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ON THE COMPLEXITY OF COMPUTABLE REAL SEQUENCES (*)

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Abstract. – *We present a definition of complexity of a computable real sequence. We also introduce a definition of effective convergence of sequences and obtain different results relating the complexity of a real sequence with the complexity of its limit.*

Résumé. – *On présente une définition de la complexité d'une suite de réels calculables. Nous introduisons également une définition de convergence effective de suites et obtenons divers résultats reliant la complexité d'une suite de réels et la complexité de sa limite.*

INTRODUCTION

Since the developement of recursive analysis, many definitions of computable real numbers have been given [5, 7, 9]. Soon it was proved by Robinson [6] that all these definitions are equivalent i. e., they all generate the same class of real numbers.

The different definitions can easily be extended to definitions of computable real sequences, but already in 1957 Mostowsky [4] showed that they do not yield the same class of sequences.

Ko [3] took the different definitions of computable real numbers and studied them from the point of view of complexity theory, dividing the real field into different complexity classes. He showed that at this level some definitions of computable real numbers are more general than others.

In this paper we extend the definition of computable real numbers that Ko showed to be more general, to obtain complexity classes of real sequences. We also give a definition of convergence that assures that the limit of a

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computable sequence is a computable real number, and therefore allows us to characterize computable real numbers as limits of computable sequences. We will see that this definition of convergence does not relate the complexity of a sequence of real numbers with the complexity of its limit, finding a sequence that is computable in polynomial time, with a limit computable only in exponential time. At the end of the paper we show a way to avoid this problem by putting some restriction on the definition of convergence. With the new restriction we are able to characterize computable real numbers of a certain complexity class in terms of limits or computable sequences belonging to the same complexity class.

NOTATION

We will denote by N and R the sets of natural and real numbers respectively. In this paper we will only consider real numbers in the interval [0, 1]. Instead of approximating real numbers in [0, 1] by sequences of rational numbers, we will consider for its simplicity the set of diadic numbers $D=\{m.2^{-n} : m, n \in N, m \leqq 2^n\}$.

We define $D_n=\{m.2^{-n} : m \in N, m \leqq 2^n\}$.

We identify in a natural way the set of finite strings over the alphabet $\{0, 1\}$ with D. String $x_1 x_2 \ldots x_n \in \{0, 1\}^*$ will represent the number 0. $x_1 x_2 \ldots x_n$ in binary or $x_1 2^{-1} + \ldots + x_n 2^{-n}$ in D. If there is no confusion, we will use string $x_1 x_2 \ldots x_n \in \{0, 1\}^*$ to denote number $x_1 2^{-1} + \ldots + x_n 2^{-n}$.

The definition of computable sequences that we are going to use is an extension of the following definition of computable real number given by Ko [2, 3].

Definition: Let x be a real number in [0, 1] we say that the function φ binary converges to x, and we write $A(x, \varphi)$, if $\varphi(n) \in D_n$ and $|\varphi(n)-x| \leqq 2^{-n}$ for all $n \in N$.

Let t be a function $t : N \to N$. We define the class of computable real numbers in time t by

$\text{RTIME}(t)=\{x \in [0, 1] : \exists \varphi A(x, \varphi)$ and the computation time of φ is bounded above by $t\}$.

If τ is a class of functions we define $\text{RTIME}(\tau)=\bigcup_{t \in \tau} \text{RTIME}(t)$.

Some of the results obtained in this article, require certain closure properties for the classes of functions that we use. In order to be able to talk about classes of functions in general, controling at the same time their closure properties, we introduce the following definition:

DEFINITION: Let τ be a class of functions $t : N \to N$. We say that τ is admissible if

(i) $\forall t \in \tau \, \exists t' \in \tau$ that bounds the computation time of t.

(ii) The constants and identity functions are in τ.

(iii) τ is closed under composition and addition of functions.

COMPUTABLE SEQUENCES OF REAL NUMBERS

We intoduce now the definition of computable real sequences, and convergence of these sequences.

DEFINITION: Let $t : N \to N$. A sequence $\{a_n\}$ of real numbers is said to be *computable in time t* if there exists a recursive function ψ such that

(i) for any k, for any $n \geqq 1$ $\psi(n, k) \in D_n$ and $|a_k - \psi(n, k)| \leqq 2^{-n}$.

(ii) for any k, for any $n \geqq 1$ $\psi(n, k)$ is computable in $t(n+k)$ steps at most.

We will call SRTIME(t) the set of sequences of real numbers computable in time t, and SRTIME(τ) the set of sequences of real numbers computable in time bounded by a function from τ, being τ a class of functions.

DEFINITION: We say that the sequence $\{a_k\}$ *effectively converges* to the real number l if there is a computable function ξ such that

$$\forall n, m \in N, \quad m \geqq \xi(n) \quad \Rightarrow \quad |a_m - l| \leqq 2^{-n}.$$

If a computable sequence effectively converges to a limit l, it is clear that this limit must be a computable real number since we can approximate l by $\psi(n, \xi(n))$, being ψ the approximation function of the sequence and ξ the function of convergence. This result does not hold if we do not require ξ to be a recursive function (*see* for example [1]). On the other hand the limit of an effectively convergent computable sequence can be more complex than the sequence, as we will see in the next theorem.

THEOREM 1: *There is a sequence* $\{a_k\} \in$ SRTIME(POLY) *that effectively converges to l, and* $l \notin$ RTIME(POLY).

Proof: Let EXP be the class of functions 2^p, being p a polynomial and let $l \in$ RTIME(EXP)$\setminus$RTIME(POLY). There is a function φ, such that $A(l, \varphi)$, and for each n, $\varphi(n)$ can be computed in $2^{p(n)}$ steps for some polynomial p. We will construct a sequence that converges to l very slowly (with many identical terms). The idea is to force a padding of the input using the index of the terms of the sequence. In this way we will be able to calculate the function ψ that aproximates the sequence, in a polynomial number of steps.

Let $\{a_k\}$ be the sequence

$$a_i=0 \quad \text{for} \quad i<2^{p(1)}$$
$$a_i=\varphi(j) \quad \text{for} \quad 2^{p(j)}\leqq i<2^{p(j+1)}$$

Observe that the components of the sequence change only after an exponential number of terms. To show that $\{a_k\}\in \text{SRTIME(POLY)}$ we define $\psi(n, k)$ as the first n digits of $\varphi(j)$, with

$$2^{p(j)}\leqq k<2^{p(j+1)}$$
$$\forall n, \quad \forall k, \quad |\psi(n, k)-a_k|\leqq|\psi(n, k)-\varphi(j)|+|\varphi(j)-a_k|\leqq 2^{-n}$$

with $2^{p(j)}\leqq k<2^{p(j+1)}$. Since $2^{p(n)}$ is time-constructible we can find j such that $2^{p(j)}\leqq k<2^{p(j+1)}$ in

$$c_1 \sum_{i=1}^{j} 2^{p(i)} \text{ steps}$$

and therefore this search can be done in $c_2 . k$ steps for some constant c_2. In order to compute $\varphi(j)$, $2^{p(j)}$ steps are needed. Since $2^{p(j)}\leqq k$, the whole procedure can be done in $O(k)$ steps; therefore $\psi(n, k)$ can be computed in $q(n+k)$ steps for some polynomial q, and $\{a_k\}\in\text{SRTIME(POLY)}$. The limit of the sequence is not computable in polynomial time because

$$\lim_{k\to\infty} a_k=\lim_{k\to\infty}\varphi(k)=l \quad \text{and} \quad l\in\text{RTIME(EXP)}\setminus\text{RTIME(POLY)}. \quad \square$$

This theorem can easily be extended to more complex numbers, i. e., there are computable sequences whose effective limit is arbitrarily complex.

COROLLARY 2: *Let* $l\in\text{RTIME}(t)$ *with* t *time-constructible, there is a sequence* $\{a_k\}\in\text{SRTIME(POLY)}$ *with effective limit* l. $\square$

To avoid this unconfortable result and to be able to control the complexity of the limit of a sequence, we put a restriction to our definition of convergence.

DEFINITION: We say that the sequence $\{a_k\}$ *converges to the limit* l *at speed* τ if there exixts a function $\xi\in\tau$ such that

$$\forall n, \quad \forall m, \quad m\geqq\xi(n) \Rightarrow |a_m-l|\leqq 2^{-n}.$$

As we will see, the complexity of a convergent sequence and its speed of convergence determine the complexity of its limit.

THEOREM 3: *Let* τ *be an admissible class of functions and* $\{a_k\} \in \mathrm{SRTIME}(\tau)$. *If* $\{a_k\}$ *converges to* l *at speed* τ *then* $l \in \mathrm{RTIME}(\tau)$.

Proof: We can approximate l by $\psi(n, \xi(n))$, being ψ the approximation function of the sequence and ξ the function of convergence. Since ψ and ξ are in τ and τ is closed under composition, l ERTIME(τ). □

Therefore we can consider that a convergent computable sequence of a certain complexity behaves well if it converges to its limit at the speed defined by its complexity. We formalize this idea in the next definition.

DEFINITION: Let τ be an admissible class of functions and $\{a_k\}$ a convergent sequence of real numbers. We say that $\{a_k\}$ *behaves well with respect to* τ if $\{a_k\} \in \mathrm{SRTIME}(\tau)$ and converges to its limit at speed τ.

This definition allows us to characterize computable real numbers of a certain complexity class in terms of limits of computable real sequences.

THEOREM 4: *Let* τ *be an admissible class of functions and* l *a real number in* $[0, 1]$. $l \in \mathrm{RTIME}(\tau)$ *if and only if* l *is the limit of a convergent real sequence* $\{a_k\}$, *that behaves well with respect to* τ.

Proof: From right to left is already shown in theorem 3.

From left to right; let $l \in \mathrm{RTIME}(\tau)$ and $\varphi \in \tau$ such that $A(l, \varphi)$. Consider the sequence $\{a_k\}$ defined by $a_k = \varphi(k)$, for each k. We will see that the sequence behaves well with respect to τ.

Define $\psi(n, k) =$ first n digits of $\varphi(k)$, and $\xi(n) = n$, for each n and k.

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \varphi(k) = l.$$

It is clear that the function ψ satisfies conditions (i) and (ii) from the difinition of computable real sequence. Also since τ is admissible, function ξ defining the speed of convergence of $\{a_k\}$ is in τ and the sequence behaves well with respect to τ. □

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