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INEVITABILITY IN DIAMOND PROCESSES (*)

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Abstract. – *This paper deals with the concurrent systems viewed as partially ordered sets. A set of system states is called inevitable if each execution of the system sooner of later meets this set. Single executions of concurrent system are represented by maximal directed subsets of this system; they are called processes. A distinguished class of processes, called diamond processes, is defined and investigated. Inevitable subsets of diamond processes are characterized.*

Résumé. – *Dans cet article, on considère les systèmes concurrents comme des ensembles partiellement ordonnés. Un ensemble d'événements est inévitable si chaque exécution du système rencontre au moins une fois cet ensemble. Une exécution séquentielle d'un système concurrent est représentée par un sous ensemble dirigé maximal de ce système, ces sous ensembles sont appelés processus. Une classe particulière de processus, les processus losanges est définie et étudiée et les ensembles inévitables de processus losanges sont caractérisés.*

INTRODUCTION

The behaviour of a concurrent system is represented here by the set of its states. We assume that states include their past histories. A state s' precedes a state s'' if the past of s' is an initial part of the past of s''. Two states are incomparable if neither of them precedes the second. This approach leads to modelling behaviours of concurrent systems as partially ordered sets of their states. If it is so, maximal directed subsets of these posets represent single executions of concurrent systems. They are called concurrent processes. A set of system states is inevitable if each execution of the system sooner or later meets this set. Maximal (with respect to the set inclusion) sequences of process states, called lines, are used for reasoning about concurrent processes.

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A process is called observable if there is a line cofinal with it, and unobservable otherwise; a system is observable if all its processes are observable. A particular property of processes, called diamond property, is defined in this paper. It seems to the authors that the set of states (in the above informally described sense) of any "real" concurrent process has the diamond property. Inevitable subsets of diamond processes (observable and unobservable) are investigated in this paper.

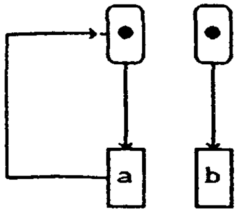
A lot of informations about posets and their properties from the viewpoint of concurrency is contained in [1]; however, the present approach is different. The notion of inevitability was formalized in [2] and broadly discussed in [3] for observable concurrent systems. The generalization of the inevitability for arbitrary concurrent systems was presented in [4]. This paper is based on [5], where the diamond property was introduced and studied.

1. CONCURRENT SYSTEMS AND PROCESSES

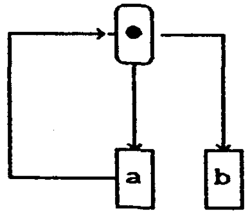
A pair $(S, \leqq)$, where S is a set and $\leqq \subseteq S \times S$ is an ordering (*i.e.* a reflexive, transitive and asymmetric relation) of S, is called a *partially ordered set* (or *poset*, for short). Posets in this paper are considered as mathematical models of behaviours of concurrent systems. For this reason they are called *concurrent systems* (or simply *systems*) and their members are called *states*. The ordering relation represents the time successiveness of states in possible executions of the system. A poset $(S, \leqq)$ is *directed* if for every $s, t \in S$ there is $u \in S$ such that $s \leqq u$ and $t \leqq u$. A poset $(S', \leqq')$ is a *subposet* of $(S, \leqq)$ if $S' \subseteq S$ and $\leqq' = \leqq \cap S' \times S'$. Any totally ordered subposet of $(S, \leqq)$ is called a *chain* in $(S, \leqq)$. A chain in $(S, \leqq)$ is called a *line* in $(S, \leqq)$ if it is a maximal (with respect to set inclusion) chain in $(S, \leqq)$. Any maximal (with respect to set inclusion) directed subposet of a system $(S, \leqq)$ is called a *process* in $(S, \leqq)$.

Processes are mathematical models of single executions of concurrent systems. A system may contain many processes. Consider the following Petri nets (*see* [6] for basic notions).

N1 :



N2:



Only one full execution of $N1$ is possible; namely, infinite number successive occurrences of a and a single occurrence of b, concurrently with all occurrences of a. This execution is expressed by the following diagram:

$S1$:

$$\begin{array}{ccccc} 1 & \xrightarrow{a} & 3 & \xrightarrow{a} & \dots \\ \downarrow b & & \downarrow b & & \\ 2 & \xrightarrow[a]{} & 4 & \xrightarrow[a]{} & \dots \end{array}$$

The only process (maximal directed subposet) in $S1$ is $S1$ itself.

On the contrary, $N2$ admits infinite number of executions. Namely, n successive occurrences of a followed by a single occurrence of b are a single execution in $N2$ for $n=0$, 1, 2, ...; additionally, infinite number of successive occurrences of a is a single execution, too. All these executions are expressed by the following diagram:

$S2$:

$$\begin{array}{ccccc} 1 & \xrightarrow{a} & 3 & \xrightarrow{a} & \dots \\ \downarrow b & & \downarrow b & & \\ 2 & & 4 & & \dots \end{array}$$

Processes (maximal directed subposets) in $S2$ are sets of the form

$$\{2n-1, 2m \mid 1 \leqq n \leqq m\} \qquad \text{for} \quad m=1, 2, 3, \dots\}.$$

and also the set $\{2n-1 \mid n=1, 2, 3, \dots\}$.

Let $(P, \leqq)$ be a process. The operations $^{\uparrow}$ and $_{\downarrow}$ are defined for subsets of P as follows:

$$X^{\uparrow} = \{z \in P \mid (\exists x \in X)\, x \leqq z\}, \qquad X_{\downarrow} = \{z \in P \mid (\exists x \in X)\, z \leqq x\}.$$

We say that X *dominates* Y (or that Y is *dominated* by X) if $Y \subseteq X_{\downarrow}$.

We say that X *intersects* Y if $X \cap Y \neq \emptyset$.

Fact 1.1: Any chain in $(P, \leqq)$ is included in some line in $(P, \leqq)$.

Proof: From the Kuratowski-Zorn Lemma. ■

2. OBSERVATIONS AND INEVITABILITY

DEFINITION 2.1 [2]: Let $(P, \leqq)$ be a process. A line L in $(P, \leqq)$ is an *observation* of $(P, \leqq)$ if L dominates P. A process is *observable* if there exists an observation of it, and *unobservable* otherwise.

The following facts hold:

Fact 2.2 [2]: Any countable process is observable.

Fact 2.3: If a process $(P, \leqq)$ is observable, then for each $p \in P$ there is an observation of $(P, \leqq)$ containing p.

Proof: Let L be an observation of $(P, \leqq)$ and $p \in P$. There is $x \in L$ such that $p \leqq x$. The chain $(L \cap \{x\}^{\uparrow}) \cup \{p\}$ is included in some line L' (Fact 1.1). Since L is an observation, L' is an observation, too. ■

It would be very convenient, if every process were observable. Unfortunately, unobservable processes exist, which is demonstrated by the following example:

Example 2.4 [2]: Let R_f be the set of all finite subsets of an uncountable set. The process $(R_f, \subseteq)$ is unobservable, since $L_{\downarrow}$ is countable for any line L in $(R_f, \subseteq)$, whereas R_f is uncountable.

As it was mentioned in the Introduction, a subset of system states will be called inevitable iff each execution of the system sooner or later meets this set. Since executions are modelled by processes, we have the following definition:

DEFINITION 2.5: Let $(S, \leqq)$ be a concurrent system and let $X \subseteq S$. X is *inevitable* in $(S, \leqq)$ iff for any process $(P, \leqq_P)$ in $(S, \leqq)$ the set $X \cap P$ is inevitable in $(P, \leqq_P)$.

The above definition is not complete; the inevitability in a process remains to be defined. So now we want to express formally the meaning of the sentence "a process sooner or later meets a set". The following definition is given in [2, 3] with comprehensive motivation:

DEFINITION 2.6: Let $(P, \leqq)$ be an observable process and $X \subseteq P$. X is *inevitable* in $(P, \leqq)$ iff any observation of $(P, \leqq)$ intersects X.

Let $(P, \leqq)$ be a process and $X \subseteq P$. Let us denote by $C1$-$C3$ the following conditions:

$C1(P, X)$: Any line in $(P, \leqq)$ dominating P intersects X.

$C2(P, X)$: Any line in $(P, \leqq)$ undominated by X intersects X.

$C3(P, X)$: Any line in $(P, \leqq)$ dominating X intersects X.

Note that $C1$ is exactly the condition defining inevitability in observable processes (Definition 2.6).

The following facts hold:

Fact 2.7: Let $(P, \leqq)$ be an observable process and $X \subseteq P$.

If $C1(P, X)$ then $C2(P, X)$.

Proof: Let L be a line undominated by X. Let $y \in L$ and $y \notin X_{\downarrow}$. It follows from the Fact 2.3 that the chain $L \cap \{y\}_{\downarrow}$ is included in some observation L' of $(P, \leqq)$. We have $L' \cap X \neq \emptyset$, since $C1(P, X)$. But $(L' \cap \{y\}^{\uparrow}) \cap X = \emptyset$, thus $(L' \cap \{y\}_{\downarrow}) \cap X \neq \emptyset$. Since $L' \cap \{y\}_{\downarrow} \subseteq L$, we have $L \cap X \neq \emptyset$.

Fact 2.8: Let $(P, \leqq)$ be an arbitrary process and $X \subseteq P$.

If $C3(P, X)$ then $C1(P, X)$.

Proof: Obvious. ■

The condition $C1$, defining inevitability in observable processes, cannot define inevitability in unobservable processes. The intuitive requirements would not be satisfied; namely, every subset would be then inevitable. Moreover, the Facts 2.7 and 2.8 cannot be converted (*see* [4] for counterexamples), thus also $C2$ and $C3$ cannot define inevitability for arbitrary processes. However, the following definition seems to be proper:

DEFINITION 2.9: Let $(P, \leqq)$ be an unobservable process and $X \subseteq P$.

$$X \text{ is } \textit{inevitable} \text{ in } (P, \leqq) \text{ iff } C2(P, X).$$

Let $(P, \leqq)$ be an arbitrary process and $X \subseteq P$. Let $I(P, X)$ denote the condition "X is inevitable in (P, X)". The definitions of inevitability in observable $(I(P, X) \equiv C1(P, X)$, def. 2.6) and unobservable $(I(P, X) \equiv C2(P, X)$, def. 2.9) processes are different. However, the following general characterization of inevitability makes the above definitions justified:

PROPOSITION 2.10: *Let* $(P, \leqq)$ *be an arbitrary process and let* $X \subseteq P$.

$$I(P, X) \quad \textit{if and only if} \quad C1(P, X) \,\&\, C2(P, X).$$

Proof: 1° $(P, \leqq)$ is observable; $I(P, X)=C1(P, X)$.
"if": obvious: "only if": by Fact 2.7.

2° $(P, \leqq)$ is unobservable; $I(P, X)=C2(P, X)$.
"If": obvious; "only if": because then $C1(P, X)\equiv$TRUE. ■

Note that the conditions $C2$ and $C3$ essentially differ from the condition $C1$. Namely, $C2$ and $C3$ are local with respect to X (they say on a behaviour of lines with respect to X only), whereas $C1$ is not local (it says on a behaviour of lines with respect to X and P). It will be proved in section 5 that inevitability is locally characterizable in the class of diamond processes, defined in section 4.

Now we give a small illustration of introduced notions. Let us look on the systems $S1$ and $S2$ of Petri nets $N1$ and $N2$ in the first section. The set $X=\{2,4,6, \ldots\}$ is inevitable in $S1$ but not in $S2$, since $\{1,3,5, \ldots\}$ is a process in $S2$. This example illustrates difference between concurrency $(N1/S1)$ and indeterminism $(N2/S2)$.

3. SYNCHRONIZED PROCESSES

DEFINITION 3.1: A process $(P, \leqq)$ is *synchronized* iff any line in $(P, \leqq)$ is an observation of P.

It directly follows from the above definition that synchronized processes are observable. Let $(P, \leqq)$ be a synchronized process and let $X \subseteq P$. Let $C0$ denote the following condition:

$$C0(P, X)\text{: Any line in } (P, \leqq) \text{ intersects } X.$$

Let $C1$ and $C3$ be the conditions defined in section 2. The following obvious fact characterizes inevitability in the class of synchronized processes:

Fact 3.2: $C0(P, X)$ iff $C1(P, X)$ iff $C3(P, X)$.

A process $(P, \leqq)$ is said to be *bounded* if there is $z\in P$, such that $x\leqq z$ for any $x\in P$, and *unbounded* if such z does not exist.

Fact 3.3: Any bounded process is synchronized.

Proof: If $(P, \leqq)$ is bounded by z then any line in $(P, \leqq)$ contains z.

4. DIAMOND PROCESSES

Let $(P, \leqq)$ be a process. Let $\langle$ denote $\leqq\setminus\mathrm{id}$. The relation $\rightarrow = \langle\setminus\langle^2$ is called the *succession* relation. Let $X \subseteq P$; the set $X^{\bullet} = \{y \in P \mid (\exists x \in X)\, x \rightarrow y\}$ is the set of *successors* of X. We denote by $\overset{*}{\rightarrow}$ the reflexive and transitive closure of $\rightarrow$ and by $\overset{n}{\rightarrow}$ the n-th power of $\rightarrow$. A process $(P, \leqq)$ is *discrete* iff $\leqq = \overset{*}{\rightarrow}$. We write $(P, \rightarrow)$ rather than $(P, \leqq)$ if $(P, \leqq)$ is discrete. A discrete process $(P, \rightarrow)$ has the *diamond property* iff $(\forall x, y, z \in P)$ if $x \rightarrow y$ and $x \rightarrow z$ with $y \neq z$ then $(\exists t \in P)$ such that $y \rightarrow t$ and $z \rightarrow t$.

The diamond property is very natural in discrete concurrent systems. It seems to the authors that all "real" discrete concurrent processes have the diamond property. We prove this opinion: Let $(P, \rightarrow)$ be a discrete process. The situation $x \rightarrow y$ and $x \rightarrow z$ with $y \neq z$ represents the fact that the states y and z directly follow the state x in the process $(P, \rightarrow)$. If it is so, then y and z are both concurrently enabled in the state x (if a conflict between y and z arises in the state x, then y and z are in different processes). If it is so, the state t reachable from x directly after concurrent execution of y and z should be in this process.

The part "with $y \neq z$" in the definition of diamond property is needed only for bounded processes. Inevitability in bounded processes is characterized in section 3 (Facts 3.3 and 3.2). Now we are interested in characterization of inevitability in unbounded processes. It will be convenient for this purpose to define diamond processes in the following way:

DEFINITION 4.1: A discrete process $(P, \rightarrow)$ is said to be a *diamond* process iff $(\forall x, y, z \in P)$ if $x \rightarrow y$ and $x \rightarrow z$ then $(\exists t \in P)$ such that $y \rightarrow t$ and $z \rightarrow t$.

Note that diamond processes (except singletons) are unbounded. Thanks to this, the following very useful lemma holds:

LEMMA 4.2: *If a discrete process $(P, \rightarrow)$ is diamond then $(\forall x, y, z \in P)$ $(\forall n, m \in \mathbb{N})$ if $x \overset{n}{\rightarrow} y$ and $x \overset{m}{\rightarrow} z$ then $(\exists t \in P)$ such that $y \overset{m}{\rightarrow} t$ and $z \overset{n}{\rightarrow} t$.*

Proof: We prove the lemma by induction on (n, m). Definition 4.1 gives the base $(n, m) = (1, 1)$. Suppose that the lemma holds for all $(k, l) \leqq (n, m)$ (*i.e.*, $k \leqq n$ and $l \leqq m$). Let $x \overset{n+1}{\longrightarrow} y$ and $x \overset{m}{\rightarrow} z$. There is $y' \in P$ such that $x \overset{n}{\rightarrow} y' \rightarrow y$. By the inductive hypothesis there is $t' \in P$ such that $y \overset{m}{\rightarrow} t'$ and

$z \xrightarrow{n} t'$. Now we conclude (again by the inductive hypothesis) that there is $t \in P$ such that $y \xrightarrow{m} t$ and $t' \to t$. Since $z \xrightarrow{n} t'$ and $t' \to t$, we have $z \xrightarrow{n+1} t$. ■

LEMMA 4.3: *Let $(P, \to)$ be a diamond process and $X \subseteq P$.*
If X dominates $X^{\bullet}$ then X dominates $X^{\uparrow}$ (*i.e.*, $X^{\bullet} \subseteq X_{\downarrow} \Rightarrow X^{\uparrow} \subseteq X_{\downarrow}$).

Proof: Let $y \in X^{\uparrow}$; thus $x \xrightarrow{n} y$ for some $x \in X$, $n \in \mathbb{N}$. We prove the lemma by induction on n. If $n=0$ then $y=x \in X \subseteq X_{\downarrow}$. If $n=1$ then $y \in X^{\bullet} \subseteq X_{\downarrow}$. Suppose that if $x \xrightarrow{n-1} z$ then $z \in X_{\downarrow}$. Since $x \xrightarrow{n} y$ there is $z \in X^{\uparrow}$ such that $x \xrightarrow{n-1} z \to y$. By inductive hypothesis $z \in X_{\downarrow}$, thus $z \xrightarrow{*} v$ for some $v \in X$. By Lemma 4.2 there is $w \in P$ such that $y \xrightarrow{*} w$ and $v \to w$. Thus $y \xrightarrow{*} w \in X^{\bullet} \subseteq X_{\downarrow}$, so $y \in X_{\downarrow}$. ■

THEOREM 4.4: *Let $(P, \to)$ be a diamond process and $\varnothing \neq X \subseteq P$.*
X dominates P iff X dominates $X^{\bullet}$ (*i.e.*, $P = X_{\downarrow}$ iff $X^{\bullet} \subseteq X_{\downarrow}$).

Proof: "Only if": obvious; "if": Let $p \in P$ and $x \in X$. Since P is a process there exists $q \in P$ such that $p \xrightarrow{*} q$ and $x \xrightarrow{*} q$. By Lemma 4.3 we have $p \xrightarrow{*} q \in X_{\downarrow}$, so $p \in X_{\downarrow}$. ■

The above theorem says that in diamond processes the condition "X dominates P" can be expressed locally as "X dominates $X^{\bullet}$". It enables us to give a local characterization of inevitability in diamond processes. This will be done in the next section.

5. INEVITABILITY IN DIAMOND PROCESSES

Now we can characterize inevitable subsets in observable diamond processes:

THEOREM 5.1: *Let $(P, \leqq)$ be an observable diamond process and $X \subseteq P$. X is inevitable in $(P, \leqq)$ if and only if $C3(P, X)$.*

Proof: "if": obvious, since any observation dominates X. "only if": Let $L \subseteq P$ be a line dominating X. If L is an observation then L intersects X, since X is inevitable. Let L be a line but not an observation. By Theorem 4.4 there is $y \in L^{\bullet}$ such that $y \notin L_{\downarrow}$. It follows from Fact 2.3 that the chain

$\{u \in L \mid u<y\} \cup \{y\}$ is included in some observation L'. Since X is inevitable, L' intersects X. Let $x \in L' \cap X$; if $y \leqq x$ and $x \in X$ then $y \leqq x \in L_{\downarrow}$, thus $y \in L_{\downarrow}$. It is impossible, thus $x<y$. But $\{x \in L' \mid x<y\} \subseteq L$, hence L intersects X. ■

The following theorem holds, thanks to Theorem 4.4:

THEOREM 5.2: *Let $(P, \leqq)$ be an unobservable diamond process and $X \subseteq P$. If X is inevitable in $(P, \leqq)$ then $C3(P, X)$.*

Proof: Let L be a line dominating X. Since L does not dominate P ($(P, \leqq)$ is unobservable), X does not dominate P. By Theorem 4.4 there are $x \in X$ and $y \in X^{\bullet}$ such that $x \rightarrow y$ and $y \notin X_{\downarrow}$. Since L dominates X, there is $z \in L$ such that $x \overset{*}{\rightarrow} z$. By Lemma 4.2 there is $w \in L^{\bullet}$ such that $y \overset{*}{\rightarrow} w$ and $z \rightarrow w$. By Fact 1.1 the chain $\{u \in L \mid u<w\} \cup \{w\}$ is included in some line L'. Since L' is undominated by X ($w \in L'$ and $w \notin X_{\downarrow}$ (since $y \overset{*}{\rightarrow} w$ and $y \notin X_{\downarrow}$)), L' intersects X. Let $v \in L' \cap X$; obviously, $v<w$ (since $w \notin X_{\downarrow}$). But $\{v \in L' \mid v<w\} \subseteq L$, hence L intersects X. ■

Now we are ready to formulate the general characterization of inevitability in the class of diamond processes:

PROPOSITION 5.3: *Let $(P, \leqq)$ be a diamond process and let $X \subseteq P$. $I(P, X)$ if and only if $C2(P, X)$ & $C3(P, X)$.*

Proof: 1° $(P, \leqq)$ is observable; $I(P, X) = C1(P, X)$.

"if": by Fact 2.8; "only if": by Fact 2.7 and Th. 5.1.

2° $(P, \leqq)$ is unobservable; $I(P, X) = C2(P, X)$.

"if": obvious; "only if": by Theorem 5.2. ■

Let us compare two characterizations of inevitability: in the class of all processes (Proposition 2.10) and in the class of diamond processes (Proposition 5.3). The first one is not local (since the condition $C1$ is not local), whereas the second one is local (since both of the conditions $C2$ and $C3$ are local).

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REFERENCES

1. E. Best, C. Fernandez and H. Plünnecke, *Concurrent Systems & Processes*, GMD-Studien 104, 1985.
2. A. Mazurkiewicz, *Complete Processes and Inevitability*, University of Leiden Report, 86-06, 1986.
3. A. Mazurkiewicz, E. Ochmanski and W. Penczek, Concurrent Systems and Inevitability, *Th. Comp. Science*, Vol. 64, 1989.
4. E. Ochmański, Inevitability in Concurrent Systems, *Information Processing Letters*, Vol. 25, No. 4, 1987.
5. W. Penczek, *Inevitability in Diamond-Discrete Systems*, I.C.S. P.A.S. Report 598, Warsaw, 1987.
6. W. Reisig, *Petri Nets*, Springer-Verlag, 1985.