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EACH REGULAR CODE IS INCLUDED IN A MAXIMAL REGULAR CODE (*)

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Abstract. — It is proved that each regular code is included in a maximal regular code. A corollary of this result settles an open question from [R].

Résumé. — On prouve que tout code rationnel est contenu dans un code rationnel maximal. Un corollaire de ce résultat répond à une question ouverte posée dans [R].

INTRODUCTION

A language $C \subseteq \Sigma^+$ is called a *code* if C^* is a free submonoid of Σ^* with base C . The theory of codes initiated by M. Schützenberger [Sch] forms an interesting fragment of formal language theory. A code $C \subseteq \Sigma^+$ is called *maximal* if, for any $x \in \Sigma^* - C$, $C \cup \{x\}$ is not a code. All codes are subsets of maximal codes and the investigation of maximal codes forms an active research area within the theory of codes (see, e. g., [BPS], [P1], [R] and [SM]). In particular one is often interested in the problem of the following kind: given a code C of type X (e. g. finite or regular) is it possible to find a maximal code D of type X such that $C \subseteq D$?

It was shown in [R] that for finite codes this question gets a negative answer. Since then the following question remained open: is every finite code included in a maximal regular code? Obviously any finite (resp. regular) prefix code is included in a finite (resp. regular) maximal prefix code. Recently it was shown in [P2] that every *finite biprefix* code is included in a maximal biprefix regular code.

In this paper we provide a positive answer to the above question. As a matter of fact we prove a more general result (theorem 5): each *regular* code is included in a regular maximal code. We would like to emphasize the

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following: *the new result presented in this paper is theorem 5*; most of the other results is in one form or the other (and perhaps in a different terminology) retrievable from the literature. However we have decided to make this paper rather self-contained and to provide all the needed results with their (sometimes different from the literature) proofs carried out in a “uniform manner”.

We assume the reader to be familiar with basic formal language theory—in particular with rudimentary theory of regular languages (*see, e. g., [S]*).

PRELIMINARIES

We use mostly standard language theoretic notation and terminology.

For a set A , $\#A$ denotes the cardinality of A .

For sets A, B , $A - B$ denotes the set theoretic difference of A and B .

For a word x , $|x|$ denotes its length and $first(x)$ denotes the first letter of x ; if $x = x_1 y x_2$ then y is called a *subword* of x (also referred to as a *segment* or a *factor* of x). The set of all subwords of x is denoted by $sub(x)$ and for a language K , $sub(K) = \bigcup_{x \in K} sub(x)$.

A nonempty word x is called *bordered* if $x = yzy$ for a nonempty word y ; otherwise x is called *unbordered*.

A language $C \subseteq \Sigma^+$ is called a *code* if every word $y \in C^+$ satisfies the following condition:

if $y = u_1 \dots u_n$ and $y = x_1 \dots x_m$ for $n, m \geq 1$ and $u_1, \dots, u_n, x_1, \dots, x_m \in C$ then $n = m$ and $u_i = x_i$ for $1 \leq i \leq n$. (In other words, y has a unique representation in C ; subwords u_1, \dots, u_n of this representation are referred to as *C-blocks* of y).

A code $C \subseteq \Sigma^+$ is called *maximal* if, for each $x \in \Sigma^* - C$, $C \cup \{x\}$ is not a code.

In the sequel of this paper we consider an arbitrary but fixed alphabet Σ where $\sigma = \#\Sigma > 1$; all languages we will consider are over Σ .

For a language K and a positive integer n , $L_n(K) = \{w \in K : |w| = n\}$ and $\alpha_n(K) = \#L_n(K)$.

We will define now and recall a number of notions concerning languages—they will be central to our paper.

Let $K \subseteq \Sigma^+$.

(1) K is *dense* if $x \in sub(K^*)$ for each $x \in \Sigma^*$.

(2) K is *fast* if there exists a positive integer n such that for each $w \in \text{sub}(K^*)$ there exist $x, y \in \Sigma^*$ such that $|xy| \leq n$ and $xwy \in K^*$.

(3) K is *rich* if there exists a positive integer e such that $\alpha_m(K^*) \geq \sigma^m/e$ for infinitely many positive integers m .

RESULTS

In this section we investigate the problem how various properties of a code (such as: fast, dense, rich, regular and maximal) influence each other. Once this relationship is explored we can settle the problem of completing a regular code to a regular maximal code.

Our first result is known (see [SM]). However for the sake of completeness we provide its proof (which is different from the proof in [SM]).

THEOREM 1: *Each maximal code is dense.*

Proof: First we prove the following result.

CLAIM 1: *Let C be a code that is not dense. There exists an unbordered word w_c such that $w_c \notin \text{sub}(C^*)$.*

Proof of Claim 1: Since C is not dense, there exists a word $z \notin \text{sub}(C^*)$. Let $b \in \Sigma$ be such that $b \neq \text{first}(z)$ and let $w_c = zb^{|\text{z}|}$. Clearly w_c is unbordered. Moreover $w_c \notin \text{sub}(C^*)$, because $z \notin \text{sub}(C^*)$.

Thus claim 1 holds. ■

Now we prove theorem 1 as follows.

Let C be a maximal code.

Assume to the contrary that C is not dense. Then let w_c be an unbordered word satisfying the statement of claim 1.

Consider $D = C \cup \{w_c\}$. Let y be an arbitrary word in D^+ . Since w_c is unbordered, y has a unique representation of the form $y = x_0 w_c x_1 w_c \dots w_c x_n$, where $n \geq 0$ (that is if $y = u_0 w_c u_1 w_c \dots w_c u_m$ where $m \geq 0$ then $m = n$ and $u_i = x_i$ for $1 \leq i \leq n$). Since C is a code and $w_c \notin \text{sub}(C^*)$, y has a unique representation in D . Thus D is a code.

Since $C \subseteq D$ and $w_c \notin \text{sub}(C^*)$ we get a contradiction (to the fact that C is maximal).

Consequently C must be dense and theorem 1 holds. ■

THEOREM 2: *Each rich code is maximal.*

Proof: Let C be a rich code and let e be a positive integer constant satisfying the definition of richness for C .

Assume to the contrary that C is not maximal. Let z be a word such that $B = C \cup \{z\}$ is a code; let $|z| = t$.

Let k be a positive integer. Let n_1, \dots, n_k be a sequence of positive integers such that:

$$n_1 < n_2 < \dots < n_k \quad \text{and} \quad \alpha_{n_i}(C^*) \geq \frac{\sigma^{n_i}}{e}. \quad (1)$$

(Since C is rich and e satisfies the definition of richness of C , such a sequence exists.)

Consider $r = n_1 + n_2 + \dots + n_k + kt$. Clearly:

$$\alpha_r(B^*) \leq \sigma^r. \quad (2)$$

On the other hand let us consider an arbitrary permutation i_1, \dots, i_k of the set $\{1, \dots, k\}$. Let $y_{i_1} \in L_{n_{i_1}}(C^*), \dots, y_{i_k} \in L_{n_{i_k}}(C^*)$ and let $\gamma(i_1, \dots, i_k) = y_{i_1} z y_{i_2} z \dots y_{i_k} z$. Since B is a code, if (j_1, \dots, j_k) is a permutation of $\{1, \dots, k\}$ different from (i_1, \dots, i_k) , then $\gamma(i_1, \dots, i_k) \neq \gamma(j_1, \dots, j_k)$. Consequently from (1) it follows that:

$$\frac{\sigma^{n_1}}{e} \frac{\sigma^{n_2}}{e} \dots \frac{\sigma^{n_k}}{e} k! \leq \alpha_r(B^*). \quad (3)$$

From (2) and (3) it follows that:

$$k! \leq e^k \sigma^{tk} = (e \sigma^t)^k. \quad (4)$$

Since $e \sigma^t$ is a constant (independent of k), there exists a positive integer k_0 such that, for all $s > k_0$, $s! > (e \sigma^t)^s$. Consequently (4) yields a contradiction (k was chosen to be an arbitrary positive integer).

Thus C must be maximal and theorem 2 holds. ■

THEOREM 3: *Each regular code is fast.*

Proof: Obvious. ■

THEOREM 4: *Each dense and fast code is rich.*

Proof: Let C be a code that is dense and fast. Then there exists a finite set F of ordered pairs of words from Σ^* such that for each $w \in \Sigma^*$ there exists $(x, y) \in F$ such that $xwy \in C^*$. Let $q = \max\{|xy| : (x, y) \in F\}$, $f = \#F$ and $d = f \sigma^q$.

CLAIM 2: *For each positive integer n there exists a positive integer $m \leq n + q$ such that $\alpha_m(C^*) \geq \sigma^m/d$.*

Proof of claim 2: Let for each $w \in \Sigma^*$, $pair(w)$ be a fixed element (x, y) of F such that $xwy \in C^*$.

Let n be a positive integer. Let:

$$E(n, x, y) = \{w \in L_n(\Sigma^*) : pair(w) = (x, y)\}.$$

Clearly for some $(x_0, y_0) \in F$, $\# E(n, x_0, y_0) \geq \sigma^n/f$. Let $p = |x_0 y_0|$. Then $\alpha_{n+p}(C^*) \geq \# E(n, x_0, y_0) \geq \sigma^n/f$.

Hence:

$$\alpha_{n+p}(C^*) \geq \frac{\sigma^n}{f} = \frac{\sigma^{n+p}}{f\sigma^p} \geq \frac{\sigma^{n+p}}{f\sigma^q} \geq \frac{\sigma^{n+p}}{d}.$$

Thus if we choose $m = n + p$ we get $m \leq n + q$ and claim 2 holds. ■

Now theorem 4 follows directly from claim 2. ■

REMARK: Theorems 2 and 4 together are more general than theorem 7.4 (due to Schutzenberger) from [E]. However, it is pointed out by D. Perrin in [P3] that a proof of the general case can be retrieved from the proof of theorem 9.3 in [E]. ■

THEOREM 5: *Let C be a regular code. There exists a code D which is dense, fast, regular and such that $C \subseteq D$.*

Proof: Let C be a regular code.

We consider separately two cases.

(i) C is dense.

Then the theorem follows from theorem 3 (take $D = C$).

(ii) C is not dense.

Then, by claim 1, there exists an unbordered word w_c such that $w_c \notin \text{sub}(C^*)$.

Let:

$$A = \{w_c x_1 w_c x_2 \dots w_c x_n w_c : n \geq 1, x_i \notin C^* \text{ and } w_c \notin \text{sub}(x_i)\}$$

and let $D = C \cup \{w_c\} \cup A$.

CLAIM 3: D is a code.

Proof of Claim 3: Let $y \in D^+$. Since w_c is unbordered, y has a unique representation of the form $y = x_1 w_c x_2 w_c \dots w_c x_n$ (that is we can uniquely distinguish all occurrences of w_c in y).

This representation provides the basis for the division of y into D -blocks which is obtained as follows:

(1) A subword $w_c x_j w_c x_{j+1} \dots w_c x_{j+l} w_c$ constitutes a D -block (corresponding to A) if $2 \leq j \leq n-1$, $j+l \leq n-1$, $x_j, \dots, x_{j+l} \notin C^*$ and $x_{j-1}, x_{j+l+1} \in C^*$; such a D -block is referred to as an A -block.

(2) All occurrences of w_c not involved in A -blocks are also D -blocks.

(3) All x_i 's which are not involved in A -blocks must be in C^* and so they are uniquely divisible in D -blocks (really C -blocks).

The definition of A and the fact that $w_c \notin \text{sub}(C^*)$ and w_c is unbordered guarantee that such a division is unique.

Hence D is a code and claim 3 holds. ■

CLAIM 4: D is dense.

Proof of claim 4: Let $u \in \Sigma^*$.

Consider $y = w_c u w_c$. Reasoning as in the proof of claim 3 we get a (unique) representation of y in D^+ .

Thus D is dense and claim 4 holds. ■

CLAIM 5: D is regular.

Proof: Obvious. ■

CLAIM 6: D is fast.

Proof: This follows from claim 5 and theorem 3. ■

Now theorem 5 follows from claims 3 through 5. ■

Our results yield two interesting corollaries. The first one solves an open problem from the theory of codes (see, e. g., [R] and [P2]). As a matter of fact it provides a more general result: Restivo has asked ([R]) whether an arbitrary *finite* code can be completed to a maximal regular code—we show that even an arbitrary *regular* code can be completed to a maximal regular code.

COROLLARY 1: *Let C be a code. If C is regular, then there exists a code D such that $C \subseteq D$, D is maximal and D is regular.*

Proof: Let C be a regular code.

By theorem 5 there exists a regular code D such that $C \subseteq D$, D is fast and dense.

Thus, by theorem 4, D is rich and so, by theorem 2, D is maximal.

Hence corollary 1 holds. ■

Secondly, we notice that theorems 1 through 4 provide an alternative proof of the theorem by Schutzenberger (see [E], p. 94).

COROLLARY 2: *Let C be a regular code. Then C is maximal if and only if C is dense.*

Proof: It follows directly from theorems 1 through 4. ■

DISCUSSION

We have established a number of relationships between dense, fast, rich, maximal and regular codes. Using these relationships we were able to demonstrate that each regular code is included in a maximal regular code.

In particular we have demonstrated that each rich code is maximal and each maximal code is dense. Hence each rich code is dense. We provide now a “direct” proof of this result—we believe it sheds a different light on this relationship.

COROLLARY 3: *Each rich code is dense.*

Proof: Let C be a rich code.

Assume that C is not dense. Hence there exists a word $z \notin \text{sub}(C^*)$; let $|z|=t$. Let n be an arbitrary positive integer; n can be represented in the form $n=k_1 t+k_2$ for some $k \geq 0$ and $k_2 < t$. An arbitrary word from $L_n(C^+)$ can be (starting from the left end) divided into k_1 consecutive subwords of length t leaving a suffix of length k_2 . Thus:

$$\alpha_n(C^+) < (\sigma^t - 1)^{k_1} \sigma^{k_2}.$$

Consequently:

$$\frac{\alpha_n(C^+)}{\sigma^n} < \frac{(\sigma^t - 1)^{k_1} \sigma^{k_2}}{\sigma^n} = \frac{(\sigma^t - 1)^{k_1} \sigma^{k_2}}{\sigma^{tk_1} \sigma^{k_2}} = \left(1 - \frac{1}{\sigma^t}\right)^{k_1}.$$

Hence:

$$\lim_{n \rightarrow \infty} \frac{\alpha_n(C^+)}{\sigma^n} = 0,$$

which contradicts the fact that C is rich.

Consequently C must be dense and the result holds. ■

To put some of the dependencies we have demonstrated in a better perspective we provide now the following result.

THEOREM 6: *There exists a maximal code which is not rich.*

Proof: Consider the family of all full binary trees in which leafs are labelled by a and all inner nodes are labelled by b . Consider now all postfix notations for these trees—in this way we get the language $P \subseteq \{a, b\}^+$. It is well known that P is a code (every forest of full binary trees has a unique representation in the postfix notation).

Consider an arbitrary word $z \in \{a, b\}^+ - P$. Clearly $a^{|z|+1} z \in P^+$ (we parse $a^{|z|+1} z$ from right to left assigning $+1$ to a and -1 to b ; then each subword yielding by summation weight $+1$ is a tree corresponding to an element of

P). Hence $P \cup \{z\}$ is not a code, because $a^{|z|+1}z$ would have two different representations in P^+ . Thus P is a maximal code.

On the other hand it is known (see, e. g., [F], ch. III, sect. 3) that:

$$\lim_{n \rightarrow \infty} \frac{\alpha_n(P^+)}{2^n} = 0.$$

(Here one considers random walks on the line of positive integers where a represents a "step up" and b represents a "step down". It turns out that the probability of starting in 0 and not returning to 1 in up to n steps equals 1 in the limit.)

Hence P is not rich and the theorem holds. ■

Perhaps the most significant open question in the area of "extending codes to their maximal counterparts" is (see [P2]): can every biprefix regular code be extended to a maximal biprefix regular code? An answer to this question will certainly make the picture of the whole area clearer.

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