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A THEORY OF COMPLEXITY OF MONADIC RECURSION SCHEMES (*)

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Abstract. — *Complexity of a monadic recursion scheme is defined through numerical characteristics of trees representing its computations. A class of such complexity characteristics of trees essentially unlike the computation time: so called mimeoinvariant complexity measures, is introduced which induce several dense hierarchies of complexity classes of monadic recursion schemes of unbounded complexity and infinite hierarchies of bounded complexity classes. Simple conditions are found under which a function is a nonreducible upper bound of complexity of a monadic recursion scheme.*

Résumé. — *On définit la complexité d'un schéma récursif monadique à l'aide de propriétés numériques des arbres qui représentent ses calculs. On introduit une classe de telles propriétés de complexité des arbres, appelées les mesures de complexité miméoinvariantes, qui sont essentiellement différentes du temps de calcul, et qui induisent plusieurs hiérarchies denses de classes de complexité pour les schémas récursifs monadiques de complexité non bornée et des hiérarchies infinies de classes de complexité bornée. On donne des conditions simples qui assurent qu'une fonction est une borne supérieure irréductible pour la complexité d'un schéma récursif monadique.*

1. INTRODUCTION

There were several recent attempts to find a reasonable computer-free concept of computational complexity for program schemes. In particular, three different definitions may be mentioned: by R. Constable [1], by K. Weihrauch [2], and by Y. Igarashi [3]. All the three definitions have an essential common feature: they model computation time. We propose a concept of complexity of absolutely different nature. Our complexity measures characterize combinatorial complexity of objects representing computations of schemes. Moreover, the computation time is an illegal measure in our model. The mentioned papers differ also by the classes of schemes under study. Constable and Weihrauch treat standard (iterative) program schemes, while in Igarashi's and our papers

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monadic recursion schemes are considered. However, we are dealing with classical monadic recursion schemes, whereas Igarashi exhibits his time hierarchy in a class of generalized monadic recursion schemes substantially broader then that of the classical schemes; furthermore, this hierarchy degenerates for classical monadic recursion schemes.

As it is well known [4, 5] there is a tight relation between monadic recursion schemes and cf-grammars: with each scheme E a cf-grammar $G(E)$ is associated in a natural way, every computation of E under an interpretation I is represented by a rightmost derivation of $G(E)$ "controlled" by I. We make next step and consider the trees of these derivations as representations of the corresponding computations. After this it is rather natural to introduce complexity measures as integer-valued functions on trees.

This relation is the base for replantation of already existing complexity theory of cf-grammars as outlined in our earlier papers [6-9] to monadic recursion schemes. In particular, we apply (with minory changes) to monadic recursion schemes a central concept of this theory: the notion of a mimeoinvariant complexity measure (section 5). The characteristic feature of these measures is their invariance under a class of transformations of trees preserving on the whole their "topology". A complexity measure m has been chosen, we associate with each monadic recursion scheme E its m-complexity function m_E and thus to any nondecreasing total function f relate the class $\mathscr{E}^m_f$ of all schemes whose complexity functions do not exeed f. Mimeoinvariance of m implies that all quasirational monadic recursion schemes fall into bounded complexity class $\mathscr{E}^m_{\text{Const.}}$. As it turns out all mimeoinvariant complexity measures have high classificational capacities. In section 6 we find simple conditions under which a function f or a constant c become nonreducible upper bounds of m-complexity of a monadic recursion scheme (thms. 6.1, 6.2). The main result of section 7 (thm. 7.1) gives a condition under which for a mimeoinvariant complexity measure m there is an infinitely decreasing sequence of functions $f_1 \succ f_2 \succ f_3 \succ \ldots$ where each f_i is a nonreducible upper bound of m-complexity of a monadic recursion scheme. The theorem 8.1, a simplified version of this hierarchy theorem, shows that for any mimeoinvariant complexity measure m there is an infinitely decreasing sequence of functions $f_1 \succ f_2 \succ f_3 \succ \ldots$ such that $\mathscr{E}^m_{f_i} - \mathscr{E}^m_{f_{i+1}} \neq \emptyset$ for all i. Hence all mimeoinvariant measures provide nondegenerate classifications of monadic recursion schemes. Finally, in the nineth section we formulate a definition of a monadic recursion scheme of maximal complexity and show that under reasonable conditions all unambiguous monadic recursion schemes are either of maximal complexity or of bounded density.

2. PRELIMINARIES

We choose for the sequel two countable disjoint alphabets Σ and W (of *terminal* and *nonterminal* symbols respectively).

DEFINITIONS AND NOTATION 2.1: Let z be a string. A *prefix* (*suffix*) of z is any string v such that $z=vu$ (respectively $z=uv$) for a string u. Let N and Z_+ denote the set of all numbers and all nonnegative integers respectively. A (*finite labelled rooted*) *tree* is a pair $T=(\Delta, l)$, where: (1) Δ is a finite nonempty subset of $(Z_+)^*$; (2) Δ is prefix closed, i.e. with every string z it containes all its prefixes; (3) l is a function (called a *labelling*) from Δ to $\Sigma \cup W \cup \{\Lambda\}$ (Λ is the empty string) ([2]).

Strings in Δ are called *nodes* of T. For a node v of T $l(v)$ is called a *label* of v.

Let v, v' be two nodes of T such that $v=v'a$ for some a in Z_+. Then v is called an *immediate successor* of v' (in T). The set of all immediate successors of a node v is denoted by $i(v)$. The cardinality of $i(v)$ is called a *width* of v. The maximal width of nodes of T is called a *width* of T. A sequence $p=(v_1, v_2, \ldots, v_n)$, $n \geqq 1$, of nodes of T is called a *path* from v_1 to v_n (n being its *length*) if v_{i+1} is in $i(v_i)$ for all $1 \leqq i < n$. A path (v_1, v_2) is called an *arrow* from v_1 to v_2. A node v' is called a *successor* of a node v if $v \neq v'$ and there is a path from v to v'.

The node Λ is called a *root* of T. A node v is *bottom* if there is no arrow from v to a node of T, and *nonbottom* otherwise. A node v is *preterminal* if $l(v)$ is in $W \cup \{\Lambda\}$ and either v is bottom or all immediate successors of v are bottom and are labelled by symbols in Σ.

We consider the following *natural* (*partial*) *order* (denoted by $\lhd$) on the set of nodes of T. $v_1 \lhd v_2$ holds iff there are nodes v, v'_1, v'_2 in Δ such that v'_1, v'_2 are in $i(v)$, $v'_1=va$, $v'_2=vb$ for some a, b in Z_+ such that $a<b$, and v_1, v'_1 (v_2, v'_2) either coincide, or v_1 (v_2) is a successor of v'_1 (v'_2). This natural order is complete on the set of bottom nodes of T. If $v_1 \lhd v_2 \lhd \ldots \lhd v_s$ is the sequence of all bottom nodes of T in their natural order then the string $l(v_1)\,l(v_2)\ldots l(v_s)$ [denoted by $t(T)$] is the *yield* of T. The *length* s of this string is denoted by $|T|$.

DEFINITION 2.2: Two trees $T_1=(\Delta_1, l_1)$ and $T_2=(\Delta_2, l_2)$ are *isomorphic* (notation $T_1 \equiv T_2$) if there is a one-to-one correspondence h between Δ_1 and Δ_2 such that for any two nodes v_1, v_2 of T_1: (1) v_2 is a successor of v_1 iff $h(v_2)$ is a successor of $h(v_1)$ in T_2, (2) $v_1 \lhd v_2$ iff $h(v_1) \lhd h(v_2)$ in T_2, (3) $l_1(v_1)=l_2(h(v_1))$.

([2]) In our definitions and theorems we admit empty labels, empty right sides of productions, empty equations, and so on. However in the proofs of theorems we dont consider such cases for the reasons of space and because of triviality or routine character of the corresponding arguments.

DEFINITION 2.3: Let $T=(\Delta, l)$ be a tree and v be a node of T. A tree $T'=(\Delta', l')$, where [3] $\Delta' \subseteq v \backslash \Delta$ and $l'(z)=l(vz)$ for all z in Δ' is called a (v-)*subtree* of T. $B(T')=v\Delta'$ is a *base set* of T'. The v-subtree (denoted by $T(v)$) whose base set $B(T(v))$ is the least subset of Δ containing v and all its successors is called a *full* (v-)*subtree* of T. The Λ-subtree of T [denoted by $CT(v)$] with the base set $(\Delta - B(T(v))) \cup \{v\}$ is called a *complementary v-subtree* of T.

DEFINITION 2.4: Let $T=(\Delta, l)$, T_1 and T_2 be some trees and v be a node of T. We say that T is a *composition* of T_1 and T_2 at v [notation $T=\mathrm{com}(T_1, v, T_2)$] if $T_1 \equiv CT(v)$ and $T_2 \equiv T(v)$.

DEFINITION 2.5: Let $T=(\Delta, l)$ be a tree. A *covering* of T is a system of its subtrees $C=\{T_1, \ldots, T_r\}$ such that (1) $B(T_i) \cap B(T_j)=\emptyset$ for all $i \neq j$, $1 \leqq i$, $j \leqq r$, and (2) $\Delta = \bigcup_{i=1}^{r} B(T_i)$.

DEFINITION 2.6: Let $\Sigma' \subseteq \Sigma$ and $W' \subseteq W$ be two alphabets. $T=(\Delta, l)$ is a (*syntactic*) *structure tree* (abbreviated *s-tree*) over Σ', W' if: (1) all nonbottom nodes of T are labelled by symbols in W', (2) all bottom nodes of T are labelled by symbols in $\Sigma' \cup W' \cup \{\Lambda\}$, (3) each nonbottom node possessing an immediate successor labelled by Λ is of width 1. A s-tree is *complete* (*cs-tree* for short) if every its bottom node is labelled by an element of $\Sigma \cup \{\Lambda\}$. A s-tree T is *linear* if every its nonbottom node has no more than one immediate successor labelled by a nonterminal, T is *trivial* if the width of T is $\leqq 1$.

We introduce several binary relations on the set of s-trees which in a sense preserve their "topology".

DEFINITION 2.7: Let $T_1=(\Delta_1, l_1)$ and $T_2=(\Delta_2, l_2)$ be two s-trees. T_2 is *mimeomorphic* (*strictly mimeomorphic*) to T_1 (notation $T_1 \leqq T_2$ and respectively $T_1 \leqq^s T_2$) if there is a covering $C=\{T_{21}, \ldots, T_{2r}\}$ of T_2 and a one-to-one correspondence (*mimeomorphism*) φ between Δ_1 and C with the following properties. Let v_1, v_2 be two nodes of T_1, $\varphi(v_1)$ be a u_1-subtree and $\varphi(v_2)$ be a u_2-subtree of T_2 for some u_1, u_2. Then: (1) if v_2 is in $i(v_1)$ in T_1, there is a node (respectively a preterminal node) u of the tree $\varphi(v_1)$ such that u_2 is in $i(u_1 u)$ in T_2, (2) if $v_1 \lhd v_2$ in T_1 then for any nodes w_1 of $\varphi(v_1)$ and w_2 of $\varphi(v_2)$ $u_1 w_1 \lhd u_2 w_2$ holds in T_2, (3) $l_1(v_1)$ is in $\Sigma \cup \{\Lambda\}$ iff $\varphi(v_1)$ is a one-node u_1-subtree and $l_2(u_1)$ is in $\Sigma \cup \{\Lambda\}$. If every tree in C is linear (trivial) we say that the mimeomorphism is *linear* (*trivial*) (notation $T_1 \leqq^l T_2$ and respectively $T_1 \leqq^t T_2$); if the mimeomorphism is both strict and linear (trivial) we say that it is *strictly linear* (*trivial*) (notation $T_1 \leqq^{sl} T_2$ and respectively $T_1 \leqq^{st} T_2$).

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[3] For a language L and a string z, $z \backslash L$ denotes the quotient language $\{w \mid zw \text{ is in } L\}$.

REMARK: If $T_1 \leqq^{st} T_2$ and all subtrees in C are either one-node or their bottom nodes are labelled by nonterminals then T_2 is homeomorphic to T_1 in the graph-theoretic sense (*cf.* [10]) (notation $T_1 \leqq^h T_2$).

NOTATION: Let $\Sigma' \subseteq \Sigma$ and $W' \subseteq W$. The set of all s-trees (cs-trees) of width $\leqq k$ over Σ', W' is denoted by $\mathscr{S}(\Sigma', W', k)$ [respectively by $\mathscr{S}^c(\Sigma', W', k)$]. Let

$$\mathscr{S}(\Sigma', W') = \bigcup_{k=0}^{\infty} \mathscr{S}(\Sigma', W', k),$$

$$\mathscr{S}^c(\Sigma', W') = \bigcup_{k=0}^{\infty} \mathscr{S}^c(\Sigma', W', k),$$

$$\mathscr{S}(\Sigma, W) = \mathscr{S}, \qquad \mathscr{S}^c(\Sigma, W) = \mathscr{S}^c.$$

DEFINITION 2.8: A set S such that $S \subseteq \mathscr{S}^c(\Sigma_1, W_1, k)$ for some k and finite $\Sigma_1 \subset \Sigma$, $W_1 \subset W$ is called a *structure set* (abbreviated *s-set*) if no two trees in S are isomorphic. $L(S) = \{ t(T) \mid T \text{ is in } S \}$ is the *language characterized* by S. S is *unambiguous* if for each z in $L(S)$ there is at most one cs-tree T in S such that $z = t(T)$; otherwise S is *ambiguous*. A s-set S is *free* if for any two s-trees $T_1 = (\Delta_1, l_1)$ and $T_2 = (\Delta_2, l_2)$ in S and any nodes v_1 of T_1 and v_2 of T_2 such that $l_1(v_1) = l_2(v_2)$ S contains a s-tree $\operatorname{com}(CT_1(v_1), v_1, T_2(v_2))$.

NOTATION: Let $\Sigma' \subseteq \Sigma$ and $W' \subseteq W$ be two alphabets. The class of all cf-grammars $G = (\Sigma_1, W_1, A, P)$ such that $\Sigma_1 \subseteq \Sigma'$ and $W_1 \subseteq W'$, is denoted by $\mathscr{G}(\Sigma', W')$. For a cf-grammar G in $\mathscr{G}(\Sigma', W')$ we denote its *structure set*, i. e. the set of all its complete phrase-structure trees, by $S(G)$.

PROPOSITION 2.1:

$$\{ S(G) \mid G \textit{ is in } \mathscr{G}(\Sigma', W') \} = \{ S \subset \mathscr{S}^c(\Sigma', W') \mid S \textit{ is a free s-set} \}$$

for all $\Sigma' \subseteq \Sigma$ *and* $W' \subseteq W$.

This well known proposition will provide a grammar-free form to our notion of complexity and to related concepts, convenient for applications to monadic recursion schemes as well as to cf-grammars.

3. COMPLEXITY MEASURES AND STRUCTURE SETS

The concepts presented in this section are introduced in [6, 7]. They form the framework within which we study there complexity of syntactic structures and derivation trees in cf-grammars. In section 5 below these concepts will be applied to monadic recursion schemes.

DEFINITION 3.1: A *complexity measure* is a computable total function m from $\mathscr{S}$ onto an infinite subset of Z_+ such that $m(T_1)=m(T_2)$ whenever $T_1 \equiv T_2$.

We cite a few examples of complexity measures (4).

Examples 1: *Density* of a s-tree [11, 12]. We define this measure by induction on full subtrees of a tree. Let T be a s-tree and v be a node of T. (1) If v is a bottom node of T then $\mu(T(v))=0$. (2) Let v be a nonbottom node and:

$$\mu_v = \max\{\mu(T(v')) \mid v' \text{ is in } i(v)\}.$$

Then:

$$\mu(T(v)) = \textit{if}\ (\exists v_1, v_2 \text{ in } i(v))\ [v_1 \neq v_2\ \&\ \mu(T(v_1)) = \mu(T(v_2)) = \mu_v],$$

then $\mu_v + 1$ *else* μ_v. $\mu(T) = \mu(T(\Lambda))$ is the the density of T.

2. *Branching* of a s-tree [6, 7] is the number $b(T)$ of preterminal nodes of T.

3. *Capacity* of a s-tree T is the number $c(T)$ of all nodes of T.

DEFINITION 3.2: Let m be a complexity measure and S be a s-set. By (m-) *complexity* of S we mean the function $\lambda n . m_S(n)$ where:

$$m_S(n) = \max\{0, m_S(T) \mid T \text{ in } S, |T| \leqq n\},$$

and:

$$m_S(T) = \min\{m(T') \mid T' \text{ in } S,\ t(T') = t(T)\} \quad \text{for all } T \text{ in } S.$$

Besides this for $z = t(T)$, T in S, we set $m_S(z) = m_S(T)$.

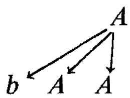
DEFINITION 3.3: Let m be a complexity measure, f be a total nondecreasing function, and S be a s-set. We say that f is *m-limiting* S if:

(a) $(\exists c)(\forall T \text{ in } S)\ [m(T) \leqq cf(|T|)]$ and

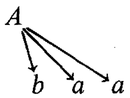
(b) there is a sequence of cs-trees $T_1, T_2, \ldots$ in S (a *fundamental sequence*) such that the set $\{|T_i| \mid i > 0\}$ is infinite and $(\exists d)(\forall i)\ [dm(T_i) \geqq f(|T_i|)]$.

This concept is very close to the notion of constructable function in automata theory and plays a similar part in our exploration.

Remark As we observed in [6] the functions $\lambda n . n$ and $\log n$ (5) are respectively b-limiting and μ-limiting the least free s-set containing:



and



(4) Some other examples of measures of importance for cf-grammar theory such as index, Yngve measures, dispersion, selfembedding index, and so on, may be found in [6, 7].

(5) Functions that we use for measuring complexity are total nondecreasing functions from Z_+ into Z_+. For example, $\log n$ denotes the function $\lambda n . [\log_2(n+1)]$.

DEFINITION 3.4: Let f be a total nondecreasing unbounded function and m be a complexity measure. We say that f is *limiting* m if:

(a) $(\forall k)(\exists c)(\forall T \text{ in } \mathscr{S}(\Sigma, W, k))\,[m(T) \leqq cf(|T|)]$ and

(b) there are k_0 and a sequence of s-tress $(T_1, \ldots, T_n, \ldots)$ in $\mathscr{S}^c(\Sigma, W, k_0)$ (a *fundamental sequence* of f) and $d>0$ such that $dm(T_i) \geqq f(|T_i|)$ for all i.

For example, $\log n$ is limiting μ [6, 9] and $\lambda n.n$ is limiting b and c.

4. MONADIC RECURSION SCHEMES

There is an unsubstantial difference between the notion of a monadic recursion scheme under study and that of [4] and [5]. Nevertheless we outline here both their syntax and semantics.

I. SYNTAX: Treating monadic recursion schemes we give to symbols of Σ and W new names: *basic* and *defined function symbols* respectively.

Let $\langle \mathscr{P}_i \rangle_{i \text{ in } N}$ be a system of countable pairwise disjoint alphabets (of *switch function symbols*) $\mathscr{P}_i = \{ p_j^i \mid j \text{ in } N \}$ such that $\mathscr{P}_i \cap (\Sigma \cup W) = \emptyset$ for all i, $\mathscr{P} = \bigcup_{i \text{ in } N} \mathscr{P}_i$, and x be a symbol not in $\mathscr{P} \cup \Sigma \cup W$ (a *variable symbol*). Let $\Sigma' \subseteqq \Sigma$, $W' \subseteqq W$. A string z in $(\Sigma' \cup W')^* x$ is a (*monadic*) *term* (over Σ', W'). A term z over Σ', W' is *basic* if it doesn't contain occurences of defined function symbols.

DEFINITION 4.1: *A monadic recursion scheme* (MR-scheme) (over Σ, W) is a system $E = (\Sigma_1, W_1, F_1, \{e_1, \ldots, e_k\})$ meeting the conditions:

(1) $\Sigma_1 \subset \Sigma$ and $W_1 = \{F_1, \ldots, F_k\} \subset W$ are finite alphabets;

(2) e_i $(1 \leqq i \leqq k)$ is a formal equation of the form:

$$e_i: \quad F_i x = (p_{m(i)}^{n(i)} x \mid u_{i1} x, \ldots, u_{in(i)} x),$$

where $p_{m(i)}^{n(i)}$ is a switch function symbol, $u_{i1} x, \ldots, u_{in(i)} x$ are monadic terms over Σ_1, W_1, $1 \leqq i \leqq k$. We say that E *defines* F_1. The set $\{ p_{m(1)}^{n(1)}, \ldots, p_{m(k)}^{n(k)} \}$ is denoted by $\mathscr{P}(E)$ and the class of all MR-schemes by $\mathscr{E}$.

With the MR-scheme E in the definition 4.1 we associate a cf-grammar $G(E)$ in the following regular way:

$$G(E) = (\Sigma_1, W_1, F_1, R), \qquad \text{where} \quad R = \bigcup_{i=1}^{k} R(e_i),$$

and

$$R(e_i) = \{ F_i \to u_{i1}, \ldots, F_i \to u_{in(i)} \}, \qquad 1 \leqq i \leqq k.$$

The associated cf-grammar will serve as a base for a semantic notion of a computation of a MR-scheme. Besides this it is a convenient means of syntactic classification of MR-schemes. For example, we call a MR-scheme E *linear* if its associated grammar $G(E)$ is linear.

II. SEMANTICS: Let $E=(\Sigma_1, W_1, F_1, \{e_1, \ldots, e_k\})$ be a MR-scheme. An *interpretation of* E is a system $I=(J, D)$, where D is a set called a *domain* of I and J is a functional on $\Sigma_1 \cup \mathscr{P}(E) \cup \{x\}$ such that:

(1) $J(f)$ is a total function from D into D for each f in Σ_1;

(2) $J(x)$ is an element of D, and (3) for each n and each p_i^n in $\mathscr{P}_n \cap \mathscr{P}(E)$ $J(p_i^n)$ is a total function from D into $\{1, \ldots, n\}$. I is naturally extendable to the set of basic terms:

$$\begin{cases} I(x)=J(x), \\ I(fvx)=J(f)(I(vx)), \end{cases}$$

for all f in Σ_1, v in Σ_1^*.

An interpretation $I=(J, D)$ of E is *free* (or Herbrand) if $D=\Sigma_1^*$, $J(x)=\Lambda$, and $J(f)(t)=ft$ for all f in Σ_1 and t in D.

A computation of a MR-scheme E may be considered as a rightmost derivation of the grammar $G(E)$ controlled by an interpretation I in the following sense.

DEFINITION 4.2: Let $I=(J, D)$ be an interpretation of the MR-scheme above. Let $X=y_1 F_i y_2$ and Y be two strings in $(\Sigma_1 \cup W_1)^*$ and y_2 be in Σ_1^*. Then:

$$X \Rightarrow_{EI} Y \quad \text{if} \quad J(p_{m(i)}^{n(i)})(I(y_2 x))=j$$

for some $j \leqq n(i)$, $Y=y_1 u_{ij} y_2$ and the equation e_i in E is of the form:

$$e_i: \quad F_i x=(p_{m(i)}^{n(i)} x \mid u_{i1} x, \ldots, u_{ij} x, \ldots, u_{in(i)} x).$$

A sequence $c(E, I)=(X_0, X_1, X_2, \ldots)$ is called an *I-computation sequence* of E if (a) $X_0=F_1$, and (b) $X_i \Rightarrow_{EI} X_{i+1}$ for all $i \geqq 0$. If the I-computation sequence $c(E, I)$ is finite, i. e. $c(E, I)=(X_0, X_1, \ldots, X_r)$ for some r, and its last string X_r is in Σ_1^* then it is called an $(I-)$ *computation* of E.

It is evident that for all interpretations I such that $c(E, I)$ is a computation it is at the same time a rightmost complete derivation of the *cf*-grammar $G(E)$. The tree of this derivation [denoted $T(E, I)$] is called a *tree of the I-computation* $c(E, I)$. The set of all trees of computations of E, i. e. the set $\{T(E, I) \mid I$ is an interpretation of $E\}$ is denoted by $S(E)$. E is *unambiguous* (*ambiguous*) if $S(E)$ is unambiguous (ambiguous). We will consider two partial value functions:

$$\mathrm{TERMVAL}(E, I)=\begin{cases} t(T(E, I)) \text{ if } c(E, I) \text{ is a computation of } E, \\ \text{undefined otherwise} \end{cases}$$

$$\mathrm{VAL}(E, I)=I(\mathrm{TERMVAL}(E, I)x).$$

The set:

$$TL(E)=\{\mathrm{TERMVAL}(E, I) \mid I \text{ is an interpretation of } E\}$$

is called a *term language* of E.

DEFINITION 4.3: We say that MR-schemes E_1, E_2 are *termally* (*strongly*) *equivalent* (notation $E_1 \equiv_t E_2$ and $E_1 \equiv_s E_2$ respectively) if $TL(E_1)=TL(E_2)$ [respectively $\lambda I.\mathrm{VAL}(E_1, I)=\lambda I.\mathrm{VAL}(E_2, I)$].

DEFINITION 4.4: An equivalence $\equiv_r$ on $\mathscr{E}$ is called *reasonable* if $E_1 \equiv_r E_2$ implies $E_1 \equiv_t E_2$ for all E_1, E_2 in $\mathscr{E}$.

Remark: Strong equivalence of MR-schemes is of course a reasonable one. This follows directly from the fact that if $E_1 \equiv_s E_2$ then $\mathrm{VAL}(E_1, I)=\mathrm{VAL}(E_2, I)$ for each free interpretation I.

DEFINITION 4.5: Let $\mathscr{E}_1$, $\mathscr{E}_2$ be two classes of MR-schemes and $\equiv_r$ be a reasonable equivalence on $\mathscr{E}$. We say that $\mathscr{E}_1$ is *termally* (*strongly*, *r*-) *translatable* into $\mathscr{E}_2$ (notation $\mathscr{E}_1 \Rightarrow_t \mathscr{E}_2$, $\mathscr{E}_1 \Rightarrow_s \mathscr{E}_2$, and $\mathscr{E}_1 \Rightarrow_r \mathscr{E}_2$ respectively) if for each MR-scheme E_1 in $\mathscr{E}_1$ there is a termally (strongly, r-) equivalent to E_1 MR-scheme E_2 in $\mathscr{E}_2$.

5. COMPLEXITY CLASSES AND MIMEOINVARIANT MEASURES

Application of complexity measures to structure sets leads in a straightforward manner to a natural notion of computation complexity of MR-schemes. In fact, we measure the complexity of trees in $S(E)$ bearing in mind that these are tree representations of the corresponding computations of E. So, we arrive at the following definition.

DEFINITION 5.1: Let m be a complexity measure and E be a MR-scheme. By *m-complexity* of E we mean the function $m_E=\lambda n.m_{S(E)}(n)$.

So that to stratify the class $\mathscr{E}$ of all MR-schemes into complexity classes we consider the following relations on the set of total functions on Z_+.

NOTATION: Let g and f be total functions. $g \prec f$ means $\lim_{n \to \infty} g(n)/f(n)=0$, $g \preceq f$ means $\overline{\lim_{n \to \infty}}\, g(n)/f(n) \leqq c$ for some c in Z_+, i.e. $(\exists c)\,(\forall^\infty n)\,[g(n) \leqq cf(n)]$, $g \asymp f$ means $g \preceq f$ & $f \preceq g$, and $g \precsim f$ means that $g \preceq f$ but not $g \prec f$.

DEFINITION 5.2: Let f be a total nondecreasing function. The set $\mathscr{E}_f^m = \{ E \mid E$ is a MR-scheme, $m_E \preceq f \}$ is a (m-) *complexity class*. Let $c \geqq 0$ be an integer. The set $\mathscr{E}_c^m = \{ E \mid E$ is a MR-scheme, $(\forall n)\,[m_E(n) \leqq c] \}$ is a *c-bounded* (m-) *complexity class*. $\mathscr{E}_{\text{Const.}}^m = \bigcup_{i=0}^{\infty} \mathscr{E}_i^m$ is called a *bounded* (m-) *complexity class*.

Of course, $\mathscr{E}_f^m = \mathscr{E}$ for any function f limiting a complexity measure m.

REMARK: All these notions can be (and they were) applied to *cf*-grammars. For example, the complexity function of a *cf*-grammar G is defined as $m_G = \lambda n . m_{S(G)}(n)$ for all complexity measures m.

DEFINITION 5.3: A complexity measure m is *nondegenerate* if there is an unambiguous MR-scheme E whose m-complexity function m_E is unbounded.

Meanwhile, the definition 5.1 is too general to be workable. We are looking for a reasonable class of measures which (1) make the complexity stratifications $\{ \mathscr{E}_f^m \}$ and $\{ \mathscr{E}_c^m \}$ nontrivial, and (2) have close values on "topologically" similar trees. We attain both objectives imposing simple conditions on complexity measures. These conditions formalize a vague formulation of our second objective in terms of mimeomorphisms. In fact we assume that a s-tree T (strictly) linear mimeomorphic to another s-tree T' is only negligibly different from it from the complexity point of view. The complexity measures meeting this condition are called mimeoinvariant.

DEFINITION 5.4 (main definition): A nondegenerate complexity measure m is *mimeoinvariant* if it satisfies the axioms:

$$\text{A} \qquad \left\{ \begin{array}{c} (\exists c_A)\,(\forall T_1, T_2 \text{ in } \mathscr{S}) \\ T_1 \leqq^{l} T_2 \supset m(T_2) \leqq c_A\, m(T_1)], \end{array} \right.$$

$$\text{B} \qquad \left\{ \begin{array}{c} (\exists d_B)\,(\forall T_1, T_2 \text{ in } \mathscr{S}) \\ {[T_1 \leqq T_2 \supset m(T_1) \leqq d_B\, m(T_2)]}, \end{array} \right.$$

and it is *asymptotically mimeoinvariant* if it meets the conditions:

$$\text{A}^{a} \qquad \left\{ \begin{array}{c} (\exists c \geqq 0)\,(\forall T_1, T_2 \text{ in } \mathscr{S}) \\ {[T_1 \leqq^{sl} T_2 \supset m(T_2) \leqq m(T_1) + c]}, \end{array} \right.$$

$$\text{B}^{a} \qquad \left\{ \begin{array}{c} (\forall T_1, T_2 \text{ in } \mathscr{S}) \\ {[T_1 \leqq T_2 \supset m(T_1) \leqq m(T_2)]}. \end{array} \right.$$

The measures μ and b are obviously mimeoinvariant and asymptotically mimeoinvariant, while c is not (it must be noted that c doesn't depend on form of s-trees). Note also that if m is (asymptotically) mimeoinvariant then all linear MR-schemes fall into $\mathscr{E}^m_{\text{Const.}}$.

6. RIGHT-NORMAL FORM *cf*-GRAMMARS AND COMPLEXITY OF INDIVIDUAL *MR*-SCHEMES

Let m be a complexity measure and f be an unbounded nondecreasing function. Let us say that a *cf*-grammar G is of *nonreducible m-complexity f* if: (1) $m_G \preceq f$ and (2) for no *cf*-grammar G' such that $m_{G'} \prec f$ $L(G) = L(G')$.

A similar notion may be introduced for MR-schemes. This notion however relates upon a choice of equivalence relation among MR-schemes. We will consider only reasonable equivalences. So let $\equiv_r$ be some reasonable equivalence relation on $\mathscr{E}$ and $\Rrightarrow_r$ be the corresponding translatability relation.

Definition 6.1: A MR-scheme E is of *r-nonreducible m-complexity f* if: (1) E is in $\mathscr{E}^m_f$ and (2) for no MR-scheme E' such that $m_{E'} \prec f$ $E' \equiv_r E$.

Remark: If there is a MR-scheme of r-nonreducible m-complexity f then the class $\mathscr{E}^m_f$ is unempty and is r-translatable neither into any class $\mathscr{E}^m_g$ such that $g \prec f$, nor into $\mathscr{E}^m_{\text{Const.}}$.

In this section we give the conditions sufficient for a function and a constant to be nonreducible MR-scheme complexity bounds.

We start with a few simple observations.

Definition 6.2: A *cf*-grammar is in a *right-normal form* if all its productions are of the form $A \to \varphi u$, u in Σ^+.

Proposition 6.1: *For each cf-grammar G in right-normal form there is a MR-scheme E such that $S(E) = S(G)$ and thus $m_E = m_G$.*

[Of course, this is a scheme such that $G(E) = G$.]

The next proposition follows directly from the proof of the theorem 2.5 in [4].

Proposition 6.2: *For every MR-scheme E there are a MR-scheme $\hat{E}$ (unambiguous if E is unambiguous) with $G(\hat{E})$ in right-normal form and a bijection $q: \mathscr{I} \to \mathscr{I}$ on the set $\mathscr{I}$ of all interpretations, such that $T_1 = T(E, I)$ exists iff $T_2 = T(\hat{E}, q(I))$ exists, and in the case they exist $t(T_1) = t(T_2)$ and $T_2 \leqq^t T_1$.*

From these propositions we have:

Corollary 6.1: *For every mimeoinvariant complexity measure m and for every MR-scheme E there is a cf-grammar G_E in right-normal form such that $m_{G_E} \asymp m_E$ and $L(G_E) = TL(E)$.*

COROLLARY 6.2: *For any mimeoinvariant complexity measure m, any reasonable equivalence relation $\equiv_r$, and any cf-grammar G in right-normal form of nonreducible m-complexity f there is a MR-scheme E of r-nonreducible m-complexity f.*

In our papers [6, 8] we have developed a technics of constructing *cf*-grammars of nonreducible *m*-complexities. The abovestated propositions permit reconstruction of these *cf*-grammars into *MR*-schemes of nonreducible *m*-complexity in the case they are in right-normal form.

DEFINITION 6.3: A function f is *semihomogeneous* if

$$(\forall c_1)(\exists c_2)(\forall n_1, n_2)$$
$$[n_1 \leqq c_1 n_2 \supset f(n_1) \leqq c_2 f(n_2)].$$

REMARK: A semihomogeneous function cannot of course be superexponential.

THEOREM 6.1: *Let $\equiv_r$ be a reasonable equivalence relation on $\mathscr{E}$, m be a mimeoinvariant complexity measure, and f be a nondecreasing unbounded semihomogeneous function m-limiting the s-set $S(E)$ for a MR-scheme E. Then there is a MR-scheme E_f of r-nonreducible m-complexity f.*

Proof: Let m, r, f and E be as above. First of all we associate with E the MR-scheme $\hat{E}$ as in the proposition 6.2 and consider the *cf*-grammar $G(\hat{E})$. Then we carry out the following construction originating from [6, 8]. Let $G(\hat{E})=(\Sigma_1, W_1, A, P)$. We choose four new symbols a, b, c, d in $\Sigma-\Sigma_1$ and choose a symbol F_π in $W-W_1$ for each production π in P. After this we set:

$$W'_1=\{F_\pi \mid \pi \text{ in } P\} \cup W_1 \quad \text{and} \quad P'=\bigcup_{\pi \text{ in } P} P',$$

where for each $\pi=F \to \varphi$ in P:

$$P'_\pi=\{F \to cF_\pi d,\ F_\pi \to aF_\pi b,\ F_\pi \to a\varphi b\}.$$

As a result, we obtain the *cf*-grammar:

$$\Gamma[G(\hat{E})]=(\Sigma_1 \cup \{a, b, c, d\}, W'_1, A, P').$$

Since m is mimeoinvariant we infere from the proposition 6.2 that the function f is m-limiting the s-set $S(\hat{E})=S(G(\hat{E}))$. This being clear, we use the following fact proven in [6] (thm. 9.4) and in [8] (thm. 1).

PROPOSITION 6.3: *Let m be a mimeoinvariant complexity measure, f be a nondecreasing semihomogeneous function m-limiting the set $S(G)$ of a cf-grammar G. Then the cf-grammar $\Gamma[G]$ associated with G as above is of nonreducible m-complexity f.*

Thus we see that $\Gamma[G(\hat{E})]$ is a *cf*-grammar in right-normal form and of nonreducible *m*-complexity *f*. Hence by the corollary 6.2 we associate with $\Gamma[G(\hat{E})]$ the needed *MR*-scheme E_f of *r*-nonreducible *m*-complexity *f*.

Q.E.D.

REMARK: Consider the *MR*-scheme:

$$E_0:\quad Fx=(px \mid FFfx,\ gx).$$

It is evident that the function $\log n$ is μ-limiting the *s*-set $S(E_0)$ and the function $\lambda n.n$ is *b*-limiting this *s*-set. Since both these measures are mimeoinvariant, both functions are semihomogeneous, and $G(E_0)$ is in right-normal form the construction of the theorem 6.1 delivers a *MR*-scheme E_1 of *r*-nonreducible μ-complexity $\log n$ and of *r*-nonreducible *b*-complexity $\lambda n.n$ for each reasonable equivalence relation $\equiv_r$. Though the proof of the theorem 6.1 defines this *MR*-scheme entirely we cite it out here:

$$E_1:\quad \begin{cases} Fx=(p_1^2 x \mid cF_1 dx,\ cF_2 dx), \\ F_1 x=(p_2^2 x \mid aF_1 bx,\ FFfx), \\ F_2 x=(p_3^2 x \mid aF_2 bx,\ gx). \end{cases}$$

The same reduction leads to an infinite hierarchy of *MR*-schemes of bounded complexity.

THEOREM 6.2: *Let m be an asymptotically mimeoinvariant complexity measure, $\equiv_r$ be some reasonable equivalence relation on $\mathscr{E}$. Then there is $c \geqq 0$ such that for any cs-tree T of m-complexity k there exists a MR-scheme E_k in $\mathscr{E}_{k+c}^m$ which is not $\equiv_r$-equivalent to any MR-scheme in any class $\mathscr{E}_l^m$ with $l<k$.*

Proof: Consider a *cs*-tree $T=(\Delta, l)$ such that $\bar{\bar{\Delta}}>1$ and $m(T)=k$. Let Σ_T be the set of all terminal labels of nodes in Δ. We add four new terminal symbols a, b, c, d in $\Sigma-\Sigma_T$ to Σ_T and set $\Sigma_1=\Sigma_T \cup \{a, b, c, d\}$. With each nonbottom node v of T we associate two nonterminals A_v, B_v in W in such a way that $\{A_v, B_v\} \cap \{A_u, B_u\}=\emptyset$ for $v \neq u$ and set $W_1=\{A_v, B_v \mid v \text{ in } \Delta\}$. To each nonbottom node v of T such that $i(v)=\{v_1, \ldots, v_n\}$ and $v_1 \lhd v_2 \lhd \ldots \lhd v_n$ we relate the system of productions $P(v)$:

$$A_v \to cB_v d,$$

$$B_v \to aB_v b,$$

$$B_v \to aX_1 X_2 \ldots X_n b,$$

where:

$$X_i = \begin{cases} A_{v_i} & \text{if } l(v_i) \text{ is in } W, \\ l(v_i) & \text{if } l(v_i) \text{ is in } \Sigma_T. \end{cases}$$

Thus we obtain the *cf*-grammar $G_T = (\Sigma_1, W_1, A_\Lambda, P)$, where $P = \bigcup_{v \text{ in } \Delta} P(v)$. In the first place we note that for each *cs*-tree T' in $S(G_T)$ $T \leqq^{sl} T'$ holds. Since m is asymptotically mimeoinvariant there is $c \geqq 0$ such that $m(T') \leqq m(T) + c = k + c$ for all T' in $S(G_T)$ and hence $m_{G_T} \leqq k + c$. Secondly we observe that G_T is in right-normal form. Thus by proposition 6.1 there is a *MR*-scheme E_T such that $S(E_T) = S(G_T)$ and hence E_T is in $\mathscr{E}^m_{k+c}$. In the paper [8] (corollary 3 from theorem 1) we prove that if a *cf*-grammar G is equivalent to G_T then there is no such $l < k$ that $m_G < l$. Assume that there is a *MR*-scheme E such that $E \equiv_r E_T$ and E is in $\mathscr{E}^m_l$ for some $l < k$. By proposition 6.2 and the axiom B[a] in the definition 5.4 there is a *cf*-grammar G_E such that $m_{G_E} \leqq m_E$ and $L(G_E) = TL(E)$. Hence we conclude that $m_{G_E} \leqq l < k$. But $L(G_T) = TL(E_T) = TL(E) = L(G_E)$, a contradiction.

Q.E.D.

Corollary 6.3: *Let m be an asymptotically mimeoinvariant complexity measure and $\equiv_r$ be a reasonable equivalence relation on $\mathscr{E}$. Then there is an infinite sequence of nonnegative integers $n_1 < n_2 < \ldots < n_k < \ldots$ and MR-schemes E_1, $E_2, \ldots, E_k, \ldots$ such that for all $k > 1$ E_k is in $\mathscr{E}^m_{n_k}$ and for no $l \leqq n_{k-1}$ and no E in $\mathscr{E}^m_l$ $E \equiv_r E_k$.*

7. INFINITE HIERARCHIES OF *MR*-SCHEMES OF NONREDUCIBLE COMPLEXITIES

Our main objective is to exibit conditions under which for a mimeoinvariant complexity measure m and for a reasonable equivalence relation $\equiv_r$ there is an infinite hierarchy of *MR*-schemes $\{E_{f_i}\}$ of r-nonreducible m-complexities $f_1 \succ f_2 \succ f_3 \succ \ldots$. This will give hierarchy of complexity classes $\mathscr{E}^m_{f_1} \supsetneqq \mathscr{E}^m_{f_2} \supsetneqq \mathscr{E}^m_{f_3} \supsetneqq \ldots$ such that for no i, j, $i < j$, $\mathscr{E}^m_{f_i} \Rightarrow_r \mathscr{E}^m_{f_j}$. A similar hierarchy $\{G_{f_i}\}$ of *cf*-grammars of nonreducible complexities is described in [6] (theorem 9.5 and its corollaries) and in [8] (theorem 2 and its corollaries). Simple reductions of the preceeding section are unfit however for reconstruction of $\{G_{f_i}\}$ into $\{E_{f_i}\}$. The reason is that the grammars G_{f_i} are not in right-normal form and even worse: the traditional reductions of G_{f_i} to right-normal form increases their complexity to the maximal. Thus we must strengthen the results of [6, 8] and expose an infinite hierarchy of *cf*-grammars in right-normal form of nonreducible complexity. To this end we need some notions and notation related to Turing machines.

NOTATION: Let $\Sigma^T \subset \Sigma$ be an infinite alphabet. We consider the class $\mathcal{M}(\Sigma^T)$ of all one-tape one-head deterministic Turing machines such that for each M in $\mathcal{M}(\Sigma^T)$ the set K_M of states of M and the set V_M of tape symbols of M are both subsets of Σ^T. We use the standard encoding of situations and computations of Turing machines. A situation of M is a string Q in $V_M^* K_M V_M^*$. If q_0 is a start and q_f is a final state of M then situations $Q_1 = q_0 z_1$ and $Q_2 = q_f z_2$, z_1 z_2 in V_M^*, are *starting* and *final* respectively. The symbol Q with or without indices we reserve for the sequel as a variable over the set of Turing machine situations or their substrings. For situations Q_1, Q_2 of M $Q_1 \underset{M}{\vdash} Q_2$ means that Q_2 immediately follows Q_1 in a computation of M. $n_M(Q)$ denotes the length of the situation Q' such that $Q \underset{M}{\vdash} Q'$. A substring of a situation Q is *active* if it containes an occurence of a state and *passive* if it doesn't. Let § be a symbol in $\Sigma - \Sigma^T$. A *x-computation record* of M is the string $P_M(x) = \S Q_1 \S Q_2 \S \ldots Q_{n-1} \S Q_n$, where $Q_1 = q_0 x$ is the starting situation with input string x, Q_n is a final situation and $Q_i \underset{M}{\vdash} Q_{i+1}$ for all $1 \leq i < n$. In the class $\mathcal{M}(\Sigma^T)$ we select the subclass $\mathcal{N}(\Sigma_T)$ of all Turing machines such that:

(1) the record function $\lambda x . P_M(x)$ is total;

(2) the function $\lambda x . |P_M(x)|$ is nondecreasing with respect to the lengths of strings x, i.e. $|x_1| \geq |x_2|$ implies $|P_M(x_1)| \geq |P_M(x_2)|$.

With each Turing machine M we associate the following integervalued function p_M which is in a sense inverse to the record-length function $\lambda x . |P_M(x)|$:

$$\begin{aligned} p_M(n) = & \ \mathit{if} (\exists x)[2 |P_M(x)| \leq n - |x|] \\ & \ \mathit{then} \max \{ r \mid (\exists x)[|x| = r \ \& \ 2 |P_M(x)| \leq n - r] \} \\ & \ \mathit{else} \ 1. \end{aligned}$$

Remark: It is easily seen that for each machine M in $\mathcal{N}(\Sigma^T)$ the function $p_M(n)$ is recursive and $p_M(n) \leq n$ for all n.

THEOREM 7.1: *Let m be a mimeoinvariant complexity measure and f be a semihomogeneous unbounded nondecreasing function m-limiting the s-set $S(G)$ of a cf-grammar $G = (\Sigma_1, W_1, I_1, P_1)$. Then for each Turing machine M in $\mathcal{N}(\Sigma^T)$ there is a right-normal form cf-grammar G_M of non-reducible m-complexity $\lambda n . f(p_M(n))$.*

Proof: To expose the needed *cf*-grammar it is convenient to describe first the language it generates. To this end we introduce several operations and predicates. Let m, f, G, and M be fixed.

NOTATION: 1. Consider $L \subset \Sigma^*$ and:

$$U_1,\ U_2 \subset \Sigma^* \times \Sigma^*.\ U_1 \blacksquare L = \{ z_1 z z_2 \mid z \text{ in } L,\ (z_1, z_2) \text{ in } U_1 \},$$

$$U_1 \bullet U_2 = \{ (z_{11} z_{21},\ z_{22} z_{12}) \mid (z_{11}, z_{12}) \text{ in } U_1,\ (z_{21}, z_{22}) \text{ in } U_2 \},$$

$$U_1^{(0)} = U_1,\ U_1^{(k+1)} = U_1^{(k)} \bullet U_1,\ U_1^{\bullet} = \bigcup_{k=0}^{\infty} U_1^{(k)}.$$

2. Let Q_1, Q_2 be two strings in $V_M^+ \cup V_M^* K_M V_M^*$. Then $ERR_M(Q_1, Q_2)$ means that either: (a) Q_1 is active, $|Q_2| = n_M(Q_1)$, but Q_2 does not coincide with the M-situation Q such that $Q_1 \underset{M}{\vdash} Q$, or (b) Q_1 is passive, $|Q_1| = |Q_2|$, but $Q_1 \neq Q_2$.

Now we proceed to the description of the grammar G_M and the language $L_M = L(G_M)$.

1. First of all we apply to the grammar $G = (\Sigma_1, W_1, I_1, P_1)$ the construction outlined above in the proof of the theorem 6.1, relating to it the grammar $\Gamma[G] = (\Sigma_0, W_0, I_0, P_0)$ with $\Sigma_0 = \Sigma_1 \cup \{a, b, c, d\}$. The language $L(\Gamma[G])$ we denote by L_0.

2. Then we introduce the following system of languages and pair languages (is a symbol in $\Sigma - (\Sigma^T \cup \{\S\})$):

$$L_1 = \{ c a^j y b^j d \mid j \geqq 0,\ y \text{ in } \{\Lambda\} \cup (\{c\} \cup \Sigma_1) \Sigma_0^* (\{d\} \cup \Sigma_1) \}.$$

$$L_2 = \{ Q^R \S x \mid Q \text{ is a situation of } M,\ x \text{ in } \Sigma_0^*,\ |x| \geqq |Q| \} \text{ [6]}.$$

$$U_0 = \{ (\Lambda, \star) \}.$$

$$U_1 = \{ (Q^R \S,\ \S Q) \mid Q \text{ is a final situation of } M \}.$$

$$U_2 = \{ (Q^R \S,\ \S Q) \mid Q \text{ is a situation of } M \}.$$

$$U_3 = \{ (\Lambda,\ \S Q) \mid Q \text{ is a situation of } M \}.$$

$$U_4 = \{ (Q_2^R \S Q_1^R \S,\ \S Q_3) \mid Q_1, Q_2, Q_3 \text{ are situations of } M,\ |Q_1 Q_2| < |Q_3| + n_M(Q_3) \}.$$

$$U_5 = \{ (Q_2^R \S Q_1^R \S,\ \S Q_3) \mid Q_1, Q_2, Q_3 \text{ are situations of } M,\ |Q_2| + n_M(Q_1) < 2|Q_3| \}.$$

$$U_6 = \{ (Q_{22}^R Q_{21}^R \S Q_1^R \S,\ \S Q_{31} Q_{32}) \mid Q_1, Q_{21} Q_{22}, Q_{31} Q_{32} \text{ are situations of } M,\ |Q_{31}| \geqq [|Q_1 Q_{21}|/3] - 1,\ ERR_M(Q_{32}, Q_{22}) \}.$$

$$U_7 = \{ (Q_2^R \S Q_{12}^R Q_{11}^R \S,\ \S Q_{31} Q_{32}) \mid Q_{11} Q_{12}, Q_2, Q_{31} Q_{32} \text{ are situations of } M,\ |Q_{32}| \geqq [|Q_2 Q_{12}|/3] - 2,\ ERR_M(Q_{11}, Q_{31}) \}.$$

$$U_8 = \{ (Q^R \S,\ \S Q) \mid Q \text{ is a starting situation of } M \}.$$

[6] z^R denotes the reversal of a string z : $[\Lambda]^R = \Lambda$, $[wa]^R = a[w]^R$.

Now the language L_M is defined as $L_M = \bigcup_{i=1}^{6} L_{Mi}$, where:

$$L_{M1} = (U_0^\bullet \bullet U_1 \bullet U_2^\bullet \bullet U_8) \blacksquare L_0,$$
$$L_{M2} = (U_0^\bullet \bullet U_2^\bullet \bullet U_3 \bullet U_4 \blacksquare U_2^\bullet) \blacksquare L_1,$$
$$L_{M3} = (U_0^\bullet \bullet U_2^\bullet \bullet U_5 \bullet U_3 \bullet U_2^\bullet) \blacksquare L_1,$$
$$L_{M4} = (U_0^\bullet \bullet U_2^\bullet \bullet U_3 \bullet U_6 \bullet U_2^\bullet) \blacksquare L_1,$$
$$L_{M5} = (U_0^\bullet \bullet U_2^\bullet \bullet U_7 \bullet U_3 \bullet U_2^\bullet) \blacksquare L_1,$$
$$L_{M6} = (U_0^\bullet \bullet U_2^\bullet \bullet U_3) \blacksquare L_2.$$

4. So that to specify the needed *cf*-grammar G_M let us notice that:

(*a*) $L_2 = U_9 \blacksquare \{\S\}$, where $U_9 = \{(Q^R, x) \mid Q$ is a situation of M, x in Σ_0^+, $|x| \geqq |Q|\}$;

(*b*) $L_1 = (U_{10} \bullet U_{11}^\bullet) \blacksquare R$, where $U_{10} = \{(c, d)\}$, $U_{11} = \{(a, b)\}$, $R = \{\Lambda\} \cup (\{c\} \cup \Sigma_1) \Sigma_0^* (\{d\} \cup \Sigma_1)$ is regular;

(*c*) for each pair language U_j, $0 \leqq j \leqq 11$, and each pair (z, w) in it $w \neq \Lambda$, so, there is a linear function g_j for each $0 \leqq j \leqq 11$ such that for all (v, u) in $U_j^\bullet$ $|v| \leqq g_j(|u|)$;

(*d*) for each regular language there is a right-linear *cf*-grammar generating it.

From (*a*)-(*d*) it follows directly that there are unambiguous right-normal form linear *cf*-grammars G_{M0}, G_{M2}, G_{M3}, G_{M4}, G_{M5}, G_{M6} generating respectively the languages:

$$L_{M0} = (U_0^\bullet \bullet U_1 \bullet U_2^\bullet \bullet U_8) \blacksquare \{I_0\}, \qquad L_{M2}, L_{M3}, L_{M4}, L_{M5}, L_{M6}.$$

Let $\overline{W}_j$ and $\overline{P}_j$ be respectively nonterminal alphabets and production sets of the grammars G_{Mj} for j in $\{0, 2, 3, 4, 5, 6\}$. We may assume without loss of generality that:

(1) all grammars G_{M0}, G_{M2}, G_{M3}, G_{M4}, G_{M5}, G_{M6} have a common axiom I;

(2) the alphabets W_0, $\overline{W}_0 - \{I\}$, $\overline{W}_2 - \{I\}$, $\overline{W}_3 - \{I\}$, $\overline{W}_4 - \{I\}$, $\overline{W}_5 - \{I\}$, $\overline{W}_6 - \{I\}$ are pairwise disjoint.

Let us denote by Σ_1^M the alphabet $K_M \cup V_M \cup \{\S, \star\} \cup \Sigma_0$ and by G_{M1} the *cf*-grammar $(\Sigma_1^M, \overline{W}_1, I, \overline{P}_1)$, where $\overline{W}_1 = W_0 \cup \overline{W}_0$ and $\overline{P}_1 = P_0 \cup \overline{P}_0$.

Finally let us set $G_M = (\Sigma_1^M, W_1^M, I, P^M)$, where $W_1^M = \bigcup_{j=1}^{6} \overline{W}_j$ and $P^M = \bigcup_{j=1}^{6} \overline{P}_j$.

It is easily seen that $L(G_{M1}) = L_{M1}$ and $L(G_M) = L_M$. By the construction all the *cf*-grammars we have described, including G_M, are in right-normal form. It remains to prove that G_M is a *cf*-grammar of nonreducible *m*-complexity $\lambda n . f(p_M(n))$.

UPPER BOUND: $m_{G_M} \preceq f(p_M)$. To establish this bound we will show that there is a $c>0$ such that $m_{G_M}(z) \leqq cf(p_M(|z|))$ for all z in L_M. Since the complexity measure m is mimeoinvariant there evidently is an integer $c_m>0$ such that for each linear s-tree $T\, m(T)<c_m$. This implies that for each cs-tree T in $S(G_{Mj})$, j in $\{0, 2, 3, 4, 5, 6\}$, $m(T)<c_m$, thus $m_{G_M}(z)<c_m$ for all z in $\bigcup_{j=2}^{6} L_{Mj}$. It suffice to establish the upper bound for all z in L_{M1}.

Now, let us take a string z in L_{M1}. z may be represented in form:

$$z=Q_n^R \,\S\, Q_{n-1}^R \,\S \ldots Q_1^R \,\S\, x \,\S\, Q_1 \ldots \S\, Q_{n-1} \,\S\, Q_n \,\star^k,$$

for some $k \geqq 0$, $n>1$, starting situation Q_1 of M, final situation Q_n of M, $1<j<n$, and x in L_0. Three alternatives arise.

1. The string $\S Q_1 \,\S Q_2 \ldots \S Q_n$ is not a computation record of M. In such a case there is a i, $1 \leqq i<n$, such that $Q_i \not\vdash_M Q_{i+1}$. Let i_0 be the least such i. The computation error $Q_{i_0} \not\vdash_M Q_{i_0+1}$ may be of the following four kinds.

(a) $|Q_{i_0+1}|>n_M(Q_{i_0})$. In this case we find that:

$$Q_{i_0-1}^R \,\S \ldots Q_2^R \,\S\, Q_1^R \,\S\, x \,\S\, Q_1 \,\S\, Q_2 \ldots \S\, Q_{i_0-1} \,\S\, Q_{i_0},$$

belongs to $(U_3 \bullet U_2^{\bullet}) \blacksquare L_1$, $([\S Q_{i_0} \,\S Q_{i_0+1}]^R, \S Q_{i_0+1})$ belongs to U_5 because:

$$2|Q_{i_0+1}|>|Q_{i_0+1}|+n_M(Q_{i_0}),\ ([\S Q_{i_0+2} \ldots \S Q_n]^R, \S Q_{i_0+2} \ldots \S Q_n \star^k)$$

is in $U_0^{\bullet} \bullet U_2^{\bullet}$, and x is in L_1 because $L_0 \subseteqq L_1$. Thus z belongs to L_{M3}, there is a linear cs-tree T in $S(G_M)$ with the yield z and therefore $m_{G_M}(z)<c_m$.

(b) $|Q_{i_0+1}|<n_M(Q_{i_0})$. In this situation we see that:

$$Q_{i_0-1}^R \,\S \ldots Q_2^R \,\S\, Q_1^R \,\S\, x \,\S\, Q_1 \,\S\, Q_2 \ldots \S\, Q_{i_0-1}$$

belongs to $U_2^{\bullet} \blacksquare L_1$, $([\S Q_{i_0} \,\S Q_{i_0+1}]^R, \S Q_{i_0})$ is in U_4 because:

$$|Q_{i_0}|+n_M(Q_{i_0})>|Q_{i_0} Q_{i_0+1}|,$$

$(\Lambda, \S Q_{i_0+1})$ is in U_3, $([\S Q_{i_0+2} \ldots \S Q_n]^R, \S Q_{i_0+2} \ldots \S Q_n \star^k)$ belongs to $U_0^{\bullet} \bullet U_2^{\bullet}$, and x belongs to L_1. Thus z is in L_{M2} and therefore $m_{G_M}(z)<c_m$.

The bounds established in (a) and (b) show that in the rest we may assume without loss of generality that $|Q_{i_0+1}|=n_M(Q_{i_0})$. We need some additional notions and notation for the analysis to follow.

NOTATION: Let $Q=z_1\, \alpha\beta\gamma\, z_2$ be the representation of a situation of M such that $|z_1|=|z_2|$, $|\beta| \leqq 1$, and α, γ are in $V_M \cup K_M$. A *central partition* of Q is the unique partition $Q=l_M(Q)\,r_M(Q)$ such that:

(a) if $z_1\,\alpha\beta$ is active then $l_M(Q)=z_1\,\alpha\beta\gamma$ and $r_M(Q)=z_2$;

(b) if $\gamma\,z_2$ is active then $l_M(Q)=z_1$ and $r_M(Q)=\alpha\beta\gamma\,z_2$.

2. Let Q_1, Q_2 be two situations of M such that $|Q_2|=n_M(Q_1)$. We call a partition $Q_2=Q'_2\,Q''_2$ of Q_2 Q_1-*derivative* if either $l_M(Q_1)$ is active and $|Q'_2|=n_M(l_M(Q_1))$, or $r_M(Q_1)$ is active and $|Q''_2|=n_M(r_M(Q_1))$.

Let us return to the proof.

Let $Q_{i_0}=Q'_{i_0}\,Q''_{i_0}$ be the central partition of Q_{i_0} and $Q_{i_0+1}=Q'_{i_0+1}\,Q''_{i_0+1}$ be the Q_{i_0}-derivative partition of Q_{i_0+1}. There are four additional cases.

(c) Q'_{i_0} is active and $Q'_{i_0}\underset{M}{\not\vdash}Q'_{i_0+1}$. First of all we have $|Q'_{i_0+1}|=n_M(Q'_{i_0})$ and $|Q''_{i_0+1}|=|Q''_{i_0}|$, From this follows:

$$|Q''_{i_0}Q_{i_0+1}|\leqq|Q''_{i_0}|+|Q_{i_0}|+1\ \leqq|Q''_{i_0}|+(2\,|Q''_{i_0}|+3)+1=3\,|Q''_{i_0+1}|+4.$$

Thus:

$$|Q''_{i_0+1}|\geqq\left[\frac{|Q_{i_0+1}\,Q''_{i_0}|-4}{3}\right]\geqq\left[\frac{|Q_{i_0+1}\,Q''_{i_0}|}{3}\right]-2.$$

Besides this we have $ERR_M(Q'_{i_0},\,Q'_{i_0+1})$. That is why in this case:

$$Q^R_{i_0-1}\S\ldots Q^R_2\,\xi\,Q^R_1\,\S\,x\,\S\,Q_1\,\S\,Q_2\ldots\S\,Q_{i_0-1}\,\S\,Q_{i_0}\quad\text{is in}\quad(U_3\bullet U_2^\bullet)\blacksquare L_1,$$

$([\S\,Q'_{i_0}\,Q''_{i_0}\,\S\,Q_{i_0+1}]^R,\ \S\,Q'_{i_0+1}\,Q''_{i_0+1})$ *is in* U_7

and:

$$([\S\,Q_{i_0+2}\ldots\S\,Q_n]^R,\ \S\,Q_{i_0+2}\ldots\S\,Q_n\,\star^k)\quad\text{is in}\quad U_0^\bullet\bullet U_2^\bullet.$$

Therefore z is in L_{M5}.

(d) Q'_{i_0} is passive and $Q'_{i_0}\neq Q'_{i_0+1}$. Since Q'_{i_0} is passive we have $|Q'_{i_0+1}|<|Q''_{i_0+1}|$, $|Q''_{i_0}|\leqq|Q''_{i_0+1}|+1$ and therefore $|Q_{i_0+1}\,Q''_{i_0}|\leqq 3\,|Q''_{i_0+1}|$. Hence

$$|Q''_{i_0+1}|>\left[\frac{|Q_{i_0+1}\,Q''_{i_0}|}{3}\right]-2.$$

Of course, $ERR_M(Q'_{i_0},\ Q'_{i_0+1})$ is true, so, as in the preceeding case $([\S\,Q'_{i_0}\,Q''_{i_0}\,\S\,Q_{i_0+1}]^R,\ \S\,Q'_{i_0+1}\,Q''_{i_0+1})$ is in U_7 and z falls into L_{M5} again.

(e) Q''_{i_0} is active and $Q''_{i_0}\underset{M}{\not\vdash}Q''_{i_0+1}$. In this case $|Q''_{i_0}|\leqq|Q'_{i_0}|+3$ and $|Q'_{i_0+1}|=|Q'_{i_0}|$, hence:

$$|Q_{i_0}\,Q'_{i_0+1}|\leqq 3\,|Q'_{i_0}|+3\qquad\text{and}\qquad|Q'_{i_0}|\geqq\left[\frac{|Q_{i_0}\,Q'_{i_0+1}|}{3}\right]-1;$$

As $ERR_M(Q''_{i_0}, Q''_{i_0+1})$ is true we infere that:

$$([\S Q_{i_0} \S Q'_{i_0+1} Q''_{i_0+1}]^R, \S Q'_{i_0} Q''_{i_0})$$

belongs to U_6. Therefore:

$$Q^R_{i_0+1} \S Q^R_{i_0} \ldots \S Q^R_1 \S x \S Q_1 \S \ldots \S Q_{i_0} \quad \text{is in } (U_6 \bullet U_2^\bullet) \blacksquare L_1,$$

$$Q^R_{i_0+1} \S \ldots Q^R_1 \S x \S Q_1 \ldots \S Q_{i_0+1} \quad \text{is in } (U_3 \bullet U_6 \bullet U_2^\bullet) \blacksquare L_1,$$

and z is in L_{M4}.

(f) The last alternative is: Q''_{i_0} is passive and $Q''_{i_0+1} \neq Q''_{i_0}$. Here we have $|Q''_{i_0}| < |Q'_{i_0}|$ and $|Q'_{i_0+1}| \leqq |Q'_{i_0}| + 1$. Hence:

$$|Q_{i_0} Q'_{i_0+1}| < 3|Q'_{i_0}| + 1 \qquad \text{and} \qquad |Q'_{i_0}| > \left[\frac{|Q_{i_0} Q'_{i_0+1}|}{3}\right] - 1.$$

As $|Q''_{i_0}| = |Q''_{i_0+1}|$ and $Q''_{i_0+1} \neq Q''_{i_0}$, $ERR_M(Q''_{i_0}, Q''_{i_0+1})$ is true and $([\S Q_{i_0} \S Q'_{i_0+1} Q''_{i_0+1}]^R, \S Q'_{i_0} Q''_{i_0})$ belongs to U_6. Therefore just as in the preceeding case z falls into L_{M4}.

Thus in all the cases (c) to (f) $m_{G_M}(z) < c_m$. This shows that $m_{G_M}(z) < c_m$ for all z in L_{M1} of the form $z = P^R x P \star^k$, where P is not a computation record of M. Now let us take the second alternative.

2. $\S Q_1 \S Q_2 \ldots \S Q_n$ is a computation record of M but $|x| \geqq |Q_1|$. Then we find that $Q_1^R \S x$ belongs to L_2, $Q_1^R \S x \S Q_1$ belongs to $U_3 \blacksquare \mathbf{L}_2$, and z falls into L_{M6}. So in this case as in all the preceeding $m_{G_M}(z) < c_m$.

The last alternative is:

3. $z = Q_n^R \S Q_{n-1}^R \S \ldots Q_1^R \S x \S Q_1 \ldots \S Q_{n-1} \S Q_n \star^k$, x in L_0, $|x| < |Q_1|$, and $\S Q_1 \ldots \S Q_n$ is a y-computation record of M for some y. Such a string z is evidently in $L_{M1} - \bigcup_{i=2}^{6} L_{Mi}$. Hence there is the only one derivation tree T in $S(G_{M1})$ such that $t(T) = z$ and therefore $m_{G_M}(z) = m(T)$. By the definition of G_{M1} this tree T can be represented in the form $T = \text{com}(T_0, v, T_1)$, where T_0 is the linear derivation tree in $S(G_{M0})$ such that $t(T_0) = Q_n \ldots Q_1^R \S I_0 \S Q_1 \ldots \S Q_n \star^k$, v is the bottom node of T_0 such that $l(v) = I_0$, and T_1 is the derivation tree in $S(\Gamma[G])$ such that $t(T_1) = x$.

Now let us notice that since $|x| \leqq |y|$ we have $|P_M(x)| \leqq |P_M(y)|$. This implies that $|P_M(x) x P_M(x)| \leqq |z| - k$ which in turn implies $|x| \leqq p_M(|z| - k) \leqq p_M(|z|)$.

Since T_1 is a tree in $S(\Gamma[G])$ there is a tree T_2 in $S(G)$ such that $t(T_2) = u$, u in $L(G)$, and $T_2 \leqq^{sl} T_1$. First of all this implies that $|u| \leqq |x|$ and hence $|u| \leqq p_M(|z|)$. Secondly, f is m-limiting the s-set $S(G)$. From this it follows that

$m(T_2) \leqq c_f f(|u|)$ for a constant $c_f > 0$ independent of T_2. Third, because m is mimeoinvariant $m(T_1) \leqq c_A m(T_2)$ is true for $c_A > 0$ independent of T_1, T_2. Finally we remark that $T = \text{com}(T_0, v, T_1)$ implies $T_1 \leqq^{sl} T$ and therefore $m(T) \leqq c_A m(T_1)$. Grouping together all these inequalities and bearing in mind that f is nondecreasing we infere:

$$m_{G_M}(z) = m(T) \leqq c_A m(T_1) \leqq c_A^2 m(T_2) \leqq c_f c_A^2 f(|u|) \leqq c_f c_A^2 f(p_M(|z|)),$$

which gives the needed upper bound.

LOWER BOUND: To establish this bound it is enough to specify for each *cf*-grammar G such that $L(G) = L_M$ an infinite sequence $(z_i \mid i \text{ in } N)$ in L_M such that for an integer $c > 0$ and for all i in N $c m_G(z_i) \geqq f(p_M(|z_i|))$.

Let G be a *cf*-grammar such that $L(G) = L_M$. The following lemma which is the main technical means of our proof of the lower bound assigns to each *cf*-grammar G a parameter $n(G)$.

LEMMA 7.1: *For each cf-grammar G with infinite language $L(G)$ there is an integer $n(G) > 0$ such that for any x in $L(G)$ and any its complete derivation $D = (I = X_1, \ldots, X_m = x)$ if $x = x_1 z x_2$ and $|z| > n(G)$ then the substring z can be decomposed into three parts $z = z_1 u z_2$ so that $|u| \neq 0$ and the derivation D is representable either in form:*

$$I \Rightarrow^* x_1 z_1 A y \Rightarrow^* x_1 z_1 u A u_1 y \Rightarrow^* x_1 z_1 u v u_1 y = x,$$

or in form:

$$I \Rightarrow^* y A z_2 x_2 \Rightarrow^* y u_1 A u z_2 x_2 \Rightarrow^* y u_1 v u z_2 x_2 = x.$$

[This result is due to A. V. Gladkii (*see* for example [13]). Some later it was reproven in a stronger form by W. Ogden [14].]

To specify the sequence $(z_i \mid i \text{ in } N)$ we need some notions and notation.

NOTATION: With each tree T in $S(G)$ we will associate an infinite sub-language of L_M whose elements we will call *T-terms*. A T-term will be defined by induction on full subtrees $T(v)$ of T:

(1) let v be a bottom node of T with $l(v) = X$. Then X is an *elementary* $T(v)$*-term;*

(2) let v be a node of T such that $i(v) = \{v_1, \ldots, v_k\}$, $v_1 \lhd v_2 \lhd \ldots \lhd v_k$ and the $T(v_i)$-terms θ_i have been defined so far for all $1 \leqq i \leqq k$. Then for all $j \geqq 0$ the string $\theta^{\langle j \rangle}(T, v) = c a^j \theta_1 \ldots \theta_k b^j d$ is a $T(v)$-term, j is a *degree* of this term, and $\theta_1, \ldots, \theta_k$ are its *subterms*;

(3) each term θ is a subterm of itself and each subterm of a subterm of θ is a subterm of θ;

(4) if v_0 is the root of T then $\theta^{\langle j \rangle}(T, v_0)$ is a *T-term* for each $j \geqq 0$.

Now we can specify the sequence $(z_j \mid j \text{ in } N)$. Let $T_1, T_2, T_3, \ldots$ be a fundamental sequence of f in $S(G)$. Let us associate with each T_j the T_j-term θ_j such that the degrees of all its nonelementary subterms are equal to $n_1 = n(G)+1$. Let e be the first symbol in V_M. Then we set:

$$z_i = [P_M(e^{r_i})]^R \theta_i P_M(e^{r_i}) \star^{v|\theta_i|},$$

where v is the width [7] of G and $r_i = (v+1)|\theta_i|$.

Let us note that the set $\{r_i \mid i \text{ in } N\}$ and hence the set $\{z_i \mid i \text{ in } N\}$ are infinite. Besides this $p_M(|z_i|) = r_i$ for each i. Therefore it is enough to show that there is an integer $c > 0$ such that for all i, $cm_G(z_i) \geqq f(r_i)$. The proof of this last statement is rather tedious and lengthy. Its essence is the following proposition.

PROPOSITION 7.1: *Let T in $S(G)$ be a cs-tree such that $t(T) = z_i$ for some i. Then T can be represented in form $T = \mathrm{com}(T_0, v, T(v))$ so that $T_i \leqq T(v)$.*

The proof of this proposition is omitted here. However it may be found in [6] (a part of the proof of theorem 9.5) and in [8] (a part of the proof of theorem 2).

So let T be the simplest tree in $S(G)$ such that $t(T) = z_i$. Then $m_G(z_i) = m(T)$. Since m is mimeoinvariant and by proposition 7.1 $d_B m(T) \geqq m(T_i)$ for a $d_B > 0$ independent of T and T_i. As $(T_i \mid i \text{ in } N)$ is a fundamental sequence of f in $S(G)$ there is a $e_f > 0$ independent of i such that $e_f m(T_i) > f(|T_i|)$. It is easy to see that $|\theta_i| < 2(v+1)(2n_1+3)|T_i|$, which implies that $r_i \leqq 2(v+1)^2(2n_1+3)|T_i|$. Since f is a semihomogeneous function there is a $c > 0$ independent of i such that $cf(|T_i|) \geqq f(r_i)$. Finally we have

$$d_B e_f cm_G(z_i) = d_B e_f cm(T) \geqq e_f cm(T_i) \geqq cf(|T_i|) \geqq f(r_i)$$

for all i.

Q.E.D.

COROLLARY 7.1: *Let m be a mimeoinvariant complexity measure and f be a nondecreasing unbounded semihomogeneous recursive function m-limiting the s-set $S(G)$ of a cf-free grammar G. Then for each unbounded and nondecreasing with respect to both arguments recursive function $\lambda m, n.h(m, n)$ there are an unbounded nondecreasing recursive function φ and a cf-grammar G in right-normal form of nonreducible m-complexity φ such that for all but finitely many n, $\varphi(n) \leqq h(n, f(n))$.*

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[7] The width of a *cf*-free grammar G is the least integer v such that $S(G) \subseteq \mathscr{S}^c(\Sigma, W, v)$.

COROLLARY 7.2: *Let m be a mimeoinvariant complexity measure, $\equiv_r$ be a reasonable equivalence relation on $\mathscr{E}$, and f be a nondecreasing unbounded semihomogeneous recursive function m-limiting the s-set $S(E)$ of a MR-scheme E. Then for each unbounded and nondecreasing with respect to both arguments recursive function $\lambda m, n . h(m, n)$ there are an unbounded nondecreasing recursive function φ and a MR-scheme E_φ of r-nonreducible m-complexity φ such that for all but finitely many n, $\varphi(n) \leq h(n, f(n))$.*

COROLLARY 7.3: *Let m be a mimeoinvariant complexity measure and $\equiv_r$ be a reasonable equivalence relation on $\mathscr{E}$. If there exist a MR-scheme E and an unbounded nondecreasing semihomogeneous function f m-limiting $S(E)$ then there is an infinite sequence of unbounded nondecreasing functions $f_1 \succ f_2 \succ f_3 \succ \ldots$ such that $f = f_1$ and for no $i<j$, $\mathscr{E}^m_{f_i} \Rrightarrow_r \mathscr{E}_{f_j}{}^m$.*

COROLLARY 7.4: *Let m be a mimeoinvariant complexity measure and $\equiv_r$ be a reasonable equivalence relation on $\mathscr{E}$. If there exist a MR-scheme E and an unbounded nondecreasing semihomogeneous recursive function f m-limiting $S(E)$ then for no nondecreasing unbounded recursive function g, $\mathscr{E}^m_g \Rrightarrow_r \mathscr{E}^m_{\text{Const.}}$.*

The proof of the corollary 7.1 may be found in [6] (corollary 1 from the theorem 9.3) and in [7] (corollary 1 from the theorem 3). Corollaries 7.2-7.4 follow from it directly.

8. ALL MIMEOINVARIANT COMPLEXITY MEASURES PROVIDE INFINITE CLASSIFICATIONS OF *MR*-SCHEMES

In sections 6, 7 we considered some simple conditions sufficient for the existence of individual MR-schemes or infinite hierarchies of MR-schemes of nonreducible complexities. It is a pity but we cannot guarantee that these conditions hold for all mimeoinvariant complexity measures. So in this section classes of MR-schemes are compaired in terms of set theoretical inclusion, and not in terms of translatability. In this much weaker sense we will show that all mimeoinvariant complexity measures provide nondegenerate classifications of MR-schemes. To this end we will simplify the construction of the theorem 7.1 so as to infere that for each mimeoinvariant complexity measure m there is an infinite hierarchy of *cf*-grammars in right-normal form of different m-complexities (these grammars however not always being of nonreducible m-complexities).

THEOREM 8.1: *For each mimeoinvariant complexity measure m there is an unbounded nondecreasing recursive function f such that for each Turing machine M in $\mathcal{N}(\Sigma^I)$ there is a right-normal form cf-grammar G_{Mm} such that $m_{G_{Mm}} \rightleftarrows f(p_M)$.*

Proof: Our mimeoinvariant complexity measure m is nondegenerate by definition. So there is an unambiguous MR-scheme E with unbounded complexity function. We apply to E the proposition 6.2 so that to obtain the MR-scheme $\hat{E}$ with the following properties:

1. $G(\hat{E})$ is a right-normal form *cf*-grammar;

2. $\hat{E}$ is unambiguous (because E is unambiguous);

3. $m_{\hat{E}}$ is recursive (because every nonbottom node of every tree T in $S(E)$ has a width no less than 2, and therefore there are only finitely many trees in $S(\hat{E})$ with n bottom nodes for all n);

4. $m_{\hat{E}}$ is unbounded (because m is mimeoinvariant and hence $m_E \asymp m_{\hat{E}}$).

Next we show that the complexity function $m_{\hat{E}}$ is m-limiting $S(\hat{E})$. First of all we infere from unambiguity of $S(\hat{E})$ that:

$$m_{S(\hat{E})}(n)=\max\{0, m(T) \mid T \text{ in } S(\hat{E}), |T| \leqq n\} \quad \text{for all } n.$$

This means that for every tree T in $S(\hat{E})$, $m(T) \leqq m_{S(\hat{E})}(|T|)$. Secondly we choose for each n a tree T_n in $S(E)$ (if any) such that:

$$|T_n| \leqq n \qquad \text{and} \qquad m(T_n)=\max\{m(T) \mid T \text{ in } S(\hat{E}), |T| \leqq n\}.$$

Since $m_{S(\hat{E})}$ is unbounded and because of the abovementioned width property of $S(\hat{E})$ the set $\{|T_n| \mid n>0\}$ is infinite. Finally we note that $m(T_n)=m_{S(\hat{E})}(|T_n|)$ for all (but finitely many) n.

This argument shows that there are a *cf*-grammar $G_0=(\Sigma_0, W_0, I_0, P_0)$ in right-normal form and an unbounded nondecreasing recursive function f m-limiting the s-set $S(G_0)$. We apply to G_0 the following construction.

Let M be a Turing machine in $\mathcal{N}(\Sigma^T)$. We associate with M and G_0 the pair languages U_1-U_8 from the proof of the theorem 7.1 and the languages:

$$L_3=\{Q^R x \mid Q \text{ is a situation of } M, x \text{ is in } \Sigma_0^*, 1+|x|>|Q|\},$$

$$L_4=\{Q_1^R x Q_2 \mid Q_1, Q_2 \text{ are situations of } M, \; x \text{ is in } \Sigma_0^*, 2|Q_2|>|Q_1|+|x|+1\},$$

and set $L_{Mm}=\bigcup_{i=1}^{7} L_{Mmi}$, where:

$$L_{Mm1}=(U_1 \bullet U_2^{\bullet} \bullet U_8) \blacksquare L(G_0),$$

$$L_{Mm2}=(U_2^{\bullet} \bullet U_3 \bullet U_4 \bullet U_2^{\bullet}) \blacksquare \Sigma_0^+,$$

$$L_{Mm3}=(U_2^{\bullet} \bullet U_5 \bullet U_3 \bullet U_2^{\bullet}) \blacksquare \Sigma_0^+,$$

$$L_{Mm4}=(U_2^{\bullet} \bullet U_3 \bullet U_6 \bullet U_2^{\bullet}) \blacksquare \Sigma_0^+,$$

$$L_{Mm5}=(U_2^{\bullet} \bullet U_7 \bullet U_3 \bullet U_2^{\bullet}) \blacksquare \Sigma_0^+,$$

$$L_{Mm6}=(U_2^{\bullet} \bullet U_3) \blacksquare L_3,$$

$$L_{Mm7}=U_2^{\bullet} \blacksquare L_4.$$

There are linear *cf*-grammars in right-normal form G_{Mm0}, G_{Mm2}, G_{Mm3}, G_{Mm4}, G_{Mm5}, G_{Mm6}, G_{Mm7} generating respectively the languages $L_{Mm0}=(U_1 \bullet U_2^{\bullet} \bullet U_8) \blacksquare \{I_0\}$, L_{Mm2}, L_{Mm3}, L_{Mm4}, L_{Mm5}, L_{Mm6}, L_{Mm7}. We assume that these grammars share the axiom I and that for any two of them I is their single common nonterminal. Besides this we denote by Σ_0^M the alphabet $K_M \cup V_M \cup \Sigma_0 \cup \{\S\}$ and by G_{Mm1} the *cf*-grammar $(\Sigma_0^M, W'_0, I, P'_0)$, where $W'_0=W_0 \cup \overline{W}_0$, $P'_0=P_0 \cup \overline{P}_0$, $\overline{W}_0$ is the nonterminal alphabet and $\overline{P}_0$ is the production set of G_{Mm0}.

Finally we set $G_{Mm}=(\Sigma_0^M, W_1, I, P)$, where W_1 is the union of nonterminal alphabets and P is the union of production sets of the grammars G_{Mmj}, $1 \leqq j \leqq 7$. Of course, $L(G_{Mm})=L_{Mm}$ and $L(G_{Mm1})=L_{Mm1}$.

UPPER BOUND: $m_{G_{Mm}} \preccurlyeq f(p_M)$. The proof of this inequality is very close to the proof of the corresponding inequality in the theorem 7.1 and is left to the reader.

LOWER BOUND: $m_{G_{Mm}} \rightrightarrows f(p_M)$. The proof of this statement is straightforward. Indeed, since f is m-limiting $S(G_0)$ we find there a fundamental sequence $(T_i^f \mid i>0)$. Let us denote by x_i the string $t(T_i^f)$ and set $z_i=[P_M(e^{|x_i|})]^R x_i P_M(e^{|x_i|})$ for each i, where e is the first symbol of V_M. The string $[P_M(e^{|x_i|})]^R I_0 P_M(e^{|x_i|})$ is the yield of a single tree T_i^M in $S(G_{Mm0})$ for each i. We denote by T_i the tree $\mathrm{com}(T_i^M, v_i, T_i^f)$, where v_i is the single bottom node of T_i^M labelled by I_0. It is obvious that T_i is in $S(G_{Mm1})$, $t(T_i)=z_i$, and T_i is the single tree in $S(G_{Mm})$ with the yield z_i. This means that for each i, $m_{G_{Mm}}(z_i)=m(T_i)$. Since $T_i^f \leqq T_i$ and m is mimeoinvariant there is an integer $d_B>0$ (not dependent on i) such that $m(T_i^f) \leqq d_B m(T_i)$. Further, $(T_i^f \mid i>0)$ is a fundamental sequence for f in $S(G_0)$, so there is an integer $c_f>0$ one for all i such that $c_f m(T_i^f) \geqq f(|x_i|)$. Finally, for all i, $|x_i|=p_M(|z_i|)$. Summarizing these inequalities we obtain for all i:

$$c_B c_f m_{G_{Mm}}(|z_i|) \geqq c_B c_f m_{G_{Mm}}(z_i)=c_b c_f m(T_i)$$
$$\geqq c_f m(T_i^f) \geqq f(|x_i|)=f(p_M(|z_i|)).$$

Q.E.D.

COROLLARY 8.1: *For any mimeoinvariant complexity measure m there is an infinite sequence of unbounded nondecreasing recursive functions* $f_1 \succ f_2 \succ f_3 \succ \dots$ *such that for all* $j>0$, $\mathscr{E}_{f_j}^m - \mathscr{E}_{f_{j+1}}^m \neq \emptyset$.

COROLLARY 8.2: *If m is a mimeoinvariant complexity measure then for any nondecreasing unbounded recursive function* φ, $\mathscr{E}_\varphi^m - \mathscr{E}_{\mathrm{Const.}}^m \neq \emptyset$.

9. COMPLEXITY OF UNAMBIGUOUS *MR*-SCHEMES

In this little section we show that under most reasonable conditions complexity of unambiguous MR-schemes is of extremal nature. We discuss first a formalization of an informal concept of extremal complexity.

DEFINITION 9.1: Let m be a complexity measure and f be a function limiting it. Then we say that a MR-scheme E is of *maximal m-complexity* if for no $g \prec f$, E is in $\mathscr{E}_g^m$.

The following proposition shows that this definition is sensible at least for mimeoinvariant complexity measures.

PROPOSITION 9.1: *Let m be a complexity measure and f be a function limiting m. Then:*

(a) *if E is a MR-scheme of maximal m-complexity then it is not in* $\mathscr{E}_{\text{Const.}}^m$;

(b) *if m is mimeoinvariant and f is semihomogeneous then there exist MR-schemes of maximal m-complexity.*

Proof: (a) Since the range of m is infinite f is unbounded; hence m_E is unbounded too.

(b) Let $T_1, T_2, T_3, \ldots$ be a fundamental sequence of f. Since m is mimeoinvariant we may assume without loss of generality that in every tree T_i in this sequence each nonbottom node is of width no less than 2 (we will refer to this condition as width condition).

Let k_0 be a number such that $\{T_1, T_2, T_3, \ldots\} \subseteq \mathscr{S}^c(\Sigma, W, k_0)$. Consider the MR-scheme:

$$E_{k_0}: \quad Fx = (px \mid cx, aFbx, \ldots, aF^{k_0}bx),$$

where p is in $\mathscr{P}_{k_0+1}$, and a, b, c are basic function symbols. E_{k_0} is unambiguous and has the following property: for each $i>0$ there is a tree T_i^* in $S(E_{k_0})$ such that $T_i \leqq^{sl} T_i^*$ while $|T_i^*| < 3|T_i|$ (this upper bound follows directly from the width condition). Since E_{k_0} is unambiguous we have $m_{E_{k_0}}(|T_i^*|) \geqq m_{E_{k_0}}(T_i^*) = m(T_i^*)$, by the axiom B in the definition 5.4 there is a $d_B > 0$ (one for all i) such that $d_B m(T_i^*) \geqq m(T_i)$. As T_i is a member of the fundamental sequence $cm(T_i) \geqq f(|T_i|)$ (c is independent of i). Finally the linear inequality $|T_i^*| < 3|T_i|$ and semihomogenity of f imply that there is a $b>0$ such that $bf(|T_i|) \geqq f(|T_i^*|)$ for all i. Hence $bc\,d_B m_{E_{k_0}}(T_i^*) \geqq f(|T_i^*|)$ for all i.

Q.E.D.

Remark: For density and branching we have $\mu_{E_2} \rightleftarrows \log n$, $b_{E_2} \rightleftarrows n$. Since μ and b are both mimeoinvariant and the functions $\log n$ and $\lambda n.n$ are both semihomogeneous E_2 is a MR-scheme of maximal density and maximal branching.

THEOREM 9.1: *Let m be a mimeoinvariant complexity measure m-limited by a semihomogeneous function. Then every unambiguous MR-scheme is either of maximal m-complexity or of bounded density (i.e. falls into* $\mathscr{E}^{\mu}_{\text{Const.}}$*).*

Proof: The proposition 6.2 guarantees that for each MR-scheme E there is a MR-scheme $\hat{E}$, unambiguous if E is unambiguous, such that $m_E \asymp m_{\hat{E}}$ and $S(E)=S(G(\hat{E}))$. This reduces our theorem to the following theorem proven in [6, 9]:

If a mimeoinvariant complexity measure m is m-limited by a semihomogeneous function then every unambiguous cf-grammar is either of maximal m-complexity or of bounded density.

Q.E.D.

Remark: From results of [11, 12] it follows that $\mathscr{E}^{\mu}_{\text{Const.}}$ coincides with the class of all quasirational ([8]) MR-schemes.

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