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**A MIXED THEORY OF INFORMATION.
I: SYMMETRIC, RECURSIVE
AND MEASURABLE ENTROPIES
OF RANDOMIZED SYSTEMS OF EVENTS (*) (1)**

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Abstract. — *The paper contains the first result in a mixed theory of information, where measures of information may depend both upon the events and their probabilities. All such entropies that are 3-symmetric, recursive and measurable are determined.*

1. In the *probabilistic* theory of information (see, e. g., [3]) the entropies and other measures of information or uncertainty are supposed to depend *solely* upon the probabilities of the events (messages, outcomes of an experiment, weather, market situations, answers to a questionnaire, etc.). On the other hand, in the *nonprobabilistic* theory of information (see, e. g., [4, 7]) these measures do *not* depend upon the probabilities *at all*, only directly upon the events themselves.

After a result of B. Forte [5] in the similar case of random variables, one of us has proposed in [1, 2] a *mixed* theory of information, where measures of information may depend *both* upon the events *and* their probabilities. The present paper contains the first result in this direction. Generalizing an important theorem of Lee [9], we determine all 3-symmetric, recursive, and measurable entropies depending upon a system of events and their probabilities, which we will call a *randomized system of events*. We will also refer to entropies of randomized systems of events in short as “inset entropies” (inset: a map set within another map; but one may also consider it (1) as “in set”). Under the above conditions, they turn out to be essentially the sum of a Shannon entropy and of the expected value of a random variable.

2. Let B be a ring of sets (containing, with any two sets also their union and their difference, thus also their intersection and the empty set 0 ; see [6]). Denote

$$\Omega_n = \{ (x_1, x_2, \dots, x_n) \mid x_i \in B, x_i \cap x_j = 0 \text{ if } i \neq j; i, j = 1, 2, \dots, n \}$$

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(1) This paper has been conceived at the meeting Colloque international du Centre national de Recherche scientifique, Les développements récents de la théorie d'information et leurs applications, organized by C.-F. Picard, July 4-8, 1977 in E.N.S.E.T. at Cachan, France.

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and

$$\Gamma_n = \left\{ (p_1, p_2, \dots, p_n) \mid \sum_{i=1}^n p_i = 1, p_i \geq 0; i = 1, 2, \dots, n \right\}$$

($n = 2, 3, \dots$). We call

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} \in \Omega_n \times \Gamma_n,$$

a randomized system of events. We use *events* x_i as name for the elements of B , while the p_i are *probabilities*.

The sequence of mappings (inset entropy) $I_n : \Omega_n \times \Gamma_n \rightarrow R$ ($n = 2, 3, \dots$; R the set of reals) is *recursive* if, for all integers $n > 2$, and all

$$\begin{aligned} \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} \in \Omega_n \times \Gamma_n, \\ I_n \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ p_1 & p_2 & p_3 & \dots & p_n \end{pmatrix} &= I_{n-1} \begin{pmatrix} x_1 \cup x_2 & x_3 & \dots & x_n \\ p_1 + p_2 & p_3 & \dots & p_n \end{pmatrix} \\ &+ (p_1 + p_2) I_2 \begin{pmatrix} x_1 & x_2 \\ p_1 & p_2 \end{pmatrix}, \end{aligned}$$

with the convention $0 \cdot I_2 \begin{pmatrix} x_1 & x_2 \\ 0/0 & 0/0 \end{pmatrix} := 0$. This states how the uncertainty changes if an event is split into two; it is also connected to Huffman codes and algorithms. The sequence $\{I_n\}$ is *k-symmetric* ($k \geq 2$) if

$$I_k \begin{pmatrix} x_1 & \dots & x_k \\ p_1 & \dots & p_k \end{pmatrix} = I_k \begin{pmatrix} x_{r(1)} & \dots & x_{r(k)} \\ p_{r(1)} & \dots & p_{r(k)} \end{pmatrix},$$

for all $\begin{pmatrix} x_1 & \dots & x_k \\ p_1 & \dots & p_k \end{pmatrix} \in \Omega_k \times \Gamma_k$ and all permutations r on $\{1, 2, \dots, k\}$ (meaning simply that the uncertainty does not depend upon the labelling of events). Finally, our inset entropy is *measurable* if the function

$$t \mapsto I_2 \begin{pmatrix} x_1 & x_2 \\ 1-t & t \end{pmatrix}, \quad (1)$$

is measurable on $]0, 1[$ for all fixed $(x_1, x_2) \in \Omega_2$.

THEOREM: *The sequence $I_n : \Omega_n \times \Gamma_n \rightarrow R$ ($n = 2, 3, \dots$) is recursive, 3-symmetric and measurable if, and only if, there exists a constant A and a*

function $g : B \rightarrow R$ such that

$$I_n \begin{pmatrix} x_1, & \dots, & x_n \\ p_1, & \dots, & p_n \end{pmatrix} = g \left(\bigcup_{i=1}^n x_i \right) - \sum_{i=1}^n p_i g(x_i) - A \sum_{i=1}^n p_i \log p_i, \quad (2)$$

for all $\begin{pmatrix} x_1, & \dots, & x_n \\ p_1, & \dots, & p_n \end{pmatrix} \in \Omega_n \times \Gamma_n$ ($n = 2, 3, \dots$) with the convention

$$0 \cdot \log 0 := 0. \quad (3)$$

3. Proof: It is obvious that any inset entropy given by (2) with arbitrary $A \in R$ and $g : B \rightarrow R$ is recursive, symmetric and measurable. Now we prove the converse.

Recursivity means, for $n = 3$,

$$I_3 \begin{pmatrix} x_1, & x_2, & x_3 \\ p_1, & p_2, & p_3 \end{pmatrix} = I_2 \begin{pmatrix} x_1 \cup x_2, & x_3 \\ p_1 + p_2, & p_3 \end{pmatrix} + (p_1 + p_2) I_2 \begin{pmatrix} x_1, & x_2 \\ p_1, & p_2 \end{pmatrix}, \quad (4)$$

for all $\begin{pmatrix} x_1, & x_2, & x_3 \\ p_1, & p_2, & p_3 \end{pmatrix} \in \Omega_3 \times \Gamma_3$. We introduce a function $f : \Omega_2 \times [0, 1] \rightarrow R$ by

$$f(x_1, x_2; t) = I_2 \begin{pmatrix} x_1, & x_2 \\ 1-t, & t \end{pmatrix}, \quad (5)$$

cf. (1).

Let $s \in [0, 1[$, $t \in [0, 1[$, $s+t \leq 1$, but s and t else arbitrary. Then, from (4) and from the 3-symmetry, we have

$$\begin{aligned} & f(x_1 \cup x_2, x_3; t) + (1-t) f \left(x_1, x_2; \frac{s}{1-t} \right) \\ &= I_3 \begin{pmatrix} x_1, & x_2, & x_3 \\ 1-s-t, & s, & t \end{pmatrix} = I_3 \begin{pmatrix} x_1, & x_3, & x_2 \\ 1-s-t, & t, & s \end{pmatrix} \\ &= f(x_1 \cup x_3, x_2; s) + (1-s) f \left(x_1, x_3; \frac{t}{1-s} \right), \end{aligned} \quad (6)$$

for all $(x_1, x_2, x_3) \in \Omega_3$ and for all

$$(s, t) \in D := \{ (s, t) \mid s \in [0, 1[, t \in [0, 1[, s+t \leq 1 \}.$$

For fixed $(x_1, x_2, x_3) \in \Omega_3$, we get from (6) with the notations

$$\left. \begin{aligned} f_1(s) &= f(x_1 \cup x_3, x_2; s), & f_2(u) &= f(x_1, x_3; u), \\ f_3(t) &= f(x_1 \cup x_2, x_3; t), & f_4(v) &= f(x_1, x_2; v), \end{aligned} \right\} \quad (7)$$

the equation

$$f_1(s) + (1-s)f_2\left(\frac{t}{1-s}\right) = f_3(t) + (1-t)f_4\left(\frac{t}{1-s}\right) \quad \text{for all } (s, t) \in D.$$

The general solutions, measurable on $]0, 1[$, have been determined for this equation in [8] (cf. [3]) as

$$f_j(t) = A \left[-t \log t - (1-t) \log(1-t) \right] + a_j t + b_j \left\{ \begin{array}{l} (t \in [0, 1[\text{ or } [0, 1]; j = 1, 2, 3, 4), \end{array} \right. \quad (8)$$

with the convention (3). (There are certain relations among b_1, b_2, b_3 and b_4 , which we will not need here. It is also unimportant how we fix the base of the logarithm.)

In the situation described by (7), when x_1, x_2 and x_3 are allowed to vary again, the coefficients A, a_j, b_j ($j = 1, 2, 3, 4$) in (8) may depend upon them. In particular, see (7),

$$\begin{aligned} f(x_1, x_2; t) &= A(x_1, x_2) \left[-t \log t - (1-t) \log(1-t) \right] + a_4(x_1, x_2)t + b_4(x_1, x_2) \quad (9) \\ f(x_1 \cup x_2, x_3; t) &= A(x_1 \cup x_2, x_3) \left[-t \log t - (1-t) \log(1-t) \right] \\ &\quad + a_3(x_1 \cup x_2, x_3)t + b_3(x_1 \cup x_2, x_3). \end{aligned}$$

But, as seen from (8), A has to be the same for f_3 and f_4 , thus

$$A(x_1, x_2) = A(x_1 \cup x_2, x_3) \quad \text{for all } (x_1, x_2, x_3) \in \Omega_3.$$

Substituting $x_1 = 0$, we get

$$A(x_2, x_3) = A(0, x_2). \quad (10)$$

So $A(x, y) = \alpha(x)$ is independent of y . Thus, combined with (10), we have that $\alpha(x_2) = \alpha(0) = \text{constant}$, that is,

$$A \text{ is constant.} \quad (11)$$

If we substitute (9), with constant A , into (6) and compare the members linear in t on the left and right hand sides, we obtain, writing simply

$$a_4 = a, \quad b_4 = b, \quad (12)$$

the equation

$$a(x_1 \cup x_2, x_3) - b(x_1, x_2) = a(x_1, x_3) \quad \text{for all } (x_1, x_2, x_3) \in \Omega_3. \quad (13)$$

[We will *not* need the other equations obtainable by comparison of the two extremities of (6).] The substitution $x_3 = 0$ now gives, with the notation

$$g(x) = a(x, 0),$$

the equation

$$b(x_1, x_2) = g(x_1 \cup x_2) - g(x_1). \tag{14}$$

Resubstituting this into (13), we get

$$a(x_1 \cup x_2, x_3) - g(x_1 \cup x_2) = a(x_1, x_3) - g(x_1)$$

and, again with $x_1 = 0$,

$$a(x_2, x_3) = g(x_2) - G(x), \tag{15}$$

where we have written

$$G(x) = g(0) - a(0, x).$$

From (5), (9), (11), (12), (14) and (15) we have now

$$I_2 \begin{pmatrix} x_1, & x_2 \\ 1-t, & t \end{pmatrix} = A [-t \log t - (1-t) \log(1-t)] \\ + g(x_1 \cup x_2) - (1-t)g(x_1) - tG(x_2), \tag{16}$$

[with (3)]. But equation (4) and the 3-symmetry

$$I_3 \begin{pmatrix} x_1, & x_2, & x_3 \\ p_1, & p_2, & p_3 \end{pmatrix} = I_3 \begin{pmatrix} x_2, & x_1, & x_3 \\ p_2, & p_1, & p_3 \end{pmatrix},$$

show that I_2 is symmetric too (that is, our inset entropies are also 2-symmetric). Thus

$$I_2 \begin{pmatrix} x_1, & x_2 \\ 1-t, & t \end{pmatrix} = I_2 \begin{pmatrix} x_2, & x_1 \\ t, & 1-t \end{pmatrix}.$$

Comparison to (16) gives immediately

$$G(x) = g(x),$$

so that (16) goes over into

$$I_2 \begin{pmatrix} x_1, & x_2 \\ 1-t, & t \end{pmatrix} = g(x_1 \cup x_2) - (1-t)g(x_1) - tg(x_2) \\ - A [(1-t) \log(1-t) + t \log t] \tag{17}$$

[with the convention (3)].

This shows that (2) holds for $n = 2$. Suppose it is true for $n - 1$ then, by the recursivity and by (17),

$$\begin{aligned}
 & I_n \left(\begin{matrix} x_1, & x_2, & \dots, & x_n \\ p_1, & p_2, & \dots, & p_n \end{matrix} \right) \\
 &= I_{n-1} \left(\begin{matrix} x_1 \cup x_2, & x_3, & \dots, & x_n \\ p_1 + p_2, & p_3, & \dots, & p_n \end{matrix} \right) + (p_1 + p_2) I_2 \left(\begin{matrix} x_1, & x_2 \\ p_1, & p_2 \end{matrix} \right) \\
 &= g(x_1 \cup x_2 \cup \dots \cup x_n) - (p_1 + p_2) g(x_1 \cup x_2) \\
 &\quad - \sum_{i=3}^n p_i g(x_i) - A \left[(p_1 + p_2) \log(p_1 + p_2) + \sum_{i=3}^n p_i \log p_i \right] \\
 &\quad + (p_1 + p_2) \left[g(x_1 \cup x_2) - \frac{p_1}{p_1 + p_2} g(x_1) - \frac{p_2}{p_1 + p_2} g(x_2) \right. \\
 &\quad \quad \left. - A \frac{p_1}{p_1 + p_2} \log \frac{p_1}{p_1 + p_2} - A \frac{p_2}{p_1 + p_2} \log \frac{p_2}{p_1 + p_2} \right] \\
 &= g \left(\bigcup_{i=1}^n x_i \right) - \sum_{i=1}^n p_i g(x_i) - A \sum_{i=1}^n p_i \log p_i
 \end{aligned}$$

(again with the convention (3), using the similar convention in the definition of recursivity), that is, (2) holds also for n . This concludes the proof.

4. REMARKS: The last member, $-\sum p_i \log p_i$ in (2) [with the convention (3)] is, of course, the *Shannon entropy* (see, e. g., [3]). If the system x_1, x_2, \dots, x_n of events is *complete*, that is, $\bigcup_{i=1}^n x_i$ is the whole space Ω (the certain event),

then $g \left(\bigcup_{i=1}^n x_i \right) = C$ is a constant and, with the notation $h(x) = C - g(x)$, the first two members in (2) reduce to

$$\sum_{i=1}^n p_i h(x_i),$$

that is, to the *expected value of a random variable* [which the second member in (2) is also in the general case]. Thus, *in this case of complete systems of events, the general recursive, 3-symmetric and measurable inset entropies are sums of the expected value of an arbitrary random variable and of an arbitrary constant multiple of the Shannon entropy,*

$$\begin{aligned}
 & I_n \left(\begin{matrix} x_1, & \dots, & x_n \\ p_1, & \dots, & p_n \end{matrix} \right) = \sum_{i=1}^n p_i h(x_i) - A \sum_{i=1}^n p_i \log p_i, \\
 & \left[(x_1, \dots, x_n) \in \Omega_n, \bigcup_{i=1}^n x_i = \Omega; (p_1, \dots, p_n) \in \Gamma_n, 0 \log 0 := 0 \right].
 \end{aligned}$$

There is a close resemblance between this representation and C. T. Ng's parallel composition law (5.8.C) in [10].

On the other hand, in the case of *incomplete* systems of events (when their union is a proper subset of the "whole space"), we may notice that the sum of probabilities is still 1. This means that we have *conditional* probabilities [observe, for instance, the probabilities assigned to x_1 and x_2 in the last member of the definition of recursivity, for instance in (4)] or measures geared to the union of the events (sets) in the inset entropy.

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