONOID OF  ${}^\infty A^\infty$ .**

We recall that a generator set  $X$  of a monoid  $M$  is minimal if  $X$  is contained in any generator set of  $M$ . Such a set, if it exists, is unique and called the base of  $M$ , denoted as  $\text{BASE}(M)$ . Every submonoid of  $A^*$  has a minimal generator set whereas there are submonoids of  ${}^\infty A^\infty$  which have no minimal generator sets. We illustrate this in the following example.

*Example 3.1:* Let  $A = \{ a, b \}$  and let  $M$  be the submonoid of  ${}^\infty A^\infty$  given by  $M = \{ \alpha \in {}^\infty A^\infty \mid |\alpha|_a = |\alpha|_b \}$  where  $|\alpha|_a$  stands for the number of occurrences of  $a$  in  $\alpha$ . This monoid has no minimal generator set.

**DEFINITION 3.1:** Let  $M$  be a submonoid of  ${}^\infty A^\infty$  and  $u, v$ , two elements of  $M_{\text{inf}}$ . We say that  $u$  precedes  $v$ , denoted by  $u < v$ , if there exists  $f \in M_{\text{fin}} - \varepsilon$  such that  $u = fv$ . An element  $u \in M_{\text{inf}}$  is called stable if  $\forall v \in M_{\text{inf}}: (u < v) \Rightarrow (u = v)$ . The set of all stable elements of  $M_{\text{inf}}$  is denoted by  $\text{STAB}(M_{\text{inf}})$ .

Let  $x, y$  be two elements of  $M_{-\text{inf}}$ . Here also we say that  $x$  precedes  $y$ , denoted by  $x < y$  if there exists  $g \in M_{\text{fin}} - \varepsilon$  such that  $x = yg$ . As before,  $x \in M_{-\text{inf}}$  is called stable if  $\forall y \in M_{-\text{inf}} : (x < y) \Rightarrow (x = y)$ . The set of all stable elements of  $M_{-\text{inf}}$  is denoted by  $\text{STAB}(M_{-\text{inf}})$ .

We say that a submonoid  $M$  satisfies the stability condition if every unstable element of  $M_{\text{inf}}$  (resp.  $M_{-\text{inf}}$ ) precedes a stable element of  $M_{\text{inf}}$  (resp.  $M_{-\text{inf}}$ ). We introduce the following two sets:

$$\begin{aligned} \text{BASE}(M_{\text{fin}}) &= (M_{\text{fin}} - \varepsilon) - (M_{\text{fin}} - \varepsilon)^2 \\ \text{UNFAC}(M_{\text{biinf}}) &= M_{\text{biinf}} - (M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}}). \end{aligned}$$

**THEOREM 3.1:** *A submonoid  $M$  of  ${}^\infty A^\infty$  has a minimal generator set iff  $M$  satisfies the stability condition and in that case, the minimal generator set of  $M$  is*

$$\begin{aligned} X &= \text{BASE}(M) \\ &= \text{BASE}(M_{\text{fin}}) \cup \text{STAB}(M_{\text{inf}}) \cup \text{STAB}(M_{-\text{inf}}) \cup \text{UNFAC}(M_{\text{biinf}}). \end{aligned}$$

*Proof:* Assume  $X$  satisfies the stability condition. Let

$$\begin{aligned} X_{\text{fin}} &= \text{BASE}(M_{\text{fin}}), & X_{\text{inf}} &= \text{STAB}(M_{\text{inf}}), & X_{-\text{inf}} &= \text{STAB}(M_{-\text{inf}}), \\ X_{\text{biinf}} &= \text{UNFAC}(M_{\text{biinf}}) & \text{and} & & X &= X_{\text{fin}} \cup X_{\text{inf}} \cup X_{-\text{inf}} \cup X_{\text{biinf}}. \end{aligned}$$

Since

$$\begin{aligned} X_{\text{fin}}^* &= M_{\text{fin}}, & M_{\text{inf}} &= \text{STAB}(M_{\text{inf}}) \cup (M_{\text{fin}} - \varepsilon) \text{STAB}(M_{\text{inf}}) \\ & & &= M_{\text{fin}} \text{STAB}(M_{\text{inf}}) = X_{\text{fin}}^* X_{\text{inf}}. \end{aligned}$$

Similarly,

$$\begin{aligned} M_{-\text{inf}} &= X_{-\text{inf}} X_{\text{fin}}^* & \text{and} & & M_{\text{biinf}} &= \text{UNFAC}(M_{\text{biinf}}) \\ & & & & \cup M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}} &= X_{\text{biinf}} \cup X_{-\text{inf}} X_{\text{fin}}^* X_{\text{inf}}. \end{aligned}$$

Therefore,

$$\begin{aligned} M &= M_{\text{fin}} \cup M_{\text{inf}} \cup M_{-\text{inf}} \cup M_{\text{biinf}} \\ &= X_{\text{fin}}^* \cup X_{\text{fin}}^* X_{\text{inf}} \cup X_{-\text{inf}} X_{\text{fin}}^* \cup X_{\text{biinf}} \\ & \quad \cup X_{-\text{inf}} X_{\text{fin}}^* X_{\text{inf}} = X^*. \end{aligned}$$

Thus  $X$  is a generator set of  $M$ . We shall prove that  $X$  is minimal. Let  $Y$  be an arbitrary generator set of  $M$ . We can assume that  $\varepsilon \notin Y$ . It is enough if

we prove that

$$X_{\text{fin}} \subseteq Y_{\text{fin}}, \quad X_{\text{inf}} \subseteq Y_{\text{inf}}, \\ X_{-\text{inf}} \subseteq Y_{-\text{inf}} \quad \text{and} \quad X_{\text{biinf}} \subseteq Y_{\text{biinf}}.$$

As  $Y_{\text{fin}}^* = M_{\text{fin}}$  and  $X_{\text{fin}}$  is the minimal generator set of  $M_{\text{fin}}$ , we have  $X_{\text{fin}} \subseteq Y_{\text{fin}}$ . Let  $u \in X_{\text{inf}}$ . Then  $u = y_1 y_2 \dots y_n$  for some  $(y_1, y_2, \dots, y_n) \in Y^{(n)}$ ,  $n \geq 1$ . If  $n = 1$ , then  $u = y_n \in Y_{\text{inf}}$ . If  $n > 1$ , we have  $u = f y_n$  with  $f = y_1 y_2 \dots y_{n-1} \in M_{\text{fin}} - \varepsilon$  i.e.,  $u < y_n$ . Since  $u$  is stable  $u = y_n \in Y_{\text{inf}}$ . Thus  $X_{\text{inf}} \subseteq Y_{\text{inf}}$ . Similarly we can show that  $X_{-\text{inf}} \subseteq Y_{-\text{inf}}$ . Let  $u \in X_{\text{biinf}}$ . Then  $u = w_1 w_2 \dots w_n$  for some  $(w_1, w_2, \dots, w_n) \in Y^{(n)}$ ,  $n \geq 1$ . If  $n = 1$ ,  $u = w_1$  where  $w_1 \in Y_{\text{biinf}}$ . If  $n \geq 2$ ,  $u = w_1 w_2 \dots w_n$  is an element of  $M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}}$  since  $w_1 \in Y_{-\text{inf}}$ ,  $Y_{\text{fin}}^* = M_{-\text{inf}}$ ,  $w_n \in Y_{\text{fin}}^*$ ,  $Y_{\text{inf}} = M_{\text{inf}}$  and  $w_2 w_3 \dots w_{n-1} \in Y_{\text{fin}}^*$ . This contradicts the choice of  $u$  since  $u \in \text{UNFAC}(M_{\text{biinf}})$ . Hence  $u \in Y_{\text{biinf}}$  and so  $X_{\text{biinf}} \subseteq Y_{\text{biinf}}$ .

We prove the converse part now. Let  $Y$  be a minimal generator set of  $M$ . Suppose  $M$  does not satisfy the stability condition. Then, there exists an unstable element of  $M_{\text{inf}}$  (resp.  $M_{-\text{inf}}$ ), which does not precede any stable element of  $M_{\text{inf}}$  (resp.  $M_{-\text{inf}}$ ). Let  $u$  be an unstable element of  $M_{\text{inf}}$  and  $v$  any element of  $M_{\text{inf}}$  such that  $v \neq u$  and  $u < v$ . If  $u \in Y_{\text{inf}}$ , then since  $Y_{\text{fin}}^* = M_{\text{fin}}$ , the set  $Y' = (Y - \{u\}) \cup \{v\}$  is a generator set of  $M$ . Since  $Y'$  does not contain  $Y$ , we get a contradiction to the minimality of  $Y$ . If  $u \notin Y_{\text{inf}}$ , then  $u = y_1 y_2 \dots y_n$  for some  $(y_1, y_2, \dots, y_n) \in Y^{(n)}$  with  $n > 1$ . Therefore  $u < y_n$ . By hypothesis,  $y_n$  is unstable. Therefore there exists  $w \in M_{\text{inf}}$  such that  $w \neq y_n$  and  $y_n < w$ . Thus, the set  $Y'' = (Y - \{y_n\}) \cup \{w\}$  is a generator set of  $M$ . Since  $Y''$  does not contain  $Y$ , we have a contradiction. Hence  $M$  satisfies the stability condition.

*Example 3.2:* Let  $A = \{a, b\}$ . Let  $M$  be the submonoid of  ${}^\omega A^\omega$  given by

$$M = \{ {}^\omega a (ab)^\omega \} \cup A^* \cup {}^\omega b A^* \cup A^* a^\omega \cup {}^\omega b A^* a^\omega.$$

Every element of  $M_{\text{inf}}$  precedes the unique stable element  $a^\omega$ . Every element of  $M_{-\text{inf}}$  precedes the unique stable element  ${}^\omega b$ .  $M$  satisfies the stability condition. By theorem 3.1,  $M$  has a minimal generator set which is  $A \cup \{a^\omega, {}^\omega b, {}^\omega a (ab)^\omega\}$ .

**DEFINITION 3.2:** Let  $M$  be a submonoid of  ${}^\omega A^\omega$ . Any increasing sequence  $u_1 < u_2 < \dots$  of elements of  $M_{\text{inf}}$  or  $M_{-\text{inf}}$  is called a chain. An infinite chain is called stationary if there exists  $n \geq 1$ , such that  $u_m = u_n$ , for all  $m \geq n$ . We say that  $M$  satisfies the stationary chain condition if every infinite chain of  $M_{\text{inf}}$  as well as  $M_{-\text{inf}}$  is stationary.

We note that stationary chain condition implies the stability condition but the converse is not true.

**DEFINITION 3.3:** A submonoid  $M$  of  ${}^\infty A^\infty$  is freeable if  $M^{-1}M \cap MM^{-1} \subseteq M$ .

The next theorem explains the existence of the minimal generator set for a freeable monoid  $M$ .

**THEOREM 3.2:** For any freeable submonoid  $M$ , the following conditions are equivalent.

- (i)  $M$  has a minimal generator set.
- (ii)  $M$  satisfies the stationary chain condition.
- (iii)  $M$  satisfies the stability condition.

Proof is similar to that of theorem 2.4 of Chapter II in [10] and is therefore omitted. The main difference is to consider infinite chains of elements of  $M_{-\text{inf}}$ .

**DEFINITION 3.4:** Let  $M$  be a submonoid of  ${}^\infty A^\infty$ . An element  $u$  of  $M_{\text{inf}}$  (resp.  $M_{-\text{inf}}$ ) is maximal if there is no element  $v$  of  $M_{\text{inf}}$  (resp.  $M_{-\text{inf}}$ ) such that  $u < v$ . The set of all maximal elements of  $M_{\text{inf}}$  (resp.  $M_{-\text{inf}}$ ) is denoted by  $\text{MAX}(M_{\text{inf}})$  [resp.  $\text{MAX}(M_{-\text{inf}})$ ]. It is evident that  $\text{MAX}(M_{\text{inf}}) \subseteq \text{STAB}(M_{\text{inf}})$  and  $\text{MAX}(M_{-\text{inf}}) \subseteq \text{STAB}(M_{-\text{inf}})$ . We say that  $M$  satisfies the maximality condition if every non maximal element of  $M_{\text{inf}}$  (resp.  $M_{-\text{inf}}$ ) precedes a maximal element of  $M_{\text{inf}}$  (resp.  $M_{-\text{inf}}$ ). Clearly, maximality condition implies stability condition but not the converse.

**DEFINITION 3.5:** Any subset  $X$  of  ${}^\infty A^\infty$  is called distinguished if  $X_{\text{inf}} \cap X_{\text{fin}}^+ X_{\text{inf}} = \Phi$ ,  $X_{-\text{inf}} \cap X_{-\text{inf}} X_{\text{fin}}^+ = \Phi$  and  $X_{\text{biinf}} \cap X_{-\text{inf}} X_{\text{fin}}^* X_{\text{inf}} = \Phi$ .

The following theorem gives the connection between maximality condition and the distinguished minimal generator set of a monoid  $M$ .

**THEOREM 3.3:** For any submonoid  $M$ , the following conditions are equivalent.

- (i)  $M$  has a distinguished minimal generator set which is

$$\begin{aligned} & \text{BASE}(M_{\text{fin}}) \cup \text{MAX}(M_{\text{inf}}) \cup \text{MAX}(M_{-\text{inf}}) \cup \text{UNFAC}(M_{\text{biinf}}) \\ &= (M - \varepsilon) - [(M_{\text{fin}} - \varepsilon)^2 \cup (M_{\text{fin}} - \varepsilon) M_{\text{inf}} \cup M_{-\text{inf}} (M_{\text{fin}} - \varepsilon) \\ & \qquad \qquad \qquad \cup M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}}] \end{aligned}$$

- (ii)  $M$  has a distinguished generator set
- (iii)  $M$  satisfies the maximality condition

*Proof:* It is clear that (i) implies (ii). We show that (ii) implies (iii). Let  $Y$  be a distinguished generator set of  $M$ . Since  $Y$  is a generator set, it is easy to see that every element of  $M_{\text{inf}} - Y_{\text{inf}}$  (resp.  $M_{-\text{inf}} - Y_{-\text{inf}}$ ) precedes an element of  $Y_{\text{inf}}$  (resp.  $Y_{-\text{inf}}$ ) and so it is enough to prove that

$$Y_{\text{inf}} \subseteq \text{MAX}(M_{\text{inf}}) [\text{resp. } Y_{-\text{inf}} \subseteq \text{MAX}(M_{-\text{inf}})].$$

We shall prove that  $Y_{\text{inf}} \subseteq \text{MAX}(M_{\text{inf}})$ . Suppose this is not true. Then, there exists  $y \in Y_{\text{inf}}$  which is not maximal. So, for some  $v \in M_{\text{inf}}$ , we have  $y < v$ . Let  $y = gv$  where  $g \in M_{\text{fin}} - \varepsilon$  and  $v = y_1 y_2 \dots y_n$  for some  $(y_1, y_2, \dots, y_n) \in Y^{(n)}$ ,  $n \geq 1$ . Since  $gy_1 y_2 \dots y_{n-1} \in Y_{\text{fin}}^+$ , we have  $y \in Y_{\text{inf}} \cap Y_{\text{fin}}^+ Y_{\text{inf}}$ . This is a contradiction since  $Y$  is distinguished. Hence (ii) implies (iii).

We now prove (iii)  $\Rightarrow$  (i). Let  $M$  satisfy the maximality condition. This means  $M$  satisfies the stability condition. By theorem 3.1,  $M$  has a minimal generator set  $X$ , namely,

$$X = \text{BASE}(M_{\text{fin}}) \cup \text{STAB}(M_{\text{inf}}) \cup \text{STAB}(M_{-\text{inf}}) \cup \text{UNFAC}(M_{\text{biinf}}).$$

Since a non maximal stable element cannot precede a maximal element,

$$\text{STAB}(M_{\text{inf}}) = \text{MAX}(M_{\text{inf}}) = M_{\text{inf}} - (M_{\text{fin}} - \varepsilon) M_{\text{inf}}$$

and

$$\text{STAB}(M_{-\text{inf}}) = \text{MAX}(M_{-\text{inf}}) = M_{-\text{inf}} - M_{-\text{inf}}(M_{\text{fin}} - \varepsilon).$$

Since

$$\text{UNFAC}(M_{\text{biinf}}) = M_{\text{biinf}} - (M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}})$$

and

$$\text{BASE}(M_{\text{fin}}) = (M_{\text{fin}} - \varepsilon) - (M_{\text{fin}} - \varepsilon)^2,$$

we have

$$X = (M - \varepsilon) - [(M_{\text{fin}} - \varepsilon)^2 \cup (M_{\text{fin}} - \varepsilon) M_{\text{inf}} \cup M_{-\text{inf}}(M_{\text{fin}} - \varepsilon) \cup M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}}].$$

Since  $X = X_{\text{fin}} \cup X_{\text{inf}} \cup X_{-\text{inf}} \cup X_{\text{biinf}}$ , let  $X_{\text{fin}} = \text{BASE}(M_{\text{fin}})$ ,  $X_{\text{inf}} = \text{MAX}(M_{\text{inf}})$ ,  $X_{-\text{inf}} = \text{MAX}(M_{-\text{inf}})$  and  $X_{\text{biinf}} = \text{UNFAC}(M_{\text{biinf}})$ . Thus  $X_{\text{inf}} \cap X_{\text{fin}}^+ X_{\text{inf}} = \Phi$ ,  $X_{-\text{inf}} \cap X_{-\text{inf}} X_{\text{fin}}^+ = \Phi$  and

$$X_{\text{biinf}} \cap X_{-\text{inf}} X_{\text{fin}}^* X_{\text{inf}} = \Phi.$$

Hence  $X$  is distinguished.

## SECTION 4

### SUBMONOID GENERATED BY CODES AND A THUE SYSTEM

In this section we introduce a bi-quasi free monoid whose underlying set is the set of all normal forms with respect to a specific Church-Rosser Thue system. We establish a characterisation of codes in terms of morphisms of monoids. We show the relation between bi-quasi free monoids, minimal generator sets and codes.

Let  $B$  be any finite alphabet. Let  $R$  be a binary relation on  $B^*$ . Elements of  $R$  are written as equations, i.e.,  $R = \{(u=v) \mid u, v \in B^*\}$ . Let  $T(B) = \langle B; R \rangle$ . We call  $T(B)$  as a Thue system associated with  $B$ . We say  $(u=v)$  is in  $T(B)$  iff  $(u=v)$  is in  $R$ .

Define the relation  $=_{T(B)}$  on elements of  $B^*$  as follows: For any  $(u=v)$  in  $T(B)$  and any  $x, y \in B^*$ , we write  $xuy =_{T(B)} xvy$ . The reflexive transitive closure of the symmetric relation  $=_{T(B)}$  is denoted as  $\equiv_{T(B)}$ . Clearly  $\equiv_{T(B)}$  is a congruence relation on  $B^*$ . If  $x \equiv_{T(B)} y$ , for any  $x, y \in B^*$ , we say that  $x$  is congruent to  $y$ . The congruence class of  $x$  is denoted by  $[x]$ .

If  $(u=v)$  is in  $T(B)$ , we write  $u \rightarrow_{T(B)} v$  if the length of  $u$  is greater than the length of  $v$ .  $\rightarrow_{T(B)}^*$  is the reflexive, transitive closure of the relation  $\rightarrow_{T(B)}$ .

When  $x \rightarrow_{T(B)}^* y$ , we say that  $x$  is an ancestor of  $y$  and  $y$  is a descendant of  $x$ .  $x$  is said to be irreducible if it has no descendant except itself. For any  $x, y \in B^*$ , if  $x \equiv_{T(B)} y$  and  $y$  is irreducible, then  $y$  is called a normal form of  $x$ .

$T(B)$  is Church-Rosser if for all  $x, y \in B^*$ , if  $x \equiv_{T(B)} y$ , then for some  $z \in B^*$ ,  $x \rightarrow_{T(B)}^* z$  and  $y \rightarrow_{T(B)}^* z$ . This means that every two congruent words have a common descendant. It is known that if  $T(B)$  is Church-Rosser, then every congruence class has a unique normal form [2]. We make use of this result in the following discussion.

We partition  $B$  into four mutually disjoint subsets  $B_1, B_2, \bar{B}_2, B_3$  and call  $B$  as a quadruple alphabet  $(B_1, B_2, \bar{B}_2, B_3)$ . With  $B$ , we associate a Thue system defined by  $T(B) = \langle B; R \rangle$  where

$$R = \{(bb' = b) \mid b \in B_2 \cup B_3, b' \in B\} \cup \{(bb' = b') \mid b \in B_1 \cup \bar{B}_2, b' \in \bar{B}_2 \cup B_3\}.$$

Now,  $\equiv_{T(B)}$  is a congruence relation on  $B^*$ . Consider the quotient monoid  $B^*/\equiv_{T(B)}$  and denote this by  $B^{[*]}$ . It is easy to see that  $T(B)$  is Church-Rosser. Hence every congruence class has a unique normal form. It is interesting to note that the set of all normal forms of elements of  $B^*$  is

$$B_1^* \cup B_1^* B_2 \cup \bar{B}_2 B_1^* \cup \bar{B}_2 B_1^* B_2 \cup B_3.$$

By a mild abuse of language, we write

$$B^{[*]} = B_1^* \cup B_1^* B_2 \cup \bar{B}_2 B_1^* \cup \bar{B}_2 B_1^* B_2 \cup B_3.$$

Define a product on  $B^{[*]}$  as follows: For  $x, y \in B^{[*]}$ ,

$$x \cdot y = \begin{cases} xy & \text{if } x \in B_1^*, \quad y \in B_1^* B_2 \cup B_1^* \\ & \text{or} \\ & x \in \bar{B}_2 B_1^*, \quad y \in B_1^* \\ x & \text{if } x \in B_1^* B_2 \cup B_3 \cup \bar{B}_2 B_1^* B_2 \\ y & \text{if } x \in B_1^* \cup \bar{B}_2 B_1^*, \\ & y \in \bar{B}_2 B_1^* \cup B_3 \cup \bar{B}_2 B_1^* B_2. \end{cases}$$

Clearly  $B^{[*]}$  is a monoid which we shall call as a bi-quasi free monoid generated by  $B$ .

LEMMA 4.1: *If  $\varphi: B^{[*]} \rightarrow {}^\infty A^\infty$  is an injective morphism and  $\varphi(B) = X$ , then  $\varphi(B_1) = X_{\text{fin}}$ ,  $\varphi(B_2) = X_{\text{inf}}$ ,  $\varphi(\bar{B}_2) = X_{-\text{inf}}$  and  $\varphi(B_3) = X_{\text{biinf}}$ .*

*Proof:* We first show that  $\varphi(B_1) \subseteq X_{\text{fin}}$ . Suppose it is not true. Then there exists  $b \in B_1$  such that

$$\varphi(b) \in X_{\text{inf}} \cup X_{-\text{inf}} \cup X_{\text{biinf}}.$$

If  $\varphi(b) \in X_{\text{inf}} \cup X_{\text{biinf}}$ , for  $b' \in B$ ,

$$\varphi(bb') = \varphi(b)\varphi(b') = \varphi(b).$$

Since  $\varphi$  is injective,  $bb' = b$  which is impossible. If  $\varphi(b) \in X_{-\text{inf}}$ , for  $b' \in \bar{B}_2 \cup B_3$ ,  $bb' = b'$ . So,  $\varphi(b)\varphi(b') = \varphi(b')$  which is impossible since  $\varphi(b) \neq \varepsilon$ . Hence  $\varphi(B_1) \subseteq X_{\text{fin}}$ .

To prove that  $\varphi(B_2) \subseteq X_{\text{inf}}$ , we suppose that it is not true. Then there exists  $b \in B_2$  such that  $\varphi(b) \in X_{\text{fin}} \cup X_{-\text{inf}} \cup X_{\text{biinf}}$ . For  $b' \in B$ ,  $bb' = b$ . So,  $\varphi(b)\varphi(b') = \varphi(b)$  and this is not possible since  $\varphi(b')$  need not be  $\varepsilon$ . Hence  $\varphi(B_2) \subseteq X_{\text{inf}}$ .

We now show that  $\varphi(\bar{B}_2) \subseteq X_{-\text{inf}}$ . If it were not so, there would exist  $b \in \bar{B}_2$  such that  $\varphi(b) \in X_{\text{fin}} \cup X_{\text{inf}} \cup X_{\text{biinf}}$ . Now, for  $b' \in \bar{B}_2 \cup B_3$ ,  $bb' = b'$  and so  $\varphi(b)\varphi(b') = \varphi(b')$ . This is not possible since  $\varphi(b) \neq \varepsilon$ .

Finally, in order to prove that  $\varphi(B_3) \subseteq X_{\text{biinf}}$ , assume that it is not true. Then there exists  $b \in B_3$  such that  $\varphi(b) \in X_{\text{fin}} \cup X_{\text{inf}} \cup X_{-\text{inf}}$ . For  $b' \in B$ ,  $bb' = b$ . Therefore  $\varphi(b)\varphi(b') = \varphi(b)$  which is not possible since  $\varphi(b') \neq \varepsilon$ .

Since  $\varphi(B) = X$ , we have,  $\varphi(B_1) = X_{\text{fin}}$ ,  $\varphi(B_2) = X_{\text{inf}}$ ,  $\varphi(\bar{B}_2) = X_{-\text{inf}}$  and  $\varphi(B_3) = X_{\text{biinf}}$ . This proves the lemma.

Given a quadruple alphabet  $B = (B_1, B_2, \bar{B}_2, B_3)$  we denote  $B^{(1)} = B$  and

$$B^{(n)} = \left\{ (b_1, b_2, \dots, b_n) / \begin{array}{l} b_1, b_2, \dots, b_{n-1} \in B_1, b_n \in B_1 \cup B_2 \\ \text{or } b_1 \in B_1 \cup \bar{B}_2, \quad b_2, b_3, \dots, b_n \in B_1 \\ \text{or } b_1 \in \bar{B}_2, b_n \in B_2, b_2, b_3, \dots, b_{n-1} \in B_1 \end{array} \right\}$$

LEMMA 4.2: (i) *If a subset  $X$  of  ${}^\infty A^\infty$  is a code, then every morphism  $\varphi: B^{[*1]} \rightarrow {}^\infty A^\infty$  which induces a bijection from  $B$  onto  $X$  with  $\varphi(B_1) \subseteq X_{\text{fin}}$ ,  $\varphi(B_2) \subseteq X_{\text{inf}}$  and  $\varphi(\bar{B}_2) \subseteq X_{-\text{inf}}$  is injective.*

(ii) *If  $\varphi: B^{[*1]} \rightarrow {}^\infty A^\infty$  is an injective morphism, then  $X = \varphi(B)$  is a code.*

Proof is on lines close to that of lemma 1.3 of Chapter III in [10] and is therefore omitted.

We now give a necessary and sufficient condition for a subset of  ${}^\infty A^\infty$  to be a code.

THEOREM 4.1: *A subset  $X$  of  ${}^\infty A^\infty$  is a code iff there exists a bi-quasi free monoid  $B^{[*1]}$  and an injective morphism  $\varphi: B^{[*1]} \rightarrow {}^\infty A^\infty$  such that  $\varphi(B) = X$ .*

Proof: Let  $X$  be a code. Let  $B = (B_1, B_2, \bar{B}_2, B_3)$  be a quadruple alphabet chosen so that  $B_1, B_2, \bar{B}_2$  and  $B_3$  are in one to one correspondence with  $X_{\text{fin}}, X_{\text{inf}}, X_{-\text{inf}}$  and  $X_{\text{biinf}}$  respectively. This correspondence shows the existence of an isomorphism

$$\begin{aligned} \varphi: B^{[*1]} &\rightarrow X^* && \text{with } \varphi(B_1) = X_{\text{fin}}, \\ & && \varphi(B_2) = X_{\text{inf}}, \\ \varphi(\bar{B}_2) &= X_{-\text{inf}} && \text{and } \varphi(B_3) = X_{\text{biinf}}. \end{aligned}$$

By lemma 4.2, the theorem holds.

DEFINITION 4.1: A submonoid  $M$  of  ${}^\infty A^\infty$  is said to be bi-quasi free if it is isomorphic to a bi-quasi free monoid  $B^{[*1]}$ .

The following theorem exhibits that the class of submonoids generated by codes coincides with the class of bi-quasi free submonoids.

**THEOREM 4.2:** (i) *Every bi-quasi free submonoid  $M$  has a minimal generator set  $X$  which is a code.*

(ii) *If  $X$  is a code, then  $X^*$  is a bi-quasi free submonoid having  $X$  as its minimal generator set.*

**Proof:** (i) Suppose  $M$  is a bi-quasi free submonoid. Then there is an isomorphism  $\varphi: B^{[*]} \rightarrow M$  from a bi-quasi free monoid onto  $M$ . By theorem 4.1,  $X = \varphi(B)$  is a code. By lemma 4.2,  $\varphi(B_1) = X_{\text{fin}}$ ,  $\varphi(B_2) = X_{\text{inf}}$ ,  $\varphi(\bar{B}_2) = X_{-\text{inf}}$  and  $\varphi(B_3) = X_{\text{biinf}}$ . We have

$$\begin{aligned} M = \varphi(B^{[*]}) &= \varphi(B_1^* \cup B_1^* B_2 \cup \bar{B}_2 B_1^* \cup \bar{B}_2 B_1^* B_2 \cup B_3) \\ &= [\varphi(B_1)]^* \cup [\varphi(B_1)]^* \varphi(B_2) \cup \varphi(\bar{B}_2) [\varphi(B_1)]^* \\ &\quad \cup \varphi(\bar{B}_2) [\varphi(B_1)]^* \varphi(B_2) \cup \varphi(B_3). \\ &= X_{\text{fin}}^* \cup X_{\text{fin}}^* X_{\text{inf}} \cup X_{-\text{inf}} X_{\text{fin}}^* \cup X_{-\text{inf}} X_{\text{fin}}^* X_{\text{inf}} \cup X_{\text{biinf}} = X^*. \end{aligned}$$

Hence  $X$  generates  $M$ . To prove the minimality of  $X$ , let  $Y$  be any generator set of  $M$  and  $x \in X$ . Then  $x = y_1 y_2 \dots y_n$  for some  $(y_1, y_2, \dots, y_n) \in Y^{(n)}$ ,  $n \geq 0$ . Since  $x \neq \varepsilon$ ,  $n \geq 1$ . Since  $X$  is a code,  $n = 1$  and so  $x = y_1$ . Hence  $X \subseteq Y$ . Thus  $X$  is minimal.

(ii) Suppose  $X$  is a code. By theorem 4.1, there exists a bi-quasi free monoid  $B^{[*]}$  and an injective morphism  $\varphi: B^{[*]} \rightarrow {}^\infty A^\infty$  such that  $\varphi(B) = X$ . Now  $\varphi$  is indeed an isomorphism from  $B^{[*]}$  onto  $\varphi(B^{[*]}) = X^*$ . Thus  $X^*$  is a bi-quasi free submonoid. By the similar argument as in (i),  $X$  is a minimal generator set of  $X^*$ .

**SECTION 5**

**A COMBINATORIAL CHARACTERIZATION OF BI-QUASI FREE SUBMONOIDS**

**LEMMA 5.1:** *Every bi-quasi free submonoid is freeable.*

**Proof:** Let  $M$  be a bi-quasi free submonoid with the minimal generator set  $X$ . By theorem 4.2,  $X$  is a code. Let  $\alpha \in M^{-1} M \cap M M^{-1}$ . Since  $\alpha \in M^{-1} M$ , there exists  $\beta \in M$  such that  $\beta \alpha \in M$ ,  $(\beta \in M_{\text{inf}} \cup M_{\text{biinf}} \Rightarrow \alpha = \varepsilon)$  and  $(\beta \in {}^\infty M \text{ and } \alpha \in A^{-N} \cup A^Z \Rightarrow \beta = \varepsilon)$ . Since  $\alpha \in M M^{-1}$ , there exists  $\mathcal{V} \in M$  such that  $\alpha \mathcal{V} \in M$ ,

$$(\alpha \in {}^\infty A \text{ and } \mathcal{V} \in M_{\swarrow \text{inf}} \cup M_{\text{biinf}} \Rightarrow \alpha = \varepsilon)$$

and

$$(\alpha \in A^N \cup A^Z \Rightarrow \mathcal{V} = \varepsilon).$$

Let

$$\begin{aligned} \beta &= x_1 x_2 \dots x_k && \text{with } (x_1, x_2, \dots, x_k) \in X^{(k)}; \\ \alpha \mathcal{V} &= x_{k+1} \dots x_n && \text{with } (x_{k+1}, x_{k+2}, \dots, x_n) \in X^{(n-k)}; \\ \beta \alpha &= x'_1 x'_2 \dots x'_l && \text{with } (x'_1, x'_2, \dots, x'_l) \in X^{(l)}; \\ \mathcal{V} &= x'_{l+1} x'_{l+2} \dots x'_m && \text{with } (x'_{l+1}, \dots, x'_m) \in X^{(m-l)}. \end{aligned}$$

If  $\beta \in M_{\text{inf}} \cup M_{\text{biinf}}$ , then  $\alpha = \varepsilon \in M$ . If  $\beta \in {}^\infty M$  and  $\alpha \in A^{-N} \cup A^Z$ , then  $\beta = \varepsilon$ . Therefore  $\beta \alpha \in M$  implies  $\alpha \in M$ . If  $\alpha \in A^N$ , then we have  $\mathcal{V} = \varepsilon$  and so  $\alpha \mathcal{V} \in M$  implies  $\alpha \in M$ . When  $\alpha \in {}^\infty A$  and  $\mathcal{V} \in M_{-\text{inf}} \cup M_{\text{biinf}}$ , then  $\alpha = \varepsilon \in M$ . We have to consider the only case when  $\beta \in {}^\infty M$ ,  $\alpha \in A^*$  and  $\mathcal{V} \in M^\infty$ . Since  $\beta(\alpha \mathcal{V}) = (\beta \alpha) \mathcal{V}$ , we get

$$x_1 x_2 \dots x_k x_{k+1} \dots x_n = x'_1 x'_2 \dots x'_l x'_{l+1} \dots x'_m.$$

Since  $X$  is a code,  $n = m$  and  $x_i = x'_i$ ,  $i = 1, 2, \dots, n$ . Since  $\beta$  is a left factor of  $\beta \alpha$ , we have  $l \geq k$ . Therefore

$$\beta \alpha = x'_1 x'_2 \dots x'_l = x_1 x_2 \dots x_k x_{k+1} \dots x_l = \beta x_{k+1} \dots x_l.$$

This implies  $\alpha = x_{k+1} \dots x_l \in M$ . Thus  $M^{-1}M \cap MM^{-1} \subseteq M$  and so  $M$  is freeable.

**DEFINITION 5.1:** We say that a submonoid  $M$  satisfies finite chain condition if all the chains in  $M_{\text{inf}}$  and  $M_{-\text{inf}}$  are finite.

The finite chain condition implies the maximality condition.

**LEMMA 5.2:** *Every bi-quasi free submonoid satisfies the finite chain condition.*

*Proof* is similar to that of proposition 3.3 of Chapter III in [10] and is omitted. The difference is to consider infinite chains in  $M_{-\text{inf}}$ .

**THEOREM 5.1:** *For any submonoid  $M$ , the following conditions are equivalent.*

- (i)  $M$  is bi-quasi free *i. e.*, generated by a code.
- (ii)  $M$  is freeable and satisfies the finite chain condition.
- (iii)  $M$  is freeable and satisfies the maximality condition
- (iv)  $M$  is freeable and has a distinguished (minimal) generator set.

*Proof:* It is clear that (iii)  $\Leftrightarrow$  (iv) by theorem 3.3. (i)  $\Rightarrow$  (ii) is by lemmas 5.1 and 5.2. (ii)  $\Rightarrow$  (iii) is evident. We have to show that (iii)  $\Rightarrow$  (i).

Suppose  $M$  is freeable and satisfies the maximality condition. By theorem 3.3,  $M$  has a distinguished minimal generator set  $X$  which is  $\text{BASE}(M_{\text{fin}}) \cup \text{MAX}(M_{\text{inf}}) \cup \text{MAX}(M_{-\text{inf}}) \cup \text{UNFAC}(M_{\text{biinf}})$ . By theorem 4.2, it is enough if we prove that  $X$  is a code. Suppose  $X$  is not a code. Then there exists a word  $\alpha$  such that it has two different factorizations on elements of  $X$ . *i. e.*,

$$\alpha = x_1 x_2 \dots x_n = x'_1 x'_2 \dots x'_m$$

where  $n, m \geq 1, (x_1, x_2, \dots, x_n) \in X^{(n)}$  and  $(x'_1, x'_2, \dots, x'_m) \in X^{(m)}$ . Clearly either  $n$ , or,  $m$  should be greater than 1. Let  $m > 1$ .

Case (a): Suppose  $\alpha \in A^*$ . We may assume that  $x_1 \neq x'_1$ . Let  $|x_1| > |x'_1|$ . Then there exists a word  $f \neq \varepsilon$  such that  $x_1 = x'_1 f$  and  $f x_2 x_3 \dots x_n = x'_2 x'_3 \dots x'_m$ . From the freeability of  $M$ , it follows that  $f \in M_{\text{fin}} - \varepsilon$ . This contradicts the hypothesis that  $x_1 \in \text{BASE}(M_{\text{fin}})$ .

Case (b): Suppose  $\alpha \in A^N$ . Then  $x_n, x'_m \in \text{MAX}(M_{\text{inf}})$ . If  $n = 1$ , then  $x_n = x'_1 x'_2 \dots x'_m$  and so  $x_n < x'_m$  which is a contradiction to the maximality of  $x_n$ . Suppose  $n \geq 2$ . If

$$|x_1 x_2 \dots x_{n-1}| = |x'_1 x'_2 \dots x'_{m-1}|,$$

then

$$x_1 x_2 \dots x_{n-1} = x'_1 x'_2 \dots x'_{m-1} \in A^*.$$

As in case (a), we get a contradiction. If not, we assume that  $|x_1 x_2 \dots x_{n-1}| > |x'_1 x'_2 \dots x'_{m-1}|$ . This implies that there exists  $f \neq \varepsilon$  with

$$x_1 x_2 \dots x_{n-1} = x'_1 x'_2 \dots x'_{m-1} f \quad \text{and} \quad f x_n = x'_m.$$

Again, by freeability of  $M$ ,  $f \in M_{\text{fin}} - \varepsilon$  and so we have  $x'_m < x_n$  which contradicts the maximality of  $x'_m$ .

Case (c): Suppose  $\alpha \in A^{-N}$ . Then  $x_1, x'_1 \in \text{MAX}(M_{-\text{inf}})$ . We can discuss as in case (b) and obtain a contradiction.

Case (d): Suppose  $\alpha \in A^Z$ . If  $n = 1$ , then  $x_1 = x'_1 x'_2 \dots x'_m$  which is a contradiction since  $X$  is a distinguished minimal generator set. Suppose  $n \geq 2$ . Then  $x_1, x'_1 \in \text{MAX}(M_{-\text{inf}})$ . There are two possibilities.

(i) If  $x_1 \neq x'_1$ , we assume  $x_1 = x'_1 f$  and so we have

$$f x_2 x_3 \dots x_n = x'_2 x'_3 \dots x'_m, m \geq 2.$$

This implies  $f \in M_{\text{fin}} - \varepsilon$  since  $M$  is freeable. Hence  $x_1 < x'_1$  which contradicts the maximality of  $x_1$ .

(ii) Suppose  $x_1 = x'_1$ . We have either

$$x_2 x_3 \dots x_n = x'_2 x'_3 \dots x'_m$$

or

$$x_2 x_3 \dots x_n \neq x'_2 x'_3 \dots x'_m.$$

If  $x_2 x_3 \dots x_n = x'_2 x'_3 \dots x'_m$ , then we assume  $x_2 \neq x'_2$  and proceed as in case (b) and get a contradiction. If  $x_2 x_3 \dots x_n \neq x'_2 x'_3 \dots x'_m$  since  $\alpha$  has two factorizations, we have either  $x_1 = x'_1 f$  and

$$f x_2 x_3 \dots x_n = x'_2 x'_3 \dots x'_m \quad \text{or} \quad x'_1 = x_1 f$$

and

$$x_2 x_3 \dots x_n = f x'_2 x'_3 \dots x'_m.$$

Since the two cases are similar, it is enough to consider any one of the possibilities, say  $x_1 = x'_1 f$  and  $f x_2 x_3 \dots x_n = x'_2 x'_3 \dots x'_m$ . Clearly  $f \in M_{\text{fin}} - \varepsilon$  as  $M$  is freeable and hence  $x_1 < x'_1$  which contradicts the maximality of  $x_1$ .

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