RAIRO. INFORMATIQUE THÉORIQUE ET APPLICATIONS

ALEX WEISS

The local and global varieties induced by nilpotent monoids

RAIRO. Informatique théorique et applications, tome 20, nº 3 (1986), p. 339-355

<http://www.numdam.org/item?id=ITA_1986__20_3_339_0>

Informatique théorique et Applications/Theoretical Informatics and Applications
(vol. 20, n° 3, 1986, p. 339 à 355)

THE LOCAL AND GLOBAL VARIETIES INDUCED BY NILPOTENT MONOIDS (*)

by Alex Weiss [1]

Communicated by J.-E. Pin

Abstract. – *It is proved that* **MNIL** $*$ **D** = **LMNIL**. *Thus the membership problems for the semigroup variety* **MNIL** $*$ **D** *and the category variety* **gMNIL** *are solved.*

Résumé. – *On démontre que* **MNIL** $*$ **D** = **LMNIL**. *Alors les problèmes d'appartenance pour la variété de semigroupes* **MNIL** $*$ **D** *et la variété de catégories* **gMNIL** *sont résolus.*

1. PRELIMINARIES

1.1. Goal

To solve the membership problem for the semigroup variety **MNIL** $*$ **D** and for the category variety induced globally by the monoid variety **MNIL**.

1.2. Introduction

This paper solves a difficult problem using the theory developped in [W & T] and [T & W]. Some familiarity with the results and terminology of these two papers is presupposed. The following proposition shows that the two membership problems are really one and the same.

1.3. Proposition: *Let* **V** *be a monoid variety. Then the following statements are equivalent.*

(*) Received in April 1985, revised in January 1986
Financial assistance from the National Science and Engineering Research Council of Canada and FONDS F.C.A.C. is gratefully acknowledged.
(1) Department of Mathematics, Concordia University, 7141 Sherbrooke St. W., Hingston Hall, Montréal, Québec, Canada H4B 1R6.

Informatique théorique et Applications/Theoretical Informatics and Applications
0296-1598 86/03 339 17/\$3.70/© Gauthier-Villars

(i) $\mathbf{V} * \mathbf{D} = \mathbf{LV}$.

(ii) $\mathbf{V}$ *induces globally and locally the same category variety.*

(iii) *Given any finite category C whose base monoids are in* $\mathbf{V}$, *there exists an injective relational morphism from C to some monoid in* $\mathbf{V}$.

(iv) *Over any graph, a local* $\mathbf{V}$*-congruence is a* $\mathbf{V}$*-congruence.*

Proof: All the equivalences are demonstrated in [T & W] and [W & T]. □

1.4. Plan

The principal result [theorem (3.1)] shows that statement (iv) in the above proposition holds for the monoid variety **MNIL**. The complete proof, however, is long and complicated. Theorem (3.1) proves the result modulu the construction of a certain function $f(n)$, to which the bulk of the paper is devoted.

The plan of the paper is as follows. Section (2) contains the basic definitions and notations. Section (3) contains the principal result. Section (4) contains a number of technical lemmas. Finally, Section (5) contains the main lemmas as well as lemma (5.6) which defines $f(n)$ and completes the proof of theorem (3.1). The next section starts with a few facts about **MNIL**.

2. MNIL-CONGRUENCES

2.1. Definition: Let A be an alphabet, $x \in A^*$. $|x|_a$ denotes the number of times the letter a appears in x. Let

$$x\gamma = \{a \in A \,\|\, |x|_a \geqq 1\}$$

that is, $x\gamma$ is the alphabet of x. Next let

$$x\gamma_n = \{a \in A \,\|\, |x|_a \geqq n\}.$$

Notice that $x\gamma = x\gamma_1$.

Let $x\delta_n$ be the subword of x obtained by erasing from x all occurences of the letters in $x\gamma_n$.

The following definition and lemma are borrowed from [S].

2.2. Definition: Define a congruence $\approx_n$ over A^* as follows. For all $x, y \in A^*$,

$$x \approx_n y \qquad \textit{iff} \quad x\gamma_n = y\gamma_n \quad \text{and} \quad x\delta_n = y\delta_n.$$

2.3. LEMMA:

(i) $\approx_n$ is a finite index congruence over A^*.

(ii) For all $n \geqq 0$, $\approx_n \supseteqq \approx_{n+1}$. □

2.4. DEFINITION: Define a family of semigroups

$$\mathbf{NIL} \hat{=} S \in \mathbf{S} \mid S \textit{ is a semigroup with a zero, this zero being the only idempotent of } S \}.$$

Given a semigroup S, let

$$S^{\cdot} = \begin{cases} S & \text{if } S \textit{ is a monoid,} \\ S \cup \{1\} & \text{if } S \textit{ is not a monoid.} \end{cases}$$

So $S^{\cdot}$ is always a monoid.

The next proposition is extracted from [S].

2.5. PROPOSITION (Straubing): The following families of monoids are all equal.

(i) The **M**-variety generated by

$$\{ S^{\cdot} \mid S \in \mathbf{NIL} \}.$$

(ii) The family

$$\{ S \in \mathbf{M} \mid \textit{there exists an } n \geqq 0 \textit{ such that} \text{ for } \textit{all } s \in S,\ s^n = s^{n+1} \text{ and for } \textit{all } s \in S \text{ and } \textit{all idempotents } e \in S,\ es = se \}.$$

(iii) The family

$$\{ S \in \mathbf{M} \mid \textit{there exists an } v \geqq 0 \textit{ such that} \text{ for } \textit{all } s \in S,\ s^n = s^{n+1} \text{ and for } \textit{all } x, y \in S,\ x^n y = y x^n \}.$$

(iv) The family

$$\{ S \in \mathbf{M} \mid \textit{there exists an } n \geqq 0 \textit{ such that} \text{ for } \textit{all } s \in S,\ s^n = s^{n+1} \text{ and for } \textit{all } x, y_0, y_1, \ldots, y_n \in S,\ y_0 x y_1 \ldots x y_n = x^n y_0 y_1 \ldots y_n = y_0 y_1 \ldots y_n x^n \}.$$

(v) The monoid variety generated by the monoids

$$\{A^*/\approx_n \mid A \text{ is an alphabet and } n \geqq 0\}. \quad \square$$

The monoid variety thus identified by the above statements in called **MNIL**. For our work, characterizations (iv) and (v) are the most useful.

2.6. DEFINITION: A (directed) graph $G=(V, E, \alpha, \omega)$ consists of a set V of vertices, a set E of edges, and two functions $\alpha, \omega : E \to V$ which assign to each edge its begin and end vertex, respectively. We denote by P the set of all (possibly empty) paths over G. We use the symbol $\sim$ to denote the coterminamity relation on paths. Let G be a graph and let $x \in E^*$. Then $x\gamma$ makes sense whether x is a path or not. Now, if $x \in P$, define

$$x\nu = \{v \in V \mid x \text{ can be factored as } x = x_0 x_1 \text{ with } x_0\omega = x_1\alpha = v\}.$$

So $x\nu$ is the set of vertices which x visits. Denote by $|x|_v$ the number of times that x visits the vertex v.

Let L denotes the set of loops in G, while L_v denotes the set of loops about the vertex v.

Let $\widetilde{\approx}_n$ be the family of **MNIL**-congruences induced by the family $\approx_n$ over E^*.

Define the relation R_n over G by

$$\begin{aligned} R_n = {} & \{(x^n, x^{n+1}) \mid x \text{ is a loop}\} \\ & \cup \{(y_0 x y_1 \ldots x y_n, y_0 y_1 \ldots y_n x^n) \mid y_0, y_1, \ldots, y_n, x \text{ are coterminal loops}\} \\ & \cup \{(y_0 x y_1 \ldots x y_n, x^n y_0 y_1 \ldots y_n) \mid y_0, y_1, \ldots, y_n, x \text{ are coterminal loops}\}. \end{aligned}$$

Note that in the definition of R_n we allow empty loops. Let β_n be the smallest congruence over G which contains R_n. Once we show [lemma (2.8) below] that the β_n's have finite index, it follows by proposition (2.5) (iv), that the β_n's are local **MNIL**-congruences.

2.7. NOTATION: Our first goal is to prove that the β_n's have finite index. To do so, we need to introduce some notation. Let

$$S_{n,1} = E \cup \{e^j \mid e \text{ is a loop-edge and } 1 \leqq j \leqq n\}.$$

For all k such that $2 \leqq k \leqq |E|$ let

$$S_{n,k} = \{x \in P \,\|\, x| < k(n|S_{n,k-1}| + 2)\}.$$

2.8. LEMMA: *For all $x \in P$ such that $|x\gamma| = k$ there exists $\bar{x} \in S_{n,k}$ such that $x \beta_n \bar{x}$.*

Proof: We proceed by induction on k where $1 \leqq k \leqq |E|$. If $|x\gamma| = 1$ then either x is an edge, in which case let $\bar{x} = x$, or $x = e^j$ where e is a loop-edge and $j \geqq 1$. In the latter case let $\bar{x} = e^{\min(j,n)}$. In either case $x \beta_n \bar{x}$.

Next suppose that the induction hypothesis holds for $k \geqq 1$, and let $|x\gamma| = k+1$. If

$$|x| < (k+1)(n|S_{n,k}|+2)$$

let $x = \bar{x}$. Else we shall construct a path $\bar{x}$ such that $x \beta_n \bar{x}$ and $|\bar{x}| < |x|$. The induction will then follow by iteration.

Put some arbitrary but fixed ordering on E. As

$$|x| \geqq (k+1)(n|S_{n,k}|+2),$$

it follows by the pigeon hole principle that there exists $e \in E$ such that $|x|_e \geqq n|S_{n,k}|+2$.

If there are several such edges, choose the first in the E-ordering. Thus x contains $n|S_{n,k}|+1$ non-overlapping segments, all of which are coterminal paths whose alphabet is of size $\leqq k$.

If one of these paths is empty, then e is a loop-edge and as $|x|_e > n$, we can construct $\bar{x}$ such that $|\bar{x}|_e = n$ and $x \beta_n \bar{x}$.

If none of the segments is empty, then, by induction, each of them is β_n-congruent to some path in $S_{n,k}$. Again, by the pigeon hole principle, at least $n+1$ of them must be β_n-congruent to the *same* path in $S_{n,k}$. Let $s_1, \ldots, s_{n+1}$ be the first $n+1$ such β_n-congruent paths and let s be the path in $S_{n,k}$ which is congruent to then. Then

$$\begin{aligned} x = x_0 s_1 x_1 \ldots s_{n+1} x_{n+1} \ & \beta_n \\ x_0 s x_1 \ldots s x_{n+1} \ & \beta_n \\ x_0 s s^n x_1 \ldots x_{n+1} \ & \beta_n \\ x_0 s^n x_1 \ldots x_{n+1} = \bar{x}. & \end{aligned}$$

As $x \beta_n \bar{x}$ and $|\bar{x}| < |x|$ we are done. □

2.9. LEMMA: *β_n is a finite index congruence over G.*

Proof: There are no more than $|S_{n,|E|}|$ β_n-classes. □

3. THE PRINCIPAL RESULT

3.1. THEOREM: *Any local* **MNIL**-*congruence is an* **MNIL**-*congruence.*

Proof: If δ is a local **MNIL**-congruence, then there exists an $n \geqq 0$ such that $\delta \supseteqq R_n$, and thus $\delta \supseteqq \beta_n$. But by definition (2.6) and lemma (2.9), β_n is a local **MNIL**-congruence. By proposition (2.5) (v), $\widetilde{\approx}_n$ is an **MNIL**-congruence. Thus if we can find a function $f=f(n)$ such that $\beta_n \supseteqq \widetilde{\approx}_{f(n)}$, then this would prove that β_n is an **MNIL**-congruence and thus so is δ.

We now embark on the task of constructing $f(n)$. $\square$

4. TECHNICAL LEMMAS

4.1. LEMMA: *Let* $L_{n,k,v}=S_{n,k} \cap L_v$. *Let* $L_{n,k}$ *be any element of* $\{L_{n,k,v} \mid v \in V\}$ *of maximal cardinality. Let* $g(n)=n|L_{n,|E|}|+1$. *Then for all* $x \in P$ *and for all* $e \in E$, $|x|_e \geqq g(n)$ *implies that* x *contains* n *non-overlapping segments, all of which are* β_n*-congruent loops whose first edge is* e.

Proof: As $|x|_e \geqq g(n)$, x can be factorized as $x=x_0 ex_1 \dots ex_{g(n)}$. So x contains $n|L_{n,|E|}|$ loops about $e\alpha$, namely, $ex_1, \dots, ex_{n|L_{n,|E|}|}$. By lemma (2.8), there are

$$\bar{x}_1, \dots, \bar{x}_{n|L_{n,|E|}|} \in L_{n,|E|,e\alpha}$$

such that for all $1 \leqq i \leqq n|L_{n,|E|}|$, $ex_i \beta_n \bar{x}_i$. By the pigeon hole principle, there exists at least n of the $\bar{x}_i$'s which are β_n-congruent to each other. Thus at least n of $ex_1, \dots, ex_{n|L_{n,|E|}|}$ are β_n-congruent to each other as well. $\square$

4.2. *Remark:* Notice that $|S_{n,|E|}|$ is a constructible upper bound to the index of β_n. Let k_n be the cardinality of a maximal cardinality base monoid of G^*/β_n. Then $|L_{n,|E|}|$ is a constructible upper bound to k_n. Lemma (4.1) would still be true with $g(n)$ defined as nk_n+1. However, while we do not have an algorithm to decide for any paths x and y whether $x \beta_n y$ is true, we do have an algorithm to decide whether x and y are β_n-congruent to the same path in $S_{n,|E|}$. If they are then $x \beta_n y$ is true, but even if not it may still be the case that $x \beta_n y$. Similarly, if x and y are coterminal loops, we can decide whether x and y are β_n-congruent to the same loop in $S_{n,|E|}$. Thus to make the proofs to come algorithmic, we must define $g(n)$ as in lemma (4.1).

We now proceed with the lemmas.

4.3. LEMMA: *For all* $n \geqq 0$, $\beta_n \supseteqq \beta_{n+1}$, and $\beta_n \subseteqq \widetilde{\approx}_n$.

Proof: The first statement follows from $\beta_n \supseteqq R_{n+1}$. The second follows from $R_n \subseteqq \widetilde{\approx}_n$. □

4.4. LEMMA: *Let* $x = ul^n v \in P$ *with* $l \in L$. *Let* $u = u_1 u_2$ *with* $u_1 \omega = l \alpha$. *Then*

$$x \, \beta_n \, u_1 \, l^n u_2 \, v.$$

Similarly, if $v = v_1 v_2$ *with* $v_1 \omega = l \alpha$ *then* $x \, \beta_n \, uv_1 \, l^n v_2$.

Proof: In fact, $ul^n v \, \beta_n \, u_1 \, l^n u_2 \, v$, since in the definition of R_n we allowed empty loops. □

4.5. LEMMA: *Let* $x \in P$ *and* $e \in E$ *be such that* $|x|_e \geqq g(n)$. *Let* l *be any of the* n β_n*-congruent non-overlapping loops starting with* e *which occur in* x *by lemma* (4.1). *Let* $x = u_1 u_2$ *be any factorization of* x *such that* $u_1 \omega = l \alpha$. *Then for all* $k \geqq n$

$$x \, \beta_n \, ul^k v.$$

Proof: Write $x = x_0 \, es_1 \, x_1 \dots es_n \, x_n$ where $es_1, \dots, es_n$ are the β_n-congruent loops. If l is any of them, we have

$$\begin{aligned} x = x_0 \, es_1 \, x_1 \dots es_n \, x_n \, \beta_n \\ x_0 \, lx_1 \dots lx_n \, \beta_n \\ x_0 \, l^n x_1 \dots x_n \, \beta_n \\ x_0 \, l^n l^n x_1 \dots x_n \, \beta_n \\ x_0 \, l^n es_1 \, x_1 \dots es_n \, x_n. \end{aligned}$$

By lemma (4.4), we conclude that $x \, \beta_n \, ul^n v$ and thus $x \, \beta_n \, ul^k v$ for all $k \geqq n$. □

4.6. DEFINITION: We introduce the notion of a *simple path.*

Let $x \in P$. x is said to be simple iff either x is an empty path, or for all $e \in x\gamma$, $|x|_e = 1$. Next we define a map $s : P \to P$ which associates to every path a simple path coterminal to it. If x is simple then $s(x) = x$. Else there exists an edge which accurs in x at least twice. Introduce an ordering on $x\gamma$ by the order in which the edges of x appear for the first time as x is scanned from left to right. Let e be the first edge in $x\gamma$ which occurs in x more than once. Then x can be factorized as $x=x_0 e x_1 e x_2$ so that x_0 is simple and $|x_2|_e=0$. Then define $s(x)=x_0 e s(x_2)$.

4.7. LEMMA: *Let x be a non-empty path. Then there exists $r>0$ such that*

$$x=x_1 e_1 \ldots x_r e_r$$

where $e_1, \ldots, e_r \in E$, $s(x)=e_1 \ldots e_r$, $x \sim e_1 \ldots e_r$ and for all $1 \leqq i \leqq r$, x_i is a (possibly empty) loop about $e_i \alpha$.

Proof: We proceed by induction on $|x|$.

If $|x|=1$ then $s(x)=x$ and $r=1$.

If $|x|>1$ then if x is simple, $r=|x|$. Else, as in (4.6), $x=x_0 e x_1 e x_2$ with x_0 being a simple path and $s(x)=x_0 e s(x_2)$. If x_2 is empty then $s(x)=x_0 e$ so $r=|x_0|+1$. Else, as $|x_2|<|x|$, by induction we may assume that there exists $u>0$ such that $x_2=z_1 e_1 \ldots z_u e_u$ where $e_1 \ldots e_u=s(x_2)$ and $x_2 \sim e_1 \ldots e_u$.

Then $x=x_0(e x_1) e z_1 e_1 \ldots z_u e_u$. So $r=|x_0|+1+u$ and $s(x)=x_0 e e_1 \ldots e_u$ and $x \sim x_0 e e_1 \ldots e_u$. □

4.8. LEMMA: *Let $u_1 v_1 u_2 \ldots v_{m-1} u_m$ be a path and let z be a simple path such that for all $e \in z\gamma$, $|u_1 \ldots u_m|_e \geqq (m+1) g(n)$. (Note that $u_1 \ldots u_m$ need not be a path). Then*

$$u_1 v_1 u_2 \ldots v_{m-1} u_m \beta_n w_1 v_1 \ldots w_{i,1} z w_{i,2} v_i \ldots w_m$$

for some $1 \leqq i \leqq m$. Furthermore, for all $e \in E$,

$$|u_1 \ldots u_m|_e = |w_1 \ldots w_{i,1} w_{i,2} \ldots w_m|_e.$$

That is, z can be created in one of $u_1, \ldots, u_m$ without affecting the v_j's.

Proof: We proceed by induction on $|z|$.

If $|z|=1$ then the result is trivial.

Else, suppose by induction that

$$u_1 v_1 u_2 \ldots v_{m-1} u_m \beta_n w_1 v_1 \ldots w_{i,1} z w_{i,2} v_i \ldots w_m$$

for a simple path z, and let $e \in E$ be such that ze is a simple path and

$$|w_1 \ldots w_{i,1} w_{i,2} \ldots w_m|_e \geqq (m+1) g(n).$$

By the pigeon hole principle, at least one of $w_1, \ldots, w_{i,1}, w_{i,2}, \ldots, w_m$ contains e at least $g(n)$ times. Call it x. By lemma (4.5), $x \beta_n u l^n v$ where l is a loop starting with e. As $z\omega = e\alpha$ the result follows by lemma (4.4). □

4.9. LEMMA: *Let $x \in P$ and let z be a simple path such that for all $e \in z\gamma$, $|x|_e \geqq (m+1)g(n)+(m-1)$. Then*

$$x\,\beta_n\, y_1 z y_2 \ldots z y_{m+1}$$

for some paths $y_1, \ldots, y_{m+1}$, and for all $e \in E$,

$$|x|_e = |y_1 z y_2 \ldots z y_{m+1}|_e.$$

Proof: We proceed by induction on i for $1 \leqq i \leqq m$. The case $i=1$ follows by lemma (4.8).

By induction suppose that after $i-1$ steps with $i>1$, we have

$$x\,\beta_n\, y_1 z y_2 \ldots z y_i.$$

Then for all $e \in z\gamma$,

$$|y_1 y_2 \ldots y_i|_e \geqq (m+1)g(n) \geqq (i+1)g(n).$$

Again using lemma (4.8), we can create another z in one of $y_1, \ldots, y_i$. □

4.10. LEMMA: *Let $h(n)=(n+1)g(n)+(n-1)$. Let $x \in P$, and let z be a simple loop about $x\omega$ such that for all $e \in z\gamma$, $|x|_e \geqq h(n)$. Then*

$$x\,\beta_n\, xz.$$

Proof: By lemma (4.9),

$$x\,\beta_n\, y_0 z y_1 \ldots z y_n.$$

As $z\alpha = y_n\omega$, y_n is a loop. Then

$$\begin{aligned} x\,&\beta_n\, y_0 y_1 \ldots y_n z^n \\ &\beta_n\, y_0 y_1 \ldots y_n z^n z \\ &\beta_n\, y_0 z y_1 \ldots z y_n z \\ &\beta_n\, xz. \quad \square \end{aligned}$$

4.11. LEMMA: *Let $x \in P$ and let z be a loop about $x\omega$ such that for all $e \in z\gamma$, $|x|_e \geqq h(n)$. Then*

$$x\,\beta_n\, xz.$$

Proof: Note that this lemma differs from the previous one in that z is no longer required to be simple.

We proceed by induction on $|z|$.

If $|z|=1$ then z is simple, so we are done by lemma (4.10).

Else suppose that $|z|>1$ and let

$$z=y_1 e_1 \ldots y_r e_r$$

where $e_1 \ldots e_r = s(z)$ and the y_i's are loops [*see* lemma (4.7)].

By lemma (4.10) we have

$$x \beta_n x e_1 \ldots e_r.$$

By induction on i for $1 \leqq i \leqq r$ we shall now prove that

$$x e_1 \ldots e_i \beta_n x y_1 e_1 \ldots y_i e_i.$$

If $i=1$ then $x \beta_n x y_1$ since y_1 is a loop and $|y_1|<|z|$.

If $i>1$ then, by induction on i, we have

$$x e_1 \ldots e_i \beta_n x y_1 e_1 \ldots y_i e_i. \tag{$*$}$$

If $i=n$ we are done. Else suppose that $i<n$. As $|y_{i+1}|<|z|$, we have

$$x e_1 \ldots e_i \beta_n x e_1 \ldots e_i y_{i+1}.$$

So

$$x e_1 \ldots e_i e_{i+1} \beta_n x e_1 \ldots e_i y_{i+1} e_{i+1}. \tag{$**$}$$

Now, using ($*$) and ($**$), we have

$$x e_1 \ldots e_i e_{i+1} \beta_n x y_1 e_1 \ldots y_i e_i y_{i+1} e_{i+1}.$$

This completes the induction on i.

Now, setting $i=r$, we obtain,

$$x e_1 \ldots e_r \beta_n x y_1 e_1 \ldots y_r e_r = xz.$$

So $x \beta_n x e_1 \ldots e_r \beta_n xz$. $\square$

5. CONCLUSION OF PROOF

5.1. *Remark:* $h(n)$ is not large enough to qualify as $f(n)$, but it is large enough to handle a special case.

5.2. LEMMA: *Let $x \approx_{h(n)} y$ and suppose that both $x\,\delta_{h(n)}$ and $y\,\delta_{h(n)}$ are empty. That is, for all $e \in x\gamma = y\gamma$, $|x|_e \geqq h(n)$ and similarly for y. Then*

$$x\,\beta_n\,y.$$

Proof: Without loss of generality, assume that $|x| \leqq |y|$. We start by proving by induction on i for $0 \leqq i \leqq |x|$ that there exist paths y_i such that $y\,\beta_n\,y_i$, y_i and x have a common prefix of length i and for all $e \in E$, $|y|_e \leqq |y_i|_e$.

For the case $i=0$ let $y_0 = y$.

Now suppose that $x = pau$ and $y_i = pbv$ where $|p| = i \geqq 0$, $a, b \in E$ and $y\,\beta_n\,pbv$. If $a=b$ we are done. Else, as $|y_i|_a \geqq |y|_a$, it follows that $|y_i|_a \geqq h(n) \geqq g(n)$. So y_i contains n non-overlapping β_n-congruent loops starting with the edge a. As $a\alpha = b\alpha$, we can use lemma (4.5) to conclude that $y_i\,\beta_n\,pl^n bv$ where l is a loop starting with a. Let $y_{i+1} = pl^n bv$. This completes the induction on i.

Now by setting $i = |x|$, we obtain $y\,\beta_n\,xz$ for some loop z. As $y\gamma = y_i\gamma$ for all $0 \leqq i \leqq |x|$, we conclude that $(xz)\gamma = y\gamma$. So $z\gamma \subseteqq x\gamma = y\gamma$. Finally using lemma (4.10), we have

$$x\,\beta_n\,xz\,\beta_n\,y. \quad \square$$

5.3. *Remark:* The next two lemmas are long and complicated, but the ideas behind them are rather simple. The reader may wish to study first the final lemma in order to see the meed for the two lemmas.

5.4. LEMMA: *Let $x \approx_{g^{|E|}(n)} y$. Write*

$$x = x_0 a_1 x_1 \ldots a_k x_k$$
$$y = y_0 a_1 y_1 \ldots a_k y_k$$

where

$$a_1 \ldots a_k = x\,\delta_{g^{|E|}(n)} = y\,\delta_{g^{|E|}(n)}.$$

Suppose that for all i such that $0 \leqq k$, $x_i\,\delta_{g^{|E|}(n)}$ is empty, and similarly for the y_i's. Then $x\,\beta_n\,x'$ and $y\,\beta_n\,y'$ where

$$= x'_0 a_1 x'_1 \ldots a_k x'_k$$
$$= y'_0 a_1 y'_1 \ldots a_k y'_k,$$

where for all $0 \leqq i \leqq k$, $x_i' \approx_n y_i'$ *and either both* x_i' *and* y_i' *are empty or both* $x_i' \delta_n$ *and* $y_i' \delta_n$ *are empty.*

Proof: Given paths p and q such that $p \delta_n$ and $q \delta_n$ are empty, we have that $p \approx_n q$ iff $p \gamma = q \gamma$ and $p \sim q$. This observation will be used later.

We start by introducing some notation. Let

$$B = (a_1 \dots a_k) \gamma.$$
$$B_0 = \{ e \in E \,\|\, x |_e < g(n) \}.$$

For all $0 < j < |E|$ let

$$B_j = \{ e \in E \,|\, g^j(n) \leqq |x|_e < g^{j+1}(n) \}.$$

Finally let

$$B_{|E|} = \{ e \in E \,\|\, x |_e \geqq g^{|E|}(n) \} = x \gamma_{g^{|E|}(n)}.$$

Observe that $B = \bigcup_{0 \leqq j \leqq |E|} B_j$. Observe further that if $B = E$ then $x = y$ and this lemma is rather trivial. Also, if $B = \emptyset$, then $k = 0$ so let $x = x'$ and $y = y'$ and the lemma follows. Thus we may assume that $0 < |B| < |E|$. This implies that there exists $0 \leqq j \leqq |E|$ such that $B_j = \emptyset$, but we shall not use this fact.

Given paths p and q, we say that q is fuller than p, written $p \leqq q$, iff $p\gamma = q\gamma$ and for all $e \in E$, $|p|_e \leqq |q|_e$. This relation satisfies $p \leqq p$ and $p \leqq q$ together with $q \leqq r$ implies $p \leqq r$. However, $p \leqq q$ and $q \leqq p$ does not imply that $p = q$. But we do not need this last property.

The proof proceeds via a certain construction. We construct two sequences of fuller and fuller paths

$$x = x^{(0)} \leqq x^{(1)} \leqq \dots \leqq x^{(|E|)}$$
$$y = y^{(0)} \leqq y^{(1)} \leqq \dots \leqq y^{(|E|)}$$

which have certain properties. These properties will now be stated for the x sequence. The corresponding properties for the y sequence can be obtained by reading x for y and y for x in the obvious places.

(i) For all l such that $0 \leqq l \leqq |E|$,

$$x^{(l)} = x_0^{(l)} a_1 x_1^{(l)} \dots a_k x_k^{(l)}$$

that is, $a_1 \dots a_k$ is a subword of every path in the x sequence. Similarly for y.

(ii) For all $0 \leqq i \leqq k$ and for all $0 \leqq l \leqq |E|$, either $x_i^{(l)} \delta_{g_{|E|-l}(n)}$ and $y_i^{(l)} \delta_{g_{|E|-l}(n)}$ are both empty or $x_i^{(l)}$ and $y_i^{(l)}$ are both empty. That is

$$x_i^{(l)} \gamma \cap \bigcup_{0 \leqq j < |E|-l} B_j = \emptyset.$$

Similarly for y.

(iii) For all $0 \leqq l < |E|$

$$x^{(l)} \beta_{g_{|E|-(l+1)}(n)} x^{(l+1)}.$$

Similarly for y.

(iv) If for some $0 \leqq l < |E|$ and for some $0 \leqq i \leqq k$

$$x_i^{(l)} \gamma \supseteqq y_i^{(l)} \gamma \textit{ then } x_i^{(l+1)} = x_i^{(l)}.$$

Similarly for y.

(v) If for some $0 \leqq l < |E|$ and for some $0 \leqq i \leqq k$ $y_i^{(l)} \gamma - x_i^{(l)} \gamma \neq \emptyset$ then $x^{(l+1)}$ will be created from $x^{(l)}$ using $R_{g_{|E|-(l+1)}(n)}$-transformations in such a way that for all $0 \leqq i \leqq k$

$$x_i^{(l+1)} \gamma \supseteqq y_i^{(l)} \gamma.$$

Similarly for y.

We defer the details of this construction to later in the proof of this lemma. In the meantime, assuming properties (i) to (v), we have the following.

LEMME: *For all* $0 \leqq l \leqq |E|$ *and for all* $0 \leqq i \leqq k$,
either $x_i^{(l)} \gamma = y_i^{(l)} \gamma$
or $|x_i^{(l)} \gamma| \geqq l$ *and* $|y_i^{(l)} \gamma| \geqq l$.

Proof: We proceed by induction on l.

If $l=0$ then the lemma is trivial. Thus assume that the lemma holds for some $0 \leqq l < |E|$. If $x_i^{(l)} \gamma = y_i^{(l)} \gamma$ then $x_i^{(l+1)} \gamma = x_i^{(l)} \gamma = y_i^{(l)} \gamma = y_i^{(l+1)} \gamma$ by (iv), so the induction follows.

Now suppose that $x_i^{(l)} \gamma \neq y_i^{(l)} \gamma$. Without loss of generality we may assume that $y_i^{(l)} \gamma - x_i^{(l)} \gamma \neq \emptyset$.

Then by (v), $x_i^{(l+1)} \gamma \supseteqq y_i^{(l)} \gamma$ so $|x_i^{(l+1)} \gamma| \geqq |x_i^{(l)} \gamma| + 1 \geqq l+1$. Next we show $|y_i^{(l)} \gamma| \geqq l+1$.

If $x_i^{(l)} \gamma - y_i^{(l)} \gamma \neq \emptyset$ we argue by symmetry. Else $y_i^{(l)} \gamma$ strictly contains $x_i^{(l)} \gamma$ so

$$|y_i^{(l)} \gamma| > l \text{ so } |y_i^{(l+1)} \gamma| = |y_i^{(l)} \gamma| \geqq l+1. \quad \square$$

Using this lemma we may conclude the proof of main lemma. From the lemma we deduce that for all $0 \leqq i \leqq k$, $x_i^{(|E|)}\gamma = y_i^{(|E|)}\gamma$. Also, by (i), $x_i^{(l)} \sim y_i^{(l)}$ for all $0 \leqq l \leqq |E|$. Thus setting $x' = x^{(|E|)}$ and $y' = y^{(|E|)}$ and remembering the observation made at the very beginning of the proof, the lemma follows.

We shall now take up the details of the construction.

We proceed by induction on $0 \leqq l \leqq |E|$.

As $x = x^{(0)}$ and $y = y^{(0)}$, the case $l = 0$ is immediate. So we pass to the induction step. Assume that for some $0 \leqq l < |E|$, $x^{(l)}$ and $y^{(l)}$ have been constructed in accordance with properties (i) to (v).

We shall now construct $x^{(l+1)}$. $y^{(l+1)}$ is constructed in an identical fashion.

Define a succession of fuller and fuller paths $z_{i,j}$ for $0 \leqq i \leqq k$ and $0 \leqq j \leqq |y_i^{(l)}\gamma - x_i^{(l)}\gamma|$, with the j index varying faster.

The $z_{i,0}$ are defined just for notational convenience.

$z_{0,0} = x^{(l)}$ and if $i \geqq 0$,

$$z_{i,0} = z_{i-1,\,|y_{i-1}^{(l)}\gamma - x_{i-1}^{(l)}\gamma|}$$

We construct the $z_{i,j}$ inductively to have the following properties.

(I) $z_{i,j} = z_0^{(i,j)} a_1 z_1^{(i,j)} \dots a_k z_k^{(i,j)}$, that is, the $z_{i,j}$ have $a_1 \dots a_k$ as a subword.

(II) $x^{(l)} \leqq z_{i,j}$ and $x_i^{(l)} \leqq z_{i,j}^{(l)}$.

(III) $x^{(l)} \beta_{g^{|E|-(l+1)}(n)} z_{i,j}$.

(*IV*) If $y_i^{(l)}\gamma - x_i^{(l)}\gamma \neq \emptyset$ let

$$\{e_j \mid 1 \leqq j \leqq |y_i^{(l)}\gamma - x_i^{(l)}\gamma|\}$$

be the edges in $y_i^{(l)}\gamma - x_i^{(l)}\gamma$ ordered by their order of appearance in $y_i^{(l)}$ as $y_i^{(l)}$ is scanned from left to right. Then

$$\{e_1, \dots, e_j\} \subseteqq z_i^{(i,j)}\gamma.$$

In fact

$$\{e_1, \dots, e_j\} \subseteqq z_i^{(i,j)}\gamma_{g^{|E|-(l+1)}(n)}.$$

Clearly $z_{0,0}$ has these properties.

Suppose by induction that so does $z_{i,j}$. If $j = |y_i^{(l)}\gamma - x_i^{(l)}\gamma|$ then the next path is $z_{i+1,0}$ and $z_{i+1,0} = z_{i,j}$, so we are done.

Else suppose that $0 \leqq j < |y_i^{(l)}\gamma - x_i^{(l)}\gamma|$. Once again we need a

LEMMA: $e_{j+1}\alpha \in z_i^{(i,j)}\nu$.

Proof: We proceed by induction on j with $0 \leqq j \leqq |y_i^{(l)}\gamma - x_i^{(l)}\gamma|$.

If $j=0$ then either $y_i^{(l)}=e_1 p$ for some $p\in P$ or $y_i^{(l)}=ue_1 w$ for some paths u and w with $u\gamma \subseteqq x_i^{(l)}\gamma$.

In the first case, $e_1\alpha = y_i^{(l)}\alpha = x_i^{(l)}\alpha = z_i^{(i,0)}\alpha$.

In the second case, $e_1\alpha = u\omega \in x_i^{(l)}\nu$. By property (II) $x_i^{(l)}\nu \subseteqq z_i^{(i,j)}\nu$, so $e_1\alpha\in z_i^{(i,0)}\nu$. This completes the $j=0$ case.

Now let $j\geqq 0$. Then either $y_i^{(l)}=ue_je_{j+1}w$ for some $u, w\in P$, or $y_i^{(l)}=ue_jpe_{j+1}w$ for some u, p, $w\in P$, where $p\gamma\subseteqq x_i^{(l)}\gamma$.

In the first case, $e_{j+1}\alpha=e_j\omega$. By property (IV) and by the induction on $z_{i,j}$, we have $e_j\in z_i^{(i,j)}\gamma$. Thus,

$$e_{j+1}\alpha = e_j\omega \in z_i^{(i,j)}\nu.$$

In the second case, $e_{j+1}\alpha = p\omega \in x_i^{(l)}\nu \subseteqq z_i^{(i,j)}\nu$. □

Thus we know that $e_{j+1}\in z_i^{(i,j)}\nu$. By the induction on l and property (ii), we have $|y_i^{(l)}|_{e_{j+1}} \geqq g^{|E|-l}(n)$. As $l<|E|$, we use lemma (4.1) to create $g^{|E|-(l+1)}(n)$ occurrences of e_{j+1} in $z_i^{(i,j)}$ using only $R_{g^{|E|-(l+1)}(n)}$-transformations, in such a way that the only segment of $z_{i,j}$ to be affected is $z_i^{(i,j)}$ itself. This new path we call $z_{i,j+1}$ which differs from $z_{i,j}$ only in that $z_i^{(i,j)}\neq z_i^{(i,j+1)}$. $z_{i,j+1}$ have been constructed in accordance with properties (I) to (IV).

This completes the induction on the $z_{i,j}$'s.

Now let $x^{(l+1)}=z_{k,|y_k^{(l)}\gamma - x_k^{(l)}\gamma|}$. Then $x^{(l+1)}$ has properties (i) to (v).

This completes the induction on l. □

5.5. LEMMA: *Let $x \approx_{g(n)} y$. Write*

$$x = x_0 a_1 x_1 \ldots a_k x_k$$

$$y = y_0 a_1 y_1 \ldots a_k y_k$$

where $a_1\ldots a_k = x\,\delta_{g(n)} = y\,\delta_{g(n)}$.

Then $x\,\beta_n\,x'$ and $y\,\beta_n\,y'$ where

$$x' \approx_n y'$$

and

$$x' = x'_0 a_1 x'_1 \ldots a_k x'_k$$

$$y' = y'_0 a_1 y'_1 \ldots a_k y'_k$$

and for all $0 \leqq i \leqq k$, $x'_i \delta_n$ *is empty, and similarly for* y'_i.

Proof: Note that $a_1 \ldots a_k = x \delta_{g(n)}$ implies that for all $0 \leqq i \leqq k$, x_i is either empty or $x_i \gamma \subseteqq x \gamma_{g(n)}$, and similarly for y_i.

Let $B = (a_1 \ldots a_k) \gamma$. As in lemma (5.4), the result is immediate unless we assume $0 < |B| < |E|$.

We proceed with a construction similar to the one used in lemma (5.4) to construct $x^{(l+1)}$ from $x^{(l)}$.

Define a succession of fuller and fuller paths $z_{i,j}$, where $0 \leqq i \leqq k$ and $0 \leqq j \leqq |x_i \gamma|$ with the j index varying faster.

We define $z_{i,0}$ for notational convenience only. We have $z_{0,0} = x$ and $z_{i,0} = z_{i-1, |x_{i-1} \gamma|}$ if $i > 0$.

The $z_{i,j}$ are constructed to conform to the following properties.

(i) $z_{i,j} = z_0^{(i,j)} a_1 z_1^{(i,j)} \ldots a_k z_k^{(i,j)}$, that is, $a_1 \ldots a_k$ is a subword of $z_{i,j}$.

(ii) $x \leqq z_{i,j}$ and $x_i \leqq z_i^{(i,j)}$.

(iii) $x \beta_n z_{i,j}$.

(iv) If $x_i \gamma \neq \emptyset$, let $\{ e_j \mid 1 \leqq j \leqq |x_i \gamma| \}$ be the edges in $x_i \gamma$ in their order of appearance in x_i as x_i is scanned from left to right. Then

$$\{ e_1, \ldots, e_j \} \subseteqq z_i^{(i,j)} \gamma_n.$$

We proceed by induction on $z_{i,j}$.

As $z_{0,0} = x$, it has the above properties. Suppose by induction that $z_{i,j}$ has the above properties. If $j = |x_i \gamma|$ then $z_{i+1,0}$ is the next path in the sequence and $z_{i+1,0} = z_{i,j}$, so the induction follows in this case.

Next suppose that $0 \leqq j < |x_i \gamma|$. If $|z_i^{(i,j)}|_{e_{j+1}} \geqq n$, let $z_{i,j+1} = z_{i,j}$. Else $|z_i^{(i,j)}|_{e_{j+1}} < n$. But as $|z_{i,j}|_{e_{j+1}} \geqq g(n)$, we may use lemma (4.5) to create in $z_i^{(i,j)}$ n occurrences of e_{j+1} using R_n-transformations, with the only segment of $z_{i,j}$ to be affected is $z_i^{(i,j)}$ itself. We call the result of these transformations $z_{i,j+1}$, which differs from $z_{i,j}$ only in that $z_i^{(i,j)} \neq z_i^{(i,j+1)}$. Thus $z_{i,j+1}$ has properties (i) to (iv).

This completes the induction on the $z_{i,j}$.

Let $x' = z_{k, |x_k \gamma|}$. In an identical fashion one obtains y'.

Note that x' and y' may contain edges of B, but those B-edges must appear in x and y at least n times. The edges which appear in x and y less than n

times can not be affected by an R_n-transformation. We thus conclude that

$$x' \bar{\approx}_n y'. \quad \square$$

5.6. FINAL LEMMA: *Let*

$$f(n) = g(g^{|E|}(h(n))).$$

Then

$$\beta_n \supseteq \bar{\approx}_{f(n)}.$$

Proof: Suppose that $x \bar{\approx}_{f(n)} y$. Write

$$x = x_0 a_1 x_1 \ldots a_k x_k$$

$$y = y_0 a_1 y_1 \ldots a_k y_k$$

where

$$a_1 \ldots a_k = x \delta_{f(n)} = y \delta_{f(n)}.$$

By lemma (5.5) $x \beta_{g^{|E|}(h(n))} x'$ and $y \beta_{g^{|E|}(h(n))} y'$ such that $x' \bar{\approx}_{g^{|g|}(h(n))} y'$ and $x' = x'_0 a_1 x'_1 \ldots a_k x'_k$ and $y' = y'_0 a_1 y'_1 \ldots a_k y'_k$ where for all $0 \leqq i \leqq k$ $x'_i \delta_{g^{|E|}(h(n))}$ is empty, and similarly for $y'_i \delta_{g^{|E|}(h(n))}$.

Now, by lemma (5.4) we conclude that $x' \beta_{h(n)} x''$ and $y' \beta_{h(n)} y''$ where $x'' = x''_0 a_1 x''_1 \ldots a_k x''_k$ and $y'' = y''_0 a_1 y''_1 \ldots a_k y''_k$ and for all $0 \leqq i \leqq k$, $x''_i \bar{\approx}_{h(n)} y''$ and $x''_i \delta_{h(n)}$ is empty and similarly for $y''_i \delta_{h(n)}$.
Thus, by lemma (5.2), for all $0 \leqq i \leqq k$, $x''_i \beta_n y''_i$. So $x'' \beta_n y''$.

So we have the chain

$$x \beta_{g^{|E|}(h(n))} x' \beta_{h(n)} x'' \beta_n y'' \beta_{h(n)} y' \beta_{g^{|E|}(h(n))} y. \quad \square$$

REFERENCES

[S] H. STRAUBING, *The Variety Generated by Finite Nilpotent Monoids*, Semigroup Forum, Vol. 24, 1982, pp. 25-38.

[T & W] D. THÉRIEN and A. WEISS, *Graph Congruences and Wreath Products*, J.P. Ap. Alg., Vol. 36, 1985, pp. 205-215.

[W & T] A. WEISS and D. THÉRIEN, *Varieties of Finite Categories*, R.A.I.R.O. Informatique Théorique, Vol. 20, n° 3, 1986, pp. 357-366.