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EVERY COMMUTATIVE QUASIRATIONAL LANGUAGE IS REGULAR (*)

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Abstract. – *A nonregular language L is minimal with respect to a language family $\mathscr{L}$ if, for each nonregular language L_1 in $\mathscr{L}$, L is in the trio generated by L_1. We show that the language $\bar{D}_1^* = \{x \in \{a_1, a_2\}^* \mid |x|_{a_1} \neq |x|_{a_2}\}$ is minimal with respect to $c(\mathscr{R})$, the family of languages consisting of the commutative closures of all regular languages. This then implies that each commutative quasirational language is regular.*

Résumé. – *Un langage L non rationnel est minimal dans une famille $\mathscr{L}$ de langages si, pour tout langage L_1 non rationnel dans $\mathscr{L}$, L appartient au plus petit cône rationnel fidèle contenant L_1. Nous montrons que le langage $\bar{D}_1^* = \{x \in \{a_1, a_2\}^* \mid |x|_{a_1} \neq |x|_{a_2}\}$ est minimal dans $c(\mathscr{R})$ qui est l'ensemble des fermetures commutatives des langages rationnels. Ceci implique que tout langage commutatif quasirationnel est rationnel.*

1. INTRODUCTION

The minimality of languages is studied in several articles, for instance in [1], [3], [9] and [10]. Let $\mathscr{T}(\mathscr{L})$ ($\hat{\mathscr{T}}(\mathscr{L})$) denote the (full) trio generated by the language family $\mathscr{L}$. In [1], [9] and [10] we can find the following conjecture:

CONJECTURE 1: If L is a nonregular language in $c(\mathscr{R})$, then $\bar{D}_1^*$ is in $\hat{\mathscr{T}}(L)$. We show that $\bar{D}_1^*$ is in $\mathscr{T}(L)$ for each nonregular language L in $c(\mathscr{R})$ thus proving the conjecture. A result of Latteux and Leguy [11] then implies:

CONJECTURE 2: Every commutative quasirational language is regular.

Conjecture 2 was stated in [8] and [10]. It was partially proved in [5] and [11]; in [5] it was shown that every commutative linear language is regular and in [11] that every commutative quasirational language over a two-letter alphabet is regular.

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2. PRELIMINARIES

A subset S of $\mathbb{N}^n$ is *linear* if:

$$S=\{u_0+k_1 u_1+\ldots+k_r u_r \mid k_j\in\mathbb{N}, j=1, \ldots, r\},$$

for some $u_i\in\mathbb{N}^n$, $i=0, 1, \ldots, r$. We say that s is the *rank* of S if there are exactly s linearly independent elements (over Q, the rationals) in $u_1, \ldots, u_r$. The rank of S is denoted by rank(S). Naturally rank$(S)\leqq n$. If rank$(S)=r$, then S is a *proper* linear set. A subset T of $\mathbb{N}^n$ is *semilinear* if it is a finite union of linear sets. The rank of T, denoted by rank(T), is s if $T=S_1\cup\ldots\cup S_m$ where each S_i is a linear set and $\max_i \operatorname{rank}(S_i)=s$. It can be verified that the rank of each semilinear set is uniquely determined. The *convex closure* conv(S) of the linear set S is defined by:

$$\operatorname{conv}(S)=\{u_0+\alpha_1 u_1+\ldots+\alpha_r u_r \mid \alpha_j\in Q,\ \alpha_j\geqq 0, j=1, \ldots, r\}\cap\mathbb{N}^n.$$

Denote

$$\mathscr{A}(S)=\{\alpha_1 u_1+\ldots+\alpha_r u_r \mid \alpha_j\in Q \text{ for each } j\}.$$

Note that $\mathscr{A}(S)$ is a linear subspace of Q^n. All the linear spaces considered are subspaces of Q^n over Q, the rationals. Again, both conv(S) and $\mathscr{A}(S)$ are well-defined. By Lemma 1, conv(S) is a semilinear set.

A linear set $S\subseteq\mathbb{N}^n$ is *fundamental* if:

$$S=\{(r_1, \ldots, r_n)+k_1(s_1, 0, \ldots, 0)+\ldots$$
$$+k_n(0, \ldots, 0, s_n) \mid k_j\in\mathbb{N}, j=1, \ldots, n\},$$

for some r_j, $s_j\in\mathbb{N}$, $r_j<s_j$, $j=1, \ldots, n$. If S is fundamental, then obviously rank$(S)=n$. A semilinear set is called fundamental if it is a finite union of fundamental linear sets.

Let $U\subseteq\mathbb{N}^n$. The *complement* of U is the set $\bar{U}$ defined by:

$$\bar{U}=\{v\in\mathbb{N}^n \mid v\notin U\}.$$

Ginsburg proves in [6] that:

(i) the intersection of two semilinear sets is a semilinear set;

(ii) the complement of a semilinear set is a semilinear set; and

(iii) each semilinear set is a finite union of proper linear sets.

These facts are extensively used in our proofs.

Let $V, W \subseteqq \mathbb{N}^n$. Then we define:

$$V+W=\{v+w \mid v \in V, w \in W\}.$$

Let $e_i \in \mathbb{N}^n$ be the element in which the i-th coordinate is one and all the others are equal to zero, $i=1, \ldots, n$. Let $\Psi_{\langle a_1, \ldots, a_n \rangle}$ be the usual Parikh-mapping from $\{a_1, \ldots, a_n\}^*$ onto $\mathbb{N}^n$. When $\Psi_{\langle a_1, \ldots, a_n \rangle}$ is understood, it is denoted by Ψ.

Let Σ_1 be an alphabet and $x \in \Sigma_1^*$. Then $|x|_a$ denotes the number of occurrences of the symbol a in x for each a in Σ_1. The empty word is denoted by ε. Let $L \subseteqq \Sigma_1^*$ be a language. Then:

$$x^{-1} L=\{y \in \Sigma_1^* \mid xy \in L\}$$

and

$$L-\{\varepsilon\}=L \cap \Sigma_1^+.$$

Define $c(x)=\{y \in \Sigma_1^* \mid |x|_a=|y|_a \text{ for each } a \in \Sigma_1\}$. The *commutative closure* of the language L is the set

$$c(L)=\bigcup_{x \in L} c(x).$$

The language L is *commutative* if $L=c(L)$.

For a language $L \subseteqq \{a_1, \ldots, a_n\}^*$, let the *complement* of L with respect to $\{a_1, \ldots, a_n\}$ be the language $\bar{L}(a_1, \ldots, a_n)$ defined by:

$$\bar{L}(a_1, \ldots, a_n)=\{x \in \{a_1, \ldots, a_n\}^* \mid x \notin L\}.$$

We denote $\bar{L}(a_1, \ldots, a_n)$ by $\bar{L}$ when $\{a_1, \ldots, a_n\}$ is understood.

A language $L \subseteqq \{a_1, \ldots, a_n\}^*$ is a *SLIP-language* if $\Psi(L)$ is a semilinear set. If L is commutative and $\Psi(L)$ is a linear set, then the *convex closure* $\operatorname{conv}(L)$ of L is the following language:

$$\operatorname{conv}(L)=\Psi^{-1}(\operatorname{conv}(\Psi(L))).$$

A commutative language $R \subseteqq \{a_1, \ldots, a_n\}^*$ is fundamental if $\Psi(R)$ is a fundamental semilinear set. Note that if R is fundamental, it is a regular commutative SLIP-language.

It should be clear that $c(\mathscr{R})$ is exactly the family of all commutative SLIP-languages and that $c(\mathscr{R})$ is closed under union, intersection and complementation. Let $D_1^*=c((a_1 a_2)^*)$.

3. MAIN RESULTS

We now prove seven lemmas which imply the main results of this paper.

LEMMA 1: *For each linear set $S \subseteq \mathbb{N}^n$, $\operatorname{conv}(S)$ is a semilinear set.*

Proof: Assume:

$$S=\{u_0+k_1 u_1+\ldots+k_m u_m \mid k_j \in \mathbb{N}, j=1, \ldots, m\}.$$

where $u_i \in \mathbb{N}^n$, $i=0, 1, \ldots, m$. Let:

$$U_0=\{u_0+\alpha_1 u_1+\ldots+\alpha_m u_m \mid \alpha_j \in Q, 0 \leqq \alpha_j < 1, j=1, \ldots, m\} \cap \mathbb{N}^n$$

and

$$U_1=\{k_1 u_1+\ldots+k_m u_m \mid k_j \in \mathbb{N}, j=1, \ldots, m\}.$$

Obviously U_0 is finite and thus $U=U_0+U_1$ is a semilinear set. We show that $\operatorname{conv}(S)=U$.

It should be clear that $U \subseteq \operatorname{conv}(S)$. Assume $u \in \operatorname{conv}(S)$. Then:

$$u=u_0+\beta_1 u_1+\ldots+\beta_m u_m \in \mathbb{N}^n$$

for some nonnegative $\beta_j \in Q$, $j=1, \ldots, m$. Now:

$$u=u_0+\gamma_1 u_1+\ldots+\gamma_m u_m+r_1 u_1+\ldots+r_m u_m$$

for some $\gamma_j \in Q$, $0 \leqq \gamma_j < 1$, $r_j \in \mathbb{N}$, where $\beta_j=\gamma_j+r_j$, $j=1, \ldots, m$. Thus $u_0+\gamma_1 u_1+\ldots+\gamma_m u_m \in U_0$ and $r_1 u_1+\ldots+r_m u_m \in U_1$, so $u \in U$. We can deduce that $\operatorname{conv}(S) \subseteq U$. The proof is now complete. $\square$

LEMMA 2: *For each proper linear set $S \subseteq \mathbb{N}^n$, there exists a fundamental semilinear set $U \subseteq \mathbb{N}^n$ such that* $\operatorname{conv}(S) \cap U=S$.

Proof: Assume:

$$S=\{u_0+k_1 u_1+\ldots+k_m u_m \mid k_j \in \mathbb{N}, j=1, \ldots, m\},$$

where $u_i \in \mathbb{N}^n$, $i=0, 1, \ldots, m$, and the elements $u_1, \ldots, u_m$ are linearly independent. Now $m \leqq n$. If $m < n$, there are distinct numbers $i_1, \ldots, i_{n-m} \in \{1, \ldots, n\}$ such that the elements $u_1, \ldots, u_m, e_{i_1}, \ldots, e_{i_{n-m}}$ are linearly independent. In this case denote $u_{m+j}=e_{i_j}$, $j=1, \ldots, n-m$.

Let $m_i \in \mathbb{N}_+$ be the smallest number such that:

$$(1) \qquad m_i e_i=r_{i1} u_1+\ldots+r_{in} u_n$$

for some $r_{ij} \in \mathbb{Z}$, $j=1, \ldots, n$, $i=1, \ldots, n$. Here $\mathbb{Z}$ is the set of all integers. Denote:

$$U_0=\{(t_1, \ldots, t_n) \in \mathbb{N}^n \mid t_i < m_i, i=1, \ldots, n\} \cap \{u_0+\alpha_1 u_1+\ldots+\alpha_n u_n \mid \alpha_i \in \mathbb{Z}, i=1, \ldots, n\}$$

and $U_1=\{k_1 m_1 e_1+\ldots+k_n m_n e_n \mid k_i \in \mathbb{N}, i=1, \ldots, n\}$. The set $U=U_0+U_1$ is a fundamental semilinear set. We show that $\operatorname{conv}(S) \cap U=S$.

Assume $u \in S$. Then $u=u_0+k_1 u_1+\ldots+k_m u_m$ for some $k_j \in \mathbb{N}$, $j=1, \ldots, m$. We can write u in the form:

$$u=(t_1, \ldots, t_n)+l_1 m_1 e_1+\ldots+l_n m_n e_n$$

for some t_j, $l_j \in \mathbb{N}$, $0 \leqq t_j < m_j$, $j=1, \ldots, n$. By (1):

$$(t_1, \ldots, t_n)=u_0+s_1 u_1+\ldots+s_n u_n$$

for some $s_j \in \mathbb{Z}$, $j=1, \ldots, n$. This means that $(t_1, \ldots, t_n) \in U_0$ and $u \in U=U_0+U_1$. Since $u \in \operatorname{conv}(S)$, $u \in \operatorname{conv}(S) \cap U$. So $S \subseteq \operatorname{conv}(S) \cap U$.

Assume now that $u \in \operatorname{conv}(S) \cap U$. Then, since $u \in \operatorname{conv}(S)$;

$$u=u_0+\alpha_1 u_1+\ldots+\alpha_m u_m,$$

for some nonnegative $\alpha_j \in Q$, $j=1, \ldots, m$. Since $u \in U$, we have:

$$u=(t_1, \ldots, t_n)+k_1 m_1 e_1+\ldots+k_n m_n e_n,$$

for some $(t_1, \ldots, t_n) \in U_0$, $k_j \in \mathbb{N}$, $j=1, \ldots, n$. By (1):

$$\begin{aligned} u_0+\alpha_1 u_1+\ldots+\alpha_m u_m &= (t_1, \ldots, t_n) \\ &\quad +k_1(r_{11} u_1+\ldots+r_{1n} u_n)+\ldots+k_n(r_{n1} u_1+\ldots+r_{nn} u_n) \\ &= u_0+l_1 u_1+\ldots+l_n u_n+k_1(r_{11} u_1+\ldots+r_{1n} u_n)+\ldots \\ &\quad +k_n(r_{n1} u_1+\ldots+r_{nn} u_n), \end{aligned}$$

for some $l_j \in \mathbb{Z}$, $j=1, \ldots, n$. The equations above imply that:

$$\alpha_j=l_j+k_1 r_{1j}+\ldots+k_n r_{nj} \in \mathbb{Z}, \qquad j=1, \ldots, m.$$

Since $\alpha_j \geqq 0$, we have $\alpha_j \in \mathbb{N}$ for each j. Thus $u \in S$. Since u is arbitrary, $\operatorname{conv}(S) \cap U \subseteq S$. Thus $S=\operatorname{conv}(S) \cap U$. $\square$

Note: A straightforward reasoning shows that (i) the intersection of two fundamental semilinear sets is either empty or a fundamental semilinear set; and (ii) the complement of a fundamental semilinear set is either empty or a fundamental semilinear set.

Let $S \subseteq \mathbb{N}^n$ be a semilinear set. Then S is *homogenous* if there exist proper linear sets $S_1, \ldots, S_m \subseteq \mathbb{N}^n$ and a fundamental semilinear set $U \subseteq \mathbb{N}^n$ such that:

(i) $S = \bigcup_{i=1}^{m} S_i$; and

(ii) $\left(\bigcup_{i=1}^{m} \operatorname{conv}(S_i)\right) \cap U = S$.

Call a language $L \subseteq \{a_1, \ldots, a_n\}^*$ homogenous if L is a commutative SLIP-language such that $\Psi(L)$ is a homogenous semilinear set.

LEMMA 3: *Let $L \subseteq \{a_1, \ldots, a_n\}^*$ be a nonregular commutative SLIP-language. Then there exists a nonregular homogenous language $L' \subseteq \{a_1, \ldots, a_n\}^*$ in $\mathscr{T}(L)$.*

Proof: Let $L_1, \ldots, L_m \in c(\mathscr{R})$ be languages such that $\Psi(L_i)$ is a proper linear set for each i, and $L = \bigcup_{i=1}^{m} L_i$. By Lemma 2, there exists a fundamental language $R_i \subseteq \{a_1, \ldots, a_n\}^*$ such that $\operatorname{conv}(L_i) \cap R_i = L_i$, $i = 1, \ldots, m$. Let $s' \in \mathbb{N}$ be the greatest number for which there exist $i_1, \ldots, i_{s'} \in \{1, \ldots, m\}$ such that $L \cap \bar{R}_{i_1} \cap \ldots \cap \bar{R}_{i_{s'}}$ is nonregular. Since $L \cap \bar{R}_1 \cap \ldots \cap \bar{R}_m = \varnothing$, $s' < m$.

Without loss of generality we may assume that $i_j = m - s' + j$, $j = 1, \ldots, s'$. Denote $s = m - s'$. If $s < m$, we have $L \cap \bar{R}_{s+1} \cap \ldots \cap \bar{R}_m$ nonregular and the language $L \cap \bar{R}_{s+1} \cap \ldots \cap \bar{R}_m \cap \bar{R}_j$ regular for each $j \in \{1, \ldots, s\}$. If $s = m$, then $L \cap \bar{R}_j$ is regular for each $j \in \{1, \ldots, m\}$. If $s < m$, denote $R = \bar{R}_{s+1} \cap \ldots \cap \bar{R}_m$, otherwise $R = \{a_1, \ldots, a_n\}^*$. Now:

$$L \cap R = \left(\bigcup_{i=1}^{s} L_i\right) \cap R$$

is nonregular and $L \cap R \in c(\mathscr{R})$. For each $i \in \{1, \ldots, s\}$ there are $A_{i1}, \ldots, A_{ir_i} \in c(\mathscr{R})$ such that $\Psi(A_{ij})$ is a proper linear set, $j = 1, \ldots, r_i$, and $L_i \cap R = \bigcup_{j=1}^{r_i} A_{ij}$. We prove that for each $i \in \{1, \ldots, s\}$:

$$\bigcup_{j=1}^{r_i} (\operatorname{conv}(A_{ij}) \cap R \cap R_i) = \bigcup_{j=1}^{r_i} A_{ij}.$$

Obviously $A_{ij} \subseteq L_i \subseteq R_i$ and $A_{ij} \subseteq R$, so the right side of the above equation is a subset of the left side of it. On the other hand, since $\Psi(L_i)$ is a linear set and $A_{ij} \subseteqq L_i$ for each $j \in \{1, \ldots, r_i\}$, it can be verified that $\operatorname{conv}(A_{ij}) \subseteqq \operatorname{conv}(L_i)$. Thus $\operatorname{conv}(A_{ij}) \cap R_i \subseteqq \operatorname{conv}(L_i) \cap R_i = L_i$, so:

$$\operatorname{conv}(A_{ij}) \cap R \cap R_i \subseteqq L_i \cap R = \bigcup_{j=1}^{r_i} A_{ij},$$

and we can deduce that the equation is right for each $i \in \{1, \ldots, s\}$. Since $L \cap R \cap \bar{R}_i$ is regular for each $i \in \{1, \ldots, s\}$, the language $L \cap R \cap \left(\bigcup_{i=1}^{s} \bar{R}_i\right)$ is regular. Since $L \cap R$ is nonregular, the language:

$$L \cap R \cap \overline{\left(\bigcup_{i=1}^{s} \bar{R}_i\right)} = L \cap R \cap \left(\bigcap_{i=1}^{s} \bar{\bar{R}}_i\right)$$
$$= L \cap (R_1 \cap \ldots \cap R_s \cap \bar{R}_{s+1} \cap \ldots \cap \bar{R}_m),$$

in $c(\mathscr{R})$ is nonregular. Denote $R_0 = R_1 \cap \ldots \cap R_s \cap \bar{R}_{s+1} \cap \ldots \cap \bar{R}_m$. By the previous note, R_0 is fundamental. For each $i \in \{1, \ldots, s\}$, $j \in \{1, \ldots, r_i\}$, let $A_{ijp} \in c(\mathscr{R})$, $p = 1, \ldots, q_{ij}$, be such that

$$A_{ij} \cap R_0 = \bigcup_{p=1}^{q_{ij}} A_{ijp}$$

and $\Psi(A_{ijp})$ is a proper linear set. We prove that for each $i \in \{1, \ldots, s\}$:

$$(*) \qquad \bigcup_{j=1}^{r_i} \bigcup_{p=1}^{q_{ij}} (\operatorname{conv}(A_{ijp}) \cap R_0) = \bigcup_{j=1}^{r_i} \bigcup_{p=1}^{q_{ij}} A_{ijp}.$$

Obviously the right side of $(*)$ is a subset of the left side of $(*)$. On the other hand:

$$\operatorname{conv}(A_{ijp}) \cap R_0 \subseteqq \bigcup_{j=1}^{r_i} (\operatorname{conv}(A_{ijp}) \cap R_0)$$
$$= \bigcup_{j=1}^{r_i} (A_{ij} \cap R_0) = \bigcup_{j=1}^{r_i} \bigcup_{p=1}^{q_{ij}} A_{ijp}.$$

Thus $(*)$ is right. Now:

$$L' = L \cap R_0 = \bigcup_{i=1}^{s} \bigcup_{j=1}^{r_i} \bigcup_{p=1}^{q_{ij}} A_{ijp} \subseteqq \{a_1, \ldots, a_n\}^*$$

is a nonregular homogenous language in $\mathcal{T}(L)$. $\square$

LEMMA 4: *Let $S_1, \ldots, S_m \subseteq \mathbb{N}^n$ be proper linear sets such that*

$$\operatorname{rank}(\overline{\operatorname{conv}(S_1)} \cap \ldots \cap \overline{\operatorname{conv}(S_m)}) = n.$$

Then there exists a proper linear set $T \subseteq \mathbb{N}^n$ such that:

$$\operatorname{conv}(T) \subseteq \overline{\operatorname{conv}(S_1)} \cap \ldots \cap \overline{\operatorname{conv}(S_m)} \qquad \textit{and} \qquad \operatorname{rank}(T) = n.$$

Proof: Denote $T' = \overline{\operatorname{conv}(S_1)} \cap \ldots \cap \overline{\operatorname{conv}(S_m)}$. Since $\operatorname{rank}(T') = n$, there exists a linear set $T_1 \subseteq T'$ such that:

$$T_1 = \{v_0 + k_1 v_1 + \ldots + k_n v_n \mid k_j \in \mathbb{N}, j = 1, \ldots, n\}$$

where $v_i \in \mathbb{N}^n$, $i = 0, 1, \ldots, n$, and the elements $v_1, \ldots, v_n$ are linearly independent. Let:

$$S_1 = \{u_0 + k_1 u_1 + \ldots + k_s u_s \mid k_j \in \mathbb{N}, j = 1, \ldots, s\},$$

where $u_i \in \mathbb{N}^n$, $i = 0, 1, \ldots, s$, and the elements $u_1, \ldots, u_s$ are linearly independent. Let:

$$V = \{k_1 v_1 + \ldots + k_n v_n \mid k_j \in \mathbb{N}, j = 1, \ldots, n\},$$
$$U = \{k_1 u_1 + \ldots + k_s u_s \mid k_j \in \mathbb{N}, j = 1, \ldots, s\}.$$

We have two subcases: (i) $s = n$; and (ii) $s < n$.

(i) Assume there is $u \in U$ such that $u = \alpha_1 v_1 + \ldots + \alpha_n v_n$ for some positive $\alpha_i \in Q$. Then, for sufficiently large and well chosen $p, q \in \mathbb{N}_+$:

$$pu + q(u_1 + \ldots + u_n) = \beta_1 v_1 + \ldots + \beta_n v_n$$

for some $\beta_i \in \mathbb{N}_+$, $i = 1, \ldots, n$. If now $r \in \mathbb{N}$ is large enough:

$$v_0 + r(\beta_1 v_1 + \ldots + \beta_n v_n) = u_0 + \gamma_1 u_1 + \ldots + \gamma_n u_n$$

for some positive $\gamma_j \in Q$, $j = 1, \ldots, n$, contradicting the fact that $T_1 \subseteq T'$. The facts above show that, for sufficiently large $r_0 \in \mathbb{N}_+$:

$$v_0 + r_0(v_1 + \ldots + v_n) + \rho_1 v_1 + \ldots + \rho_n v_n \neq u_0 + \xi_1 u_1 + \ldots + \xi_n u_n,$$

for each nonnegative ρ_j, $\xi_j \in Q$, $j = 1, \ldots, n$. Let $w_0 = v_0 + r_0(v_1 + \ldots + v_n)$. Then $T_2 = w_0 + V$ is a proper linear set such that $\operatorname{conv}(T_2) \cap \operatorname{conv}(S_1) = \text{k}$.

(ii) By the construction of Lemma 1, $\operatorname{rank}(\operatorname{conv}(T_1) \cap \operatorname{conv}(S_1)) \leqq s < n$.

Let $W_1, \ldots, W_p \subseteq \mathbb{N}^n$ be linear sets such that $\operatorname{conv}(T_1) \cap \operatorname{conv}(S_1) = \bigcup_{i=1}^{p} W_i$ and

$$W_i = \{v_{i0} + k_1 v_{i1} + \ldots + k_{r_i} v_{ir_i} \mid k_j \in \mathbb{N}, j=1, \ldots, r_i\},$$

where $v_{ij} \in \mathbb{N}^n$, $j=0, 1, \ldots, r_i$, $i=1$, $i=1, \ldots, p$. Since each v_{ij} is obviously a linear combination of the elements $u_1, \ldots, u_s$, there are at most s linearly independent elements in $v_{11}, \ldots, v_{1r_1}, \ldots, v_{p1}, \ldots, v_{pr_p}$. Let $w_1, \ldots, w_q$ be a maximal number of linearly independent elements in the above sequence, $q \leqq s$. Thus $q < n$. Let $w_{q+1}, \ldots, w_n$ be elements in $v_1, \ldots, v_n$ such that $w_1, \ldots, w_n$ are linearly independent. Let r_0 be such that $v_0 - v_{i0} + r_0 (w_1 + \ldots + w_n)$ is a linear combination of $w_1, \ldots, w_n$ with positive rational coefficients for each $i=1, \ldots, p$:

$$w_0 = v_0 + r_0 (w_1 + \ldots + w_n),$$

and

$$T_2 = \{w_0 + k_1 w_1 + \ldots + k_n w_n \mid k_j \in \mathbb{N}, j=1, \ldots, n\}.$$

We show that $\operatorname{conv}(T_2) \cap \operatorname{conv}(S_1) = \varnothing$. Assume the contrary. Since $\operatorname{conv}(T_2) \subseteq \operatorname{conv}(T_1)$, we have $\operatorname{conv}(T_2) \cap (\operatorname{conv}(T_1) \cap \operatorname{conv}(S_1)) \neq \varnothing$ which means:

$$(1) \quad v_0 + r_0 (w_1 + \ldots + w_n) + \alpha_1 w_1 + \ldots + \alpha_n w_n = v_{i0} + \beta_1 v_{i1} + \ldots + \beta_{r_i} v_{ir_i},$$

for some $i \in \{1, \ldots, p\}$, $\alpha_j \in Q$, $\alpha_j \geqq 0$, $\beta_l \in \mathbb{N}$, $j=1, \ldots, n$, $l=1, \ldots, r_i$. Obviously:

$$\beta_1 v_{i1} + \ldots + \beta_{r_i} v_{ir_i} = \lambda_1 w_1 + \ldots + \lambda_q w_q$$

for some $\lambda_j \in Q$, $j=1, \ldots, q$. Then (1) implies that $\xi_1 w_1 + \ldots + \xi_n w_n = \bar{0}$ for some $\xi_j \in Q$, $j=1, \ldots, n$, where $\xi_{q+1} \neq 0, \ldots, \xi_n \neq 0$. Since $w_1, \ldots, w_n$ are linearly independent, we have a contradiction. Thus $T_2 \subseteq \mathbb{N}^n$ is a proper linear set such that $T_2 \subseteq T'$ and $\operatorname{conv}(T_2) \cap \operatorname{conv}(S_1) = \varnothing$.

Continuing like this for each S_j, $j=2, \ldots, m$, we can find a proper linear set T_{m+1} such that $T_{m+1} \subseteq T'$ and

$$\operatorname{conv}(T_{m+1}) \cap (\operatorname{conv}(S_1) \cup \ldots \cup \operatorname{conv}(S_m)) = \varnothing,$$

thus:

$$\operatorname{conv}(T_{m+1}) \subseteq T' = \overline{\operatorname{conv}(S_1)} \cap \ldots \cap \overline{\operatorname{conv}(S_m)}. \quad \square$$

LEMMA 5: *Let* $L \subseteqq \{a_1, \ldots, a_n\}^*$ *be a homogenous language containing* ε. *Assume* $S_1, \ldots, S_m \subseteqq \mathbb{N}^n$ *are proper linear sets and* $U \subseteqq \mathbb{N}^n$ *is a fundamental semilinear set such that:*

$$\Psi(L) = \bigcup_{i=1}^{m} S_i \quad \text{and} \quad \left(\bigcup_{i=1}^{m} \operatorname{conv}(S_i)\right) \cap U = \Psi(L).$$

If $\operatorname{rank}(\Psi(L)) = \operatorname{rank}(\overline{\operatorname{conv}(S_1)} \cap \ldots \cap \overline{\operatorname{conv}(S_m)}) = n$, *then the language* $\bar{D}_1^*$ *is in* $\mathscr{T}(L)$.

Proof: There must be S_i, say S_1, such that:

$$S_1 = \{u_0 + k_1 u_1 + \ldots + k_n u_n \mid k_j \in \mathbb{N}, j = 1, \ldots, n\},$$

where $u_j \in \mathbb{N}^n$, $j = 0, 1, \ldots, n$, with $u_1, \ldots, u_n$ linearly independent. Denote $T = \overline{\operatorname{conv}(S_1)} \cap \ldots \cap \overline{\operatorname{conv}(S_m)}$. By the previous lemma, there exists:

$$T_1 = \{v_0 + k_1 v_1 + \ldots + k_n v_n \mid k_j \in \mathbb{N}, j = 1, \ldots, n\},$$

where $v_i \in \mathbb{N}^n$, $i = 0, 1, \ldots, n$, with $v_1, \ldots, v_n$ linearly independent such that $\operatorname{conv}(T_1) \subseteqq T$. Naturally $\operatorname{conv}(S_1) \cap \operatorname{conv}(T_1) = \varnothing$. Now there must be $U_1 \subseteqq U$ such that:

$$U_1 = \{w_0 + k_1 s_1 e_1 + \ldots + k_n s_n e_n \mid k_j \in \mathbb{N}, j = 1, \ldots, n\},$$

for some $w_0 \in \mathbb{N}^n$, $s_j \in \mathbb{N}_+$, $j = 1, \ldots, n$. Denote:

$$s = s_1 \ldots s_n, \qquad u = u_1 + \ldots + u_n, \qquad v = v_1 + \ldots + v_n,$$
$$w_1 = su, \qquad w_2 = sv.$$

Obviously the set $U_2 = \{w_0 + k_1 w_1 + k_2 w_2 \mid k_1, k_2 \in \mathbb{N}\}$ is a subset of $U_1 \subseteqq U$. Now $w_2 = \alpha_1 u_1 + \ldots + \alpha_n u_n$ for some $\alpha_i \in Q$, $i = 1, \ldots, n$. Since $\operatorname{conv}(S_1) \cap \operatorname{conv}(T_1) = \varnothing$, there is at least one $j \in \{1, \ldots, n\}$ such that $\alpha_j < 0$. Let:

$$\alpha = \max\{|\alpha_j| \mid \alpha_j < 0, j = 1, \ldots, n\}.$$

Let $m_1 \in \mathbb{N}$ be the smallest integer such that $w_0 + m_1 w_1 \in \operatorname{conv}(S_1)$. Such a number m_1 clearly exists. Consider the statemant:

$$(1) \qquad w_0 + k_1 w_1 + k_2 w_2 \in \operatorname{conv}(S_1), \qquad k_1, k_2 \in \mathbb{N}.$$

Then (1) is equivalent with:

$$(2) \qquad w_0 + m_1 w_1 + (k_1 - m_1) w_1 + k_2 w_2 \in \operatorname{conv}(S_1), \qquad k_1, k_2 \in \mathbb{N}.$$

If $(s/\alpha)(k_1-m_1) > k_2$, then (1) is true. Now $(s/\alpha)(k_1-m_1) > k_2$ is equivalent with $k_1 > (\alpha/s)k_2+m_1$. It is obvious that there are arbitrarily large $k_1, k_2 \in \mathbb{N}$ such that $k_1 > (\alpha/s)k_2+m_1$ and k_1/k_2 is arbitrarily near to α/s.

The element w_1 can be written in the form $w_1=\beta_1 v_1+\ldots+\beta_n v_n$ for some $\beta_i \in Q$, $i=1, \ldots, n$. Since $\operatorname{conv}(S_1) \cap \operatorname{conv}(T_1)=\varnothing$, there is at least one $j \in \{1, \ldots, n\}$ such that $\beta_j < 0$. Define:

$$\beta=\max\{|\beta_j| \mid \beta_j < 0, j=1, \ldots, n\}.$$

Let $m_2 \in \mathbb{N}$ be the smallest integer such that $w_0+m_2 w_2 \in \operatorname{conv}(T_1)$. Consider the statement:

$$(3) \qquad w_0+k_1 w_1+k_2 w_2 \in \operatorname{conv}(T_1), \qquad k_1, k_2 \in \mathbb{N}.$$

Now (3) is equivalent with:

$$(4) \qquad w_0+m_2 w_2+k_1 w_1+(k_2-m_2) w_2 \in \operatorname{conv}(T_1), \qquad k_1, k_2 \in \mathbb{N},$$

which is true if $k_1 < (s/\beta)k_2-(s/\beta)m_2$.

Let $x_i \in \{a_1, \ldots, a_n\}^*$ be words such that $\Psi(x_i)=w_i$, $i=0, 1, 2$. It should be clear that the language $x_0^{-1}L \subseteq \{a_1, \ldots, a_n\}^*$ is a commutative SLIP-language in $\mathcal{T}(L)$. Let $h: \{a_1, a_2\}^* \to \{a_1, \ldots, a_n\}^*$ be a morphism defined by $h(a_i)=x_i$, $i=1, 2$. Then $L_1=h^{-1}(x_0^{-1}L) \subseteq \{a_1, a_2\}^*$ is a commutative SLIP-language by the results in [7]. Obviously $L_1 \in \mathcal{T}(L)$. By the results of [4] and [9] it suffices to show that L_1 is nonregular.

Assume that $x \in L_1$. Then $x \in c(a_1^p a_2^q) \subseteq h^{-1}(x_0^{-1}L)$ for some $p, q \in \mathbb{N}$. Then $h(a_1^p a_2^q) \in x_0^{-1}L$ which implies that $x_0 h(a_1^p a_2^q)=x_0 x_1^p x_2^q \in L$. Now:

$$\Psi(x_0 x_1^p x_2^q)=w_0+pw_1+qw_2 \in \Psi(L).$$

Since $(\operatorname{conv}(S_1) \cup \ldots \cup \operatorname{conv}(S_m)) \cap T=\varnothing$, we must have $p \geqq (s/\beta)q-(s/\beta)m_2$.

On the other hand we can find arbitrarily large $p', q' \in \mathbb{N}$ such that p'/q' is arbitrarily near to α/s and

$$w_3=w_0+p' w_1+q' w_2 \in \operatorname{conv}(S_1).$$

Since $w_3 \in U$, $w_3 \in \Psi(L)$. Obviously $c(x_1^{p'} x_2^{q'}) \subseteq x_0^{-1}L$ and thus:

$$c(a_1^{p'} a_2^{q'}) \subseteq L_1=h^{-1}(x_0^{-1}L).$$

Now, if L_1 were regular, then we could find (by the pumping properties of regular languages), $r_j \in \mathbb{N}$, $j=1, 2, 3, 4$, r_2, $r_3 \neq 0$, such that $a_1^{r_1}(a_1^{r_2})^*(a_2^{r_3})^* a_2^{r_4} \subseteq L_1$. This contradicts the fact that $p \geqq (s/\beta)q-(s/\beta)m_2$ for each $a_1^p a_2^q \in L_1$. $\square$

A semilinear set $S \subseteqq \mathbb{N}^n$ is *unlimited* if for each $m \in \mathbb{N}$ there exists $(m_1, \ldots, m_n) \in S$ such that $m_j > m$, $j = 1, \ldots, n$.

LEMMA 6: *Assume $L \subseteqq \{a_1, \ldots, a_n\}^*$ is a commutative SLIP-language containing ε such that the rank of $S = \Psi(L)$ is s, $s < n$, and S is unlimited. Then $D_1^* \in \mathscr{T}(L)$.*

Proof: Assume $S_1, \ldots, S_m \subseteqq \mathbb{N}^n$ are proper linear sets such that $S = \bigcup_{i=1} S_i$ and:

$$S_i = \{u_{i0} + k_1 u_{i1} + \ldots + k_{r_i} u_{ir_i} \mid k_j \in \mathbb{N}, j = 1, \ldots, r_i\},$$

$u_{i_j} \in \mathbb{N}^n$, $j = 0, 1, \ldots, r_i$, with the vectors $u_{i1}, \ldots, u_{ir_i}$ linearly independent, $r_i \leqq s$, $i = 1, \ldots, m$. Since S is unlimited, there exists $q \in \{1, \ldots, m\}$ such that $u_{q1} + \ldots + u_{qr_q} \in \mathbb{N}_+^n$. Choose q in such a way that $\mathscr{A}(S_q)$ is not a proper subset of $\mathscr{A}(S_j)$ for any $j \in \{1, \ldots, m\}$. Let K be the set of all $k \in \{1, \ldots, m\}$ such that either:

(i) $\mathscr{A}(S_k)$ is not a subset of $\mathscr{A}(S_q)$; or

(ii) $\mathscr{A}(S_k) \subseteqq \mathscr{A}(S_q)$ and $u_{k0} \notin u_{q0} + \mathscr{A}(S_q)$.

Now there must be $w_0 \in S_q$ such that $w_0 \notin u_{k0} + \mathscr{A}(S_k)$ for any $k \in K$. For S_k satisfying (ii) this is certainly true since $(u_{q0} + \mathscr{A}(S_q)) \cap (u_{k0} + \mathscr{A}(S_k)) = \emptyset$. Let $K' \subseteqq K$ be the set of all k such that S_k satisfies (i). Assume for each $w_0 \in S_q$, $w_0 \in u_{k0} + \mathscr{A}(S_k)$ for some $k \in K'$. Then $\mathscr{A}(S_q) \subseteqq \bigcup_{k \in K'} \mathscr{A}(S_k)$ and $\mathscr{A}(S_q) = \bigcup_{k \in K'} (\mathscr{A}(S_k) \cap \mathscr{A}(S_q))$. The elementary results of linear algebra then imply that there exists $k' \in K'$ such that:

$$\mathscr{A}(S_q) = \mathscr{A}(S_{k'}) \cap \mathscr{A}(S_q) \subseteqq \mathscr{A}(S_{k'}).$$

Then $\mathscr{A}(S_q) \nsubseteqq \mathscr{A}(S_{k'})$ contradicting the choice of q.

Let $k \in K$ and $t \in \mathbb{N}_+$ be fixed. Consider the equation:

$$(1) \qquad w_0 + \alpha_1 t e_1 + \ldots + \alpha_n t e_n = u_{k0} + \beta_1 u_{k1} + \ldots + \beta_{r_k} u_{kr_k}$$

where α_i, $\beta_j \in \mathbb{N}$, $i = 1, \ldots, n$, $j = 1, \ldots, r_k$. The equation (1) is equivalent with:

$$(2) \qquad u_{k0} - w_0 + \beta_1 u_{k1} + \ldots + \beta_{r_k} u_{kr_k} = (\alpha_1 t, \ldots, \alpha_n t).$$

Since $w_0 \notin u_{k0} + \mathscr{A}(S_k)$, the elements $u_{k0} - w_0, u_{k1}, \ldots, u_{kr_k}$ are linearly independent. Denote $r = r_k$ and:

$$\rho_0 = (\rho_{01}, \ldots, \rho_{0n}) = u_{k0} - w_0, \qquad \rho_j = (\rho_{j1}, \ldots, \rho_{jn}) = u_{kj}, \ j = 1, \ldots, r.$$

Then (2) is equivalent with:

$$\rho_0+\beta_1\rho_1+\ldots+\beta_r\rho_r=(\alpha_1 t, \ldots, \alpha_n t) \tag{3}$$

which is equivalent with:

$$\begin{cases} \rho_{01}+\beta_1\rho_{11}+\ldots+\beta_r\rho_{r1}=\alpha_1 t, \\ \ldots, \\ \rho_{0n}+\beta_1\rho_{1n}+\ldots+\beta_r\rho_{rn}=\alpha_n t, \end{cases} \tag{4}$$

α_i, $\beta_j\in\mathbb{N}$, $i=1, \ldots, n$, $j=1, \ldots, r$. Since the elements $\rho_0, \ldots, \rho_r$ are linearly independent, there are exactly $r+1$ linearly independent elements in $(\rho_{01}, \ldots, \rho_{r1}), \ldots, (\rho_{0n}, \ldots, \rho_{rn})$. Without loss of generality we may assume that the elements:

$$\xi_1=(\rho_{01}, \ldots, \rho_{r1}), \ldots, \xi_{r+1}=(\rho_{0,r+1}, \ldots, \rho_{r,r+1})$$

are such for which $d_0=|\det(\xi_1^T, \ldots, \xi_{r+1}^T)|>0$ is the greatest (x^T meaning the vector transpose of x). Then (4) implies a new system of equations:

$$\begin{cases} \rho_{01}+\beta_1\rho_{11}+\ldots+\beta_r\rho_{r1}=\alpha_1 t, \\ \ldots, \\ \rho_{0,r+1}+\beta_1\rho_{1,r+1}+\ldots+\beta_r\rho_{r,r+1}=\alpha_{r+1} t. \end{cases} \tag{5}$$

Now (5) implies that:

$$1=\frac{\begin{vmatrix} \alpha_1 t & \rho_{11} & \cdots & \rho_{r1} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \alpha_{r+1} t & \rho_{1,r+1} & \cdots & \rho_{r,r+1} \end{vmatrix}}{\begin{vmatrix} \rho_{01} & \rho_{11} & \cdots & \rho_{r1} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \rho_{0,r+1} & \rho_{1,r+1} & \cdots & \rho_{r,r+1} \end{vmatrix}} =t\,\frac{\begin{vmatrix} \alpha_1 & \rho_{11} & \cdots & \rho_{r1} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \alpha_{r+1} & \rho_{1,r+1} & \cdots & \rho_{r,r+1} \end{vmatrix}}{\det(\xi_1^T, \ldots, \xi_{r+1}^T)}.$$

If we choose $t>d_0$, we see that (1) is not true for any α_i, $\beta_j\in\mathbb{N}$, $i=1, \ldots, n$, $j=1, \ldots, r_k$. Thus there exists $t_0\in\mathbb{N}$ such that if $t\geq t_0$, then for any $k\in K$, the inequality:

$$w_0+\alpha_1 te_1+\ldots+\alpha_n te_n\neq u_{k0}+\beta_1 u_{k1}+\ldots+\beta_{r_k} u_{kr_k}$$

Remember that $r_q \leqq s < n$ and $u_{q1} + \ldots + u_{qr_q} \in \mathbb{N}_+^n$. Since $r_q < n$, there must be $d \in \{1, \ldots, n\}$ such that $e_d \notin \mathscr{A}(S_q)$. Let $x_i \in \{a_1, \ldots, a_n\}^*$, $i=0, 1, 2$, be such that $\Psi(x_0) = w_0$, $\Psi(x_1) = t_0(u_{q1} + \ldots + u_{qr_q} - e_d)$ and $\Psi(x_2) = t_0 e_d$. Of course $x_2 = a_d^{t_0}$. Let $h: \{a_1, a_2\}^* \to \{a_1, \ldots, a_n\}^*$ be the morphism defined by $h(a_1) = x_1$ and $h(a_2) = x_2$. Obviously $x_0^{-1} L$ is a commutative language in $\mathscr{T}(L)$. We finish the proof by showing that $D_1^* = h^{-1}(x_0^{-1} L)$.

Assume $x \in D_1^*$. Then $x \in c(a_1^i a_2^i)$ for some $i \in \mathbb{N}$. Since $h^{-1}(x_0^{-1} L)$ is commutative, it suffices to show that $a_1^i a_2^i \in h^{-1}(x_0^{-1} L)$. Now $x_1^i x_2^i \in x_0^{-1} L$ since:

$$\Psi(x_1^i x_2^i) = it_0(u_{q1} + \ldots + u_{qr_q} - e_d) + it_0 e_d = it_0 u_{q1} + \ldots + it_0 u_{qr_q} \in \Psi(x_0^{-1} L).$$

On the other hand, the word $a_1^i a_2^i \in h^{-1}(x_1^i x_2^i)$.

Let $x \in h^{-1}(x_0^{-1} L)$. Then $x \in c(a_1^i a_2^j) \subseteq h^{-1}(x_0^{-1} L)$ for some $i, j \in \mathbb{N}$. Since D_1^* is commutative, it suffices to show that $i=j$. Now $a_1^i a_2^j \in h^{-1}(x_0^{-1} L)$ which implies that $h(a_1^i a_2^j) \in x_0^{-1} L$ and $x_0 h(a^i a_2^j) = x_0 x_1^i x_2^j \in L$. Thus:

$$\Psi(x_0 x_1^i x_2^j) = w_0 + it_0(u_{q1} + \ldots + u_{qr_q} - e_d) + jt_0 e_d = w_0 + t_0 \alpha_1 e_1 + \ldots + t_0 \alpha_n e_n,$$

for some $\alpha_j \in \mathbb{N}$, $j = 1, \ldots, n$. By the choice of t_0, $\Psi(x_0 x^i x_2^j)$ cannot be in S_k for any $k \in K$. For each $l \in \{1, \ldots, m\}$ such that $l \notin K$, $u_{l0} + \mathscr{A}(S_l) \subseteq u_{q0} + \mathscr{A}(S_q)$. This implies that:

$$w_0 + it_0(u_{q1} + \ldots + u_{qr_q}) + jt_0 e_d = u_{q0} + \beta_1 u_{q1} + \ldots + \beta_{r_q} u_{qr_q}$$

for some $\beta_{j'} \in Q$, $j' = 1, \ldots, r_q$. Since $w_0 \in u_{q0} + \mathscr{A}(S_q)$, we have:

$$it_0(u_{q1} + \ldots + u_{qr_q}) - jt_0 e_d \in \mathscr{A}(S_q).$$

Then $i=j$ since otherwise $e_d \in \mathscr{A}(S_q)$, which is a contradiction. The proof is now complete. □

LEMMA 7: *Let $L \subseteq \{a_1, \ldots, a_n\}^*$ be a homogenous language containing* ε. *Assume $T_1, \ldots, T_p \subseteq \mathbb{N}^n$ are proper linear sets and $U \subseteq \mathbb{N}^n$ is a fundamental semilinear set such that $\Psi(L) = \bigcup_{i=1}^{p} T_i$ and $\left(\bigcup_{i=1}^{p} \operatorname{conv}(T_i)\right) \cap U = \Psi(L)$. If the rank of the set $S = \left(\bigcap_{i=1}^{p} \overline{\operatorname{conv}(T_i))}\right) \cap U$ is smaller than n, and S is unlimited, then $\bar{D}_1^*$ is in $\mathscr{T}(L)$.*

Proof: Assume $\operatorname{rank}(S) = s$, $s < n$. The beginning of the proof is an exact copy of the proof for Lemma 6. Assume $S_1, \ldots, S_m \subseteq \mathbb{N}^n$ are proper linear sets such that $S=\bigcup_{i=1}^{m} S_i$ and for each $i\in\{1, \ldots, m\}$:

$$S_i=\{u_{i0}+k_1 u_{i1}+\ldots+k_{r_i} u_{ir_i} \mid k_j\in\mathbb{N}, j=1, \ldots, r_i\},$$

where $u_{ij}\in\mathbb{N}^n$, $j=0, 1, \ldots, r_i$, and the elements $u_{i1}, \ldots, u_{ir_i}$ are linearly independent. Let $q\in\{1, \ldots, m\}$, $K\subseteq\{1, \ldots, m\}$ and $w_0\in S_q$ be as in the proof of Lemma 6. By an analogous reasoning as in the proof of Lemma 6 we can find t_0 with the following property. If $t\geqq t_0$, then for each $k\in K$, the inequality:

$$(1) \qquad w_0+\alpha_1 te_1+\ldots+\alpha_n te_n\neq u_{k0}+\beta_1 u_{k1}+\ldots+\beta_{r_k} u_{kr_k}$$

holds for all α_i, $\beta_j\in\mathbb{N}$, $i=1, \ldots, n$, $j=1, \ldots, r_k$. Since U is fundamental, there exists $U_1\subseteq U$ such that w_0 is in U_1 and:

$$U_1=\{v_0+k_1(m_1, 0, \ldots, 0)+\ldots+k_n(0, \ldots, 0, m_n) \mid k_j\in\mathbb{N}, j=1, \ldots, n\},$$

for some $v_0\in\mathbb{N}^n$, $m_j\in\mathbb{N}_+$, $j=1, \ldots, n$. Let $t'=t_0 m_1\ldots m_n$.

Now $r_q\leqq s<n$ and $u_{q1}+\ldots+u_{qr_q}\in\mathbb{N}_+^n$. Since $r_q<n$, there must be $d\in\{1, \ldots, n\}$ such that $e_d\notin\mathscr{A}(S_q)$. Let $x_i\in\{a_1, \ldots, a_n\}^*$, $i=0, 1, 2$, be such that $\Psi(x_0)=w_0$, $\Psi(x_1)=t'(u_{q1}+\ldots+u_{qr_q}-e_d)$ and $\Psi(x_2)=t' e_d$. Obviously $x_2=a_d^{t'}$. Let h: $\{a_1, a_2\}^*\to\{a_1, \ldots, a_n\}^*$ be the morphism defined by $h(a_1)=x_1$ and $h(a_2)=x_2$. Clearly $x_0^{-1}L\subseteq\{a_1, \ldots, a_n\}^*$ is a commutative language in $\mathscr{T}(L)$. We show that $\bar{D}_1^*=h^{-1}(x_0^{-1}L)$.

Assume $a_1^i a_2^i\in h^{-1}(x_0^{-1}L)$ for some $i\in\mathbb{N}$. Then $h(a_1^i a_2^i)\in x_0^{-1}L$ which implies that $x_0 h(a_1^i a_2^i)=x_0 x_1^i x_2^i\in L$. This means that:

$$\Psi(x_0 x_1^i x_2^i)=w_0+it'(u_{q1}+\ldots+u_{qr_q}-e_d)+it' e_d=w_0+it'(u_{q1}+\ldots+u_{qr_q})$$

is in $\Psi(L)$, a contradiction, since the above element is clearly in $S_q\subseteq\overline{\Psi(L)}$. Since $h^{-1}(x_0^{-1}L)$ is commutative, we may deduce that $h^{-1}(x_0^{-1}L)\subseteq\bar{D}_1^*$.

Let $x\in\bar{D}_1^*$. Then $x\in c(a_1^i a_2^j)$ for some $i, j\in\mathbb{N}$, $i\neq j$. To prove that $x\in h^{-1}(x_0^{-1}L)$, it suffices to show that $a_1^i a_2^j\in h^{-1}(x_0^{-1}L)$. Consider the element:

$$w_1=w_0+it'(u_{q1}+\ldots+u_{qr_q}-e_d)+jt' e_d.$$

Let $w_2=w_0+it'(u_{q1}+\ldots+u_{qr_q})$. Now $w_1\notin w_0+\mathscr{A}(S_q)$, since otherwise $w_1-w_2=(i-j)e_d\in\mathscr{A}(S_q)$ and (since $i\neq j$) $e_d\in\mathscr{A}(S_q)$, a contradiction. By the choice of t', $w_1 \notin S_k$ for any $k \in K$. This means that:

$$w_1 \notin S = \left(\bigcap_{i=1}^{p} \overline{\operatorname{conv}(T_i)} \right) \cap U.$$

Since $w_1 \in U_1 \subseteqq U$, w_1 must be in $\left(\bigcup_{i=1}^{p} \operatorname{conv}(T_i) \right) \cap U = \Psi(L)$. Now $\Psi(x_0 x_1^i x_2^j) = w_1$. Since L is commutative, the word $x_0 x_1^i x_2^j \in L$, so $x_1^i x_2^j \in x_0^{-1} L$. Obviously $a_1^i a_2^j \in h^{-1}(x_0^{-1} L)$. We deduce that $\bar{D}_1^*$ is a subset of $h^{-1}(x_0^{-1} L)$. □

We are now able to give a proof to Conjecture 1.

THEOREM 1: *Let $L \in c(\mathscr{R})$ be nonregular. Then $\bar{D}_1^*$ is in $\hat{\mathscr{T}}(L)$.*

Proof: Without loss of generality we may assume that $L \subseteqq \{a_1, \ldots, a_k\}^*$, $k \in \mathbb{N}$. We first note that $k \geqq 2$ since each SLIP-language over one symbol is regular. The proof is by induction on k.

Using the results of Berstel and Boasson ([2], [4]) Latteux proves in [9] that the theorem is true when $k = 2$.

Assume that the theorem is true for each $k = 2, 3, \ldots, n-1$, $n > 2$.

Consider the case $k = n$. By Lemma 3 we may assume that L is homogenous. Since $\hat{\mathscr{T}}(L) = \hat{\mathscr{T}}(L \cup \{\varepsilon\})$, we may also assume that L contains ε. Let $S_1, \ldots, S_m$ be linear sets and U a fundamental semilinear set such that:

$$\Psi(L) = \bigcup_{i=1}^{m} S_i \qquad \text{and} \qquad \left(\bigcup_{i=1}^{m} \operatorname{conv}(S_i) \right) \cap U = \Psi(L).$$

Let $T = \bigcap_{i=1}^{m} \overline{\operatorname{conv}(S_i)}$. If If $\operatorname{rank}(\Psi(L)) = \operatorname{rank}(T) = n$, then $\bar{D}_1^* \in \hat{\mathscr{T}}(L)$ by Lemma 5.

Assume first that $\operatorname{rank}(\Psi(L)) = s$, $s < n$. If $\Psi(L)$ is unlimited, then, by Lemma 6, $D_1^* \in \hat{\mathscr{T}}(L)$ which implies that $\bar{D}_1^* \in \hat{\mathscr{T}}(L)$. So assume that $\Psi(L)$ is not unlimited. Then for each $i \in \{1, \ldots, m\}$ there exists $j_i \in \{1, \ldots, n\}$ such that:

$$S_i \subseteq w_i + \mathbb{N}^{j_i - 1} \times \{0\} \times \mathbb{N}^{n - j_i}$$

for some $w_i \in \mathbb{N}^n$. Let $x_1 \in \Psi^{-1}(w_i)$ and $L_i = \Psi^{-1}(S_i)$. Then:

$$L = \bigcup_{i=1}^{m} L_i \qquad \text{and} \qquad L_i \subseteqq c(x_i a_1^* \ldots a_{j_i - 1}^* a_{j_i + 1}^* \ldots a_n^*).$$

Denote $R_i = c(x_i a_1^* \ldots a_{j_i-1}^* a_{j_i+1}^* \ldots a_n^*)$, $i=1, \ldots, m$. Obviously, for each i, R_i is regular. Since $L \subseteq \bigcup_{i=1}^{m} R_i$ and L is nonregular, there must be $q \in \{1, \ldots, m\}$ such that $L \cap R_q$ is nonregular. Now:

$$L \cap R_q \subseteq c(x_q a_1^* \ldots a_{j_q-1}^* a_{j_q+1}^* \ldots a_n^*).$$

The language $L \cap R_q$ above is obviously commutative. Then $L' = x_q^{-1}(L \cap R_q)$ is a nonregular commutative SLIP-language in $\hat{\mathscr{T}}(L)$. On the other hand $L' \subseteq \{a_1, \ldots, a_{j_q-1}, a_{j_q+1}, \ldots, a_n\}^*$. By induction, $\bar{D}_1^*$ is in $\hat{\mathscr{T}}(L') \subseteq \hat{\mathscr{T}}(L)$.

Let now $\operatorname{rank}(T) = s'$, $s' < n$. Let L_i be as above and $R = \Psi^{-1}(U)$. Obviously $R \in c(\mathscr{R})$ is regular. Since:

$$\Psi(L) = \left(\bigcup_{i=1}^{m} \operatorname{conv}(S_i)\right) \cap U$$

and L is commutative, we have $L = \left(\bigcup_{i=1}^{m} \operatorname{conv}(L_i)\right) \cap R$. Since L is nonregular, the language $\left(\bigcap_{i=1}^{m} \overline{\operatorname{conv}(L_i)}\right) \cap R$ is nonregular. Now:

$$\Psi\left(\left(\bigcap_{i=1}^{m} \overline{\operatorname{conv}(L_i)}\right) \cap R\right) = \left(\bigcap_{i=1}^{m} \overline{\operatorname{conv}(S_i)}\right) \cap U = T \cap U.$$

Since $T \cap U \subseteq T$, $\operatorname{rank}(T \cap U) \leq s'$. If $T \cap U$ is unlimited, the theorem is true by Lemma 7. Assume $T \cap U$ is not unlimited. Let $T_1, \ldots, T_r \subseteq \mathbb{N}^n$ be proper linear sets such that $T \cap U = \bigcup_{i=1}^{r} T_i$. Then for each $i \in \{1, \ldots, r\}$ there exists $j_i \in \{1, \ldots, n\}$ such that:

$$T_i \subseteq v_i + \mathbb{N}^{j_i-1} \times \{0\} \times \mathbb{N}^{n-j_i},$$

for some $v_i \in \mathbb{N}^n$. Let $y_i \in \Psi^{-1}(v_i)$. Then:

$$\Psi^{-1}(T_i) \subseteq c(y_i a_1^* \ldots a_{j_i-1}^* a_{j_i+1}^* \ldots a_n^*).$$

Denote $R'_i = c(y_i a_1^* \ldots a_{j_i-1}^* a_{j_i+1}^* \ldots a_n^*)$.Now:

$$\left(\bigcap_{i=1}^{m} \overline{\operatorname{conv}(L_i)}\right) \cap R \subseteq \bigcup_{i=1}^{m} R'_i.$$

Since $\left(\bigcap_{i=1}^{m}\overline{\operatorname{conv}(L_i)}\right)\cap R$ is nonregular, there must be $t\in\{1, \ldots, r\}$ such that $\left(\bigcap_{i=1}^{m}\overline{\operatorname{conv}(L_i)}\right)\cap R\cap R_t$ is nonregular. This implies that the language $L\cap R_t=\left(\bigcup_{i=1}^{m}\operatorname{conv}(L_i)\right)\cap R\cap R_t$ is nonregular (and commutative). Then:

$$L\cap R_t\subseteq c(y_t a_1^*\ldots a_{j_t-1}^* a_{j_t+1}^*\ldots a_n^*).$$

It is easy to see that the language $L''=y^{-1}(L\cap R_t)$ is a nonregular commutative SLIP-language in $\hat{\mathscr{T}}(L)$ and $L''\subseteq\{a_1, \ldots, a_{j_t-1}, a_{j_t+1}, \ldots, a_n\}^*$. By induction, $\bar{D}_1^*\in\hat{\mathscr{T}}(L'')\subseteq\hat{\mathscr{T}}(L)$. $\square$

COROLLARY: *Let $L\in c(\mathscr{R})$ be nonregular. Then $\bar{D}_1^*$ is in $\mathscr{T}(L)$.*

Proof: By the results of Latteux [9], $\hat{\mathscr{T}}(L)=\mathscr{T}(L\cup\{\varepsilon\})$. If $\varepsilon\in L$, then the corollary is clearly true. Assume L does not contain ε. The fact that $\mathscr{T}(L'\cup\{\varepsilon\})=\{L'', L''\cup\{\varepsilon\}\mid L''\in\mathscr{T}(L')\}$ for each ε-free language L' then implies that $\bar{D}_1^*\in\mathscr{T}(L)$. $\square$

Note: Using the techniques of the previous corollary it is easy to see that the assumption that L contains ε in Lemma 5 and Lemma 7 can be removed.

The family QR of quasirational languages is the substitution closure of linear languages. The family QR is also called "derivation bounded languages" and "standard matching choice languages". Let $L\in QR$ be commutative. Since L is a context-free language, $L\in c(\mathscr{R})$. Latteux and Leguy prove in [11] that $\bar{D}_1^*$ is not in QR. By the previous theorem, L must be regular. We can thus state:

THEOREM 2: *Every commutative quasirational language is regular.*

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