RAIRO. INFORMATIQUE THÉORIQUE

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RAIRO. Informatique théorique, tome 12, nº 1 (1978), p. 33-48
<http://www.numdam.org/item?id=ITA_1978__12_1_33_0>

R.A.I.R.O. Informatique théorique/Theoretical Computer Science
(vol. 12, n° 1, 1978, p. 33 à 48)

COMPUTING GRAMMARS AND CONTEXT-SENSITIVE LANGUAGES (*)

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Communicated by J. F. Perrot

Summary.— *We introduce the notion of a computing grammar and we study the family of functions that can be defined by such a grammar with a linearly limited buffer size. These functions in turn define context-sensitive languages. We investigate various closure properties of this family of functions and use them to show that certain languages are context-sensitive.*

I. INTRODUCTION

It is a well known fact that every context-sensitive language is a recursive set. Using a diagonalization argument it is possible to prove that the class of context-sensitive languages is strictly included into the class of recursive languages. Moreover, the progress of Computational Complexity Theory now permits the direct specification of languages that are recursive but not context-sensitive since they require exponential space for recognition—e. g. extended regular expressions denoting the empty set ([1], chap. 11)— and are therefore outside the scope of any linear bounded automaton.

Therefore, for formal language theory it is an important matter to clarify as much as possible the border of the class of context-sensitive languages by proving that certain classes of recursive languages are composed in fact by context-sensitive languages.

By N_1 we shall denote the set of positive natural numbers $N_1 = N \setminus \{ 0 \}$. The aim of our paper is to study a certain subset of the set of functions $\{ f \mid f : N_1^h \to N_1^k, k, h \in N_1 \}$ for which the bounded languages

$$L_f = \{ i_1^{m_1} \dots i_h^{m_h} \mid (m_1, \dots, m_h) \in f(N_1^k) \}$$

are context-sensitive. Here $i_1, \dots, i_h$ are symbols belonging to an appropriate alphabet I.

(*) Received August 1977; revised October 1977. *Note*: A preliminary version of this paper was presented at the 6th Conference on Mathematical Foundations of Computer Science (Tatrouska Lomnica, Czechoslovakia, September 1977), but an unexpected delay presented its inclusion in the Proceedings.

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After studying closure properties of this class of functions with respect to common operations of recursive function theory we consider some elementary examples which justify the interest of this topic.

II. COMPUTING GRAMMARS AND FUNCTIONS COMPUTABLE WITH A LINEARLY-LIMITED BUFFER-SIZE

In the sequel we shall introduce a new revriting device.

DÉFINITION : A *computing grammar* (cg) is a 7-uple

$$K = (I_N, I_T, \{ x_1, \ldots, x_k \}, \{ i_1, \ldots, i_h \}, b, \#, F),$$

where I_N and I_T are finite nonvoid sets representing respectively the *nonterminal* and the *terminal alphabet*, $\{ x_1, \ldots, x_k \} \subseteq I_N$ is the set of *start symbols*, $I_T = \{ i_1, \ldots, i_h, \#, b \}$, where "$\#$" is the *marker* and "$b$" is the *buffer* and F is a finite set of pairs of words from $(I_N \cup I_T)^+ \times (I_N \cup I_T)^*$.

In addition, we suppose that if $(u, v) \in F$ then:

(i) u contains at least a nonterminal symbol;

(ii) if "$\#$" occurs in u then "$\#$" has exactly only one occurence both in u and v, namely in the first position of these words.

If $(u, v) \in F$ this fact will be denoted by $u \to v$.

The generation relation "$\underset{K}{\Rightarrow}$" and its reflexive and transitive closure are considered exactly as for grammars.

A cg is *length-increasing* if for every rule $u \to v$ we have $l(u) \leqq l(v)$, where $l(p)$ is the length of the word p.

If $\mathbf{n} = (n_1, \ldots, n_k) \in N_1^k$ the number $\| \mathbf{n} \|$ is the sum $\sum \{ n_j \mid 1 \leqq j \leqq k \}$.

A function $f : N_1^k \to N_1^h$ is computed by a cg $K = (I_N, I_T, \{x_1, \ldots, x_k \}, \{ i_1, \ldots, i_k \}, b, \#, F)$ if for every $\mathbf{n} = (n_1, \ldots, n_k) \in N_1^k$, there exists an unique $\mathbf{m} = (m_1, \ldots, m_h) \in N^h$ so that

$$\# x_1^{n_1} \ldots x_k^{n_k} \overset{\star}{\underset{K}{\Rightarrow}} \# i_h^{m_1} \ldots i_h^{m_h} b^{M_f^K(\mathbf{n})}.$$

The number $M_f^K(\mathbf{n})$ is the buffer sizer for the function f and the input $\mathbf{n}$ in the cg K. In this manner we obtain a function $M_f^K : N_1^k \to N_1^h$.

A function $f : N_1^k \to N_1^h$ is computable with a *linearly limited buffer size* by the length-increasing cg K if there exists $c \in R_+$ so that:

$$M_f^K(\mathbf{n}) \leqq c \| f(\mathbf{n}) \|,$$

for every $\mathbf{n} \in N_1^k \setminus Q_f$, where Q_f is a finite subset of N_1^k.

Let *LLBS* be the class of functions which are computable by *length-increasing* cgs with linearly limited buffer size.

The usefulness of this concept of cg is pointed out by:

THEOREM 1: *If a function $f : N_1^k \to N_1^h$ belongs to LLBS then the language*

$$L_f = \{ i_1^{m_1} \dots i_h^{m_h} \mid \mathbf{m} = (m_1, \dots, m_h) \in f(N_1^k) \}$$

is a context-sensitive one.

Proof: Suppose that f is computed by the length-increasing cg $K = (I_N, I_T, \{ x_1, \dots, x_k \}, \{ i_1, \dots, i_h \}, b, \#, F)$ and let us consider the length-increasing grammar

$$\begin{aligned} G = (I_N \cup \{ y_1, \dots, y_k \}, & \\ I_T, x_0, F \cup \{ x_0 \to \# y_1 \dots y_k, & \; y_1 \to y_1 x_1, \\ & y_1 \to x_1, \dots, y_k \to y_k x_k, y_k \to x_k \}), \end{aligned}$$

where $y_1, \dots, y_k$ are new symbols non belonging to the set $I_N \cup I_T$.

We infer that the language

$$\begin{aligned} L = (L(G) \cap \{ \# \} \{ i_1 \}^+ \dots \{ i_h \}^+ \{ b \}^*) \\ - \{ \# i_1^{q_1} \dots i_h^{q_h} \mid (q_1, \dots, q_h) \in f(Q_f) \} \end{aligned}$$

is again context-sensitive since the class of context-sensitive languages is closed with respect to intersection with regular languages.

Let us take $p \in L$. The derivation $x_0 \overset{\star}{\underset{G}{\Rightarrow}} p$ can be split as follows:

$$x_0 \Rightarrow \# y_1 \dots y_k \overset{\star}{\Rightarrow} \# x_1^{n_1} \dots x_k^{n_k} \overset{\star}{\Rightarrow} \# i_1^{m_1} \dots i_h^{m_h} b^{M_f^K(\mathbf{n})},$$

where the last part of the derivation uses only rules from F.

We conclude that L has the following form

$$L = \{ \# i_1^{m_1} \dots i_h^{m_h} b^{M_f^K(\mathbf{n})} \mid \mathbf{m} = f(\mathbf{n}), \mathbf{n} \in N_1^k \setminus Q_f, \mathbf{m} = (m_1, \dots, m_h) \}.$$

Let us consider now the homomorphism

$$H : \{ i_1, \dots, i_h, b, \# \} \to \{ i_1, \dots, i_h \}^*$$

given by:

$$H(y) = \begin{cases} y & \text{if } y \in \{ i_1, \dots, i_h \}, \\ e & \text{if } y \in \{ b, \# \}. \end{cases}$$

H is a $(c+2)$-linear erasing homomorphism with respect to L for, if $p = \# i_1^{m_1} \dots i_h^{m_h} b^{M_f^K(\mathbf{n})}$ we have

$$\begin{aligned} l(p) = 1 + \| f(\mathbf{n}) \| + M_f^K(\mathbf{n}) \leq 1 + \| f(\mathbf{n}) \| + c \| f(\mathbf{n}) \| \\ \leq (c+2) \| f(\mathbf{n}) \| = (c+2)\, l(H(p)), \end{aligned}$$

since $H(p) = i_1^{m_1} \dots i_h^{m_h}$ and $f(n_1, \dots, n_k) = (m_1, \dots, m_h)$.

Taking into account that the class of context-sensitive languages is closed under linear erasing [4] it follows that $H(L)$ is a context-sensitive language.

III. CLOSURE PROPERTIES OF THE CLASS OF FUNCTIONS COMPUTABLE BY LENGTH-INCREASING cgs

This part of the paper contains a study of closure properties of functions computable by length-increasing cgs.

We shall recall first the definitions of several well known operations on functions.

If $f : N_1^k \to N_1^h$ and $g : N_1^b \to N_1^c$ are two functions, the combination of f and g is the function $f \times g : N_1^k \times N_1^b \to N_1^h \times N_1^c$, given by

$$(f \times g)(\mathbf{m}, \mathbf{p}) = (f(\mathbf{m}), g(\mathbf{p})),$$

for every $\mathbf{m} \in N_1^k$ and $\mathbf{p} \in N_1^b$.

The *exponentiation* of the function $f : N_1^k \to N_1^k$ is the function $f : N_1^{k+1} \to N_1^k$ which is defined by:

$$f^{\#}(\mathbf{n}, p) = \underbrace{f \circ f \circ \ldots \circ f}_{p}(\mathbf{n}),$$

where "$\circ$ ' denotes the composition of functions.

The function $f : N_1^{k+1} \to N_1^l$ is obtained by *primitive recursion* from $g : N_1^k \to N_1^l$ and $h : N_1^{k+1+l} \to N_1^l$ if $f(\mathbf{n}, 1) = g(\mathbf{n})$ and

$$f(\mathbf{n}, p+1) = h(\mathbf{n}, p, f(\mathbf{n}, p)), \mathbf{n} \in N_1^k, p \in N_1.$$

The operations of composition, combination, exponentiation and recursion are not independent. For instance, it is possible to prove [2] that each function defined by exponentiation starting from a primitive recursive function can also be defined by primitive recursion from primitive recursive functions. In this proof the projections are inherently involved. Since these functions are obviously non-computable by a length-increasing computing grammar, it is useful to study closure properties of the set of functions with respect to the whole set of operations.

The strategy of our approach is the following. In the next four theorems prove closure properties of the class of functions computable by cgs.

After establishing evaluations for the buffer size of the computed functions we obtain in the corollaries, closure properties of the class of *LLBS* functions.

THEOREM 2: *The class of functions which are computable by length-increasing* cgs *is closed with respect to composition.*

Proof: Let $f : N_1^k \to N_1^h, g : N_1^h \to N_1^l$ be the functions computed respectively by the grammars:

$$K_f = (I_{Nf}, I_{Tf}, \{x_1, \ldots, x_k\}, \{i_1, \ldots, i_h\}, b, \#, F_f),$$

$$K_g = (I_{Ng}, I_{Tg}, \{x'_1, \ldots, x'_h\}, \{i'_1, \ldots, i'_l\}, b, \#, F_g).$$

Without loss of generality we assume that

$$I_{Nf} \cap I_{Ng} = \emptyset \quad \text{and} \quad I_{Tf} \cap I_{Tg} = \{ b, \# \}.$$

Then the composition $g \circ f : N_1^k \to N_1^l$ is computed by the cg:

$$K_{g \circ f} = (I_{Nf} \cup I_{Ng} \cup (I_{Tf} \setminus \{ b \}), I_{Tg}, \{ x_1, \ldots, x_k \},$$
$$\{ i'_1, \ldots, i'_l \}, b, \#, \{ i_m \to x'_m \mid 1 \leqq m \leqq h \} \cup F_f \cup F_g)$$

since we can write:

$$\# x_1^{n_1} \ldots x_k^{n_k} \overset{\star}{\Rightarrow} \# i_1^{m_1} \ldots i_h^{m_h} b^{M_f^{K_f}(\mathbf{n})}$$
$$\overset{\star}{\Rightarrow} \# x_1'^{m_1} \ldots x_h'^{m_h} b^{M_f^{K_f}(\mathbf{n})} \# i_1'^{p_1} \ldots i_l'^{p_l} b^{M_g^{K_g}(f(\mathbf{n}))+M_f^{K_f}(\mathbf{n})}.$$

COROLLARY 1: *If f, $g \in LLBS$, $\| g(\mathbf{n}) \| \geqq \| \mathbf{n} \|$ for almost all $n \in N_1^k$ (excepting possibly a finite set $Q \subseteq N_1^k$) and $f^{-1}(Q_g \cup Q)$ is a finite set then the composition $g \circ f$ belongs to LLBS.*

Proof: From the proof of th. 2 it follows that

$$M_{g \circ f}^{K_{g \circ f}}(\mathbf{n}) = M_f^K(\mathbf{n}) + M_g^{K_g}(f(\mathbf{n})),$$

hence it is possible to obtain the following evaluation:

$$M_{g \circ f}^{K_{g \circ f}}(\mathbf{n}) \leqq c_g \| g(f(\mathbf{n})) \| + c_f \| f(\mathbf{n}) \| \leqq c_g \| g(f(\mathbf{n})) \|$$
$$+ c_f \| g(f(\mathbf{n})) \| \leqq \max(c_g, c_f) \| g(f(\mathbf{n})) \|,$$

where $f(\mathbf{n}) \notin Q_g \cup Q$ and $\mathbf{n} \notin Q_f$. These restriction can be summarized asking $\mathbf{n} \notin Q_f \cup f^{-1}(Q_g \cup Q)$ and, since this last set is finite it follows $g \circ f \in LLBS$.

THEOREM 3: *The class of functions which are computable by length-increasing* cgs *is closed with respect to combination.*

Proof: Suppose that $f : N_1^k \to N_1^h$ and $g : N_1^j \to N_1^l$ are two functions computed by the cgs:

$$K_f = (I_{Nf}, I_{Tf}, \{ x_1, \ldots, x_k \}, \{ i_1, \ldots, i_h \}, b, \#, F_f),$$
$$K_g = (I_{Ng}, I_{Tg}, \{ x'_1, \ldots, x'_j \}, \{ i'_1, \ldots, i'_l \}, b', \#', F_g).$$

Assuming that $I_{Nf} \cap I_{Ng} = \emptyset$ and $I_{Tf} \cap I_{Nf} = \emptyset$ we shall consider the following length-increasing cg:

$$K_{f \times g} = (I_{Nf} \cup I_{Ng} \cup \{ b, \#' \}, (I_{Tf} \setminus \{ b \}) \cup (I_{Tg} \setminus \{ \#' \}),$$
$$\{ x_1, \ldots, x_k, x'_1, \ldots, x'_j \}, b', \#,$$
$$F_f \cup F_g \cup \{ bx'_1 \to b \#' x'_1, i_h x'_1 \to i_h \#' x'_1 \}$$
$$\cup \{ bi'_t \to i'_t b \mid 1 \leqq t \leqq l \} \cup \{ b' \to b \}).$$

Denoting by $c = M_f^{K_f}(\mathbf{n})$ and $d = M_g^{K_g}(\mathbf{q})$ we have in this new cg:

$$\begin{aligned}
\# x_1^{n_1} \dots x_k^{n_k} x_1'^{q_1} \dots x_j'^{q_j} &\overset{\star}{\Rightarrow} \# i_1^{m_1} \dots i_h^{m_h} b^c x_1'^{q_1} \dots x_j'^{q_j} \\
&\overset{\star}{\Rightarrow} \# i_1^{m_1} \dots i_h^{m_h} b^c \#' x_1'^{q_1} \dots x_j'^{q_j} \\
&\overset{\star}{\Rightarrow} \# i_1^{m_1} \dots i_h^{m_h} b^c i_1'^{p_1} \dots i_l'^{p_l} b'^d \\
&\overset{\star}{\Rightarrow} \# i_1^{m_1} \dots i_h^{m_h} i_1'^{p_1} \dots i_l'^{p_l} b'^{c+d}.
\end{aligned}$$

Hence, the function $f \times g$ is computed by the cg $K_{f \times g}$ and we have

$$M_{f \times g}^{K_{f \times g}}((\mathbf{n}, \mathbf{q})) = M_f^{K_f}(\mathbf{n}) + M_g^{K_g}(\mathbf{q}). \tag{1}$$

COROLLARY 2: *The combination of any two functions from LLBS is again in LLBS.*

Proof: Let $f, g \in LLBS$, where $M_g^{K_f}(\mathbf{n}) \leqq c_f \, \| f(\mathbf{n}) \|$ for $\mathbf{n} \in N_k^1 \setminus Q_f$ and $M_g^{K_g}(\mathbf{q}) \leqq c_g \, \| g(\mathbf{q}) \|$ for $\mathbf{q} \in N_1^b \setminus Q_g$. Taking into account (1) it is possible to obtain the following evaluation:

$$\begin{aligned}
M_{f \times g}^{K_{f \times g}}((\mathbf{n}, \mathbf{q})) &= M_f^{K_f}(\mathbf{n}) + M_g^{K_g}(\mathbf{q}) \leqq c_f \, \| f(\mathbf{n}) \| \\
&\quad + c_g \, \| g(\mathbf{q}) \| \leqq \max(c_f, c_g)(\| f(\mathbf{n}) \| + \| g(\mathbf{q}) \|) \\
&= \max(c_f, c_g) \, \| f \times g(\mathbf{n}, \mathbf{q}) \|,
\end{aligned}$$

for $n \notin Q_f \cup Q_g$, hence $f \times g \in LLBS$.

THEOREM 4. *If f is a function computed by a length-increasing* cg *then there exists a length-increasing* cg *which computes its iteration $f^{\#}$.*

Proof: Let us suppose that the function $f : N_1^k \to N_1^k$ is computed by the length-increasing cg $K = (I_N, I_T, \{ x_1, \dots, x_k \}, \{ i_1, \dots, i_k \}, b, \#, F)$ and let us take

$$\mathbf{q}^h = (q_1^h, \dots, q_k^h) = \underbrace{f \circ f \circ f \dots \circ f}_{h \text{ times}}(\mathbf{n}) \qquad \text{and} \qquad d^h = M_f^K(q^{h-1}).$$

We shall consider the cg:

$$\begin{aligned}
K_{\#} = (I_N \cup \{ x_{k+1}, z_1, \dots, z_k, y \}, \\
I_T, \{ z_1, \dots, z_k, x_{k+1} \}, \{ i_1, \dots, i_k \}, b, \#, F \cup F_1)
\end{aligned}$$

where F_1 consists of the following rules:

(i) $z_j \, x_{k+1} \to x_{k+1} \, x_j$, $1 \leqq j \leqq k$;

(ii) $i_j \, x_{k+1} \to x_{k+1} \, x_j$, $1 \leqq j \leqq k$;

(iii) $bx_{k+1} \longrightarrow x_{k+1}\, b$;

(iv) $\#\, x_{k+1}\, x_1 \longrightarrow \#\, y\, x_1$;

(v) $y\, i_j \longrightarrow i_j\, y$, $1 \leqq j \leqq k$;

(vi) $i_k\, yb \longrightarrow i_k\, bb$, $i_k\, y\, x_{k+1} \longrightarrow i_k\, bx_{k+1}$.

Since F consists only from length-increasing rules so $F \cup F_1$ does, hence $K_{\#}$ is indeed a length-increasing cg.

Now, we obtain in $K_{\#}$ the following derivation:

$$
\begin{aligned}
\#\, z_1^{n_1} \ldots z_k^{n_k}\, x_{k+1}^{p} \;&\overset{\star}{\underset{(\mathrm{i})}{\Rightarrow}}\; \#\, x_{k+1}\, x_1^{n_1} \ldots x_k^{n_k}\, x_{k+1}^{p-1} \\
&\underset{(\mathrm{iv})}{\Rightarrow}\; \#\, y x_1^{n_1} \ldots x_k^{n_k}\, x_{k+1}^{p-1} \;\overset{\star}{\underset{F}{\Rightarrow}}\; \#\, y i_1^{q_1^1} \ldots i_k^{q_k^1}\, b^{d_1}\, x_{k+1}^{p-1} \\
&\overset{\star}{\underset{(\mathrm{v})}{\Rightarrow}}\; \#\, i_1^{q_1^1} \ldots i_k^{q_k^1}\, y b^{d_1}\, x_{k+1}^{p-1} \;\underset{(\mathrm{iv})}{\Rightarrow}\; \#\, i_1^{q_1^1} \ldots i_k^{q_k^1}\, b^{d_1+1}\, x_{k+1}^{p-1} \\
&\overset{\star}{\underset{(\mathrm{iii})}{\Rightarrow}}\; \#\, i_1^{q_1^1} \ldots i_k^{q_k^1}\, x_{k+1}\, b^{d_1+1}\, x_{k+1}^{p-2} \\
&\overset{\star}{\underset{(\mathrm{ii})}{\Rightarrow}}\; \#\, x_{k+1}\, x_1^{q_1^1} \ldots x_k^{q_k^1}\, b^{d_1+1}\, x_{k+1}^{p-2} \\
&\underset{(\mathrm{iv})}{\Rightarrow}\; \#\, y x_1^{q_1^1} \ldots x_k^{q_k^1}\, b^{d_1+1}\, x_{k+1}^{p-2} \\
&\overset{\star}{\underset{F}{\Rightarrow}}\; \#\, y i_1^{q_1^2} \ldots i_k^{q_k^2}\, b^{d_1+d_2+1}\, x_{k+1}^{p-2} \\
&\overset{\star}{\Rightarrow}\; \ldots \;\overset{\star}{\Rightarrow}\; \#\, i_1^{q_1^p} \ldots i_k^{q_k^p}\, b^{d},
\end{aligned}
$$

where $d = p + \sum \{\, d_j \mid 1 \leqq j \leqq p \,\}$.

It is clear now that $f^{\#} : N_1^{k+1} \longrightarrow N_1^{k}$ is computed by $K_{\#}$. Moreover, we have

$$M_{f\#}^{K\#}(\mathbf{n}, p) = \sum_{j=1}^{p} M_f^K(\underbrace{f \circ f \circ \ldots \circ f}_{j-1 \text{ times}}(\mathbf{n})) + p.$$

COROLLARY 3: *Suppose that for the function* $f : N_1^k \longrightarrow N_1^k$ *there exists* $\alpha \in (0, 1)$ *so that* $\|\mathbf{n}\|\, \alpha \leqq \|f(\mathbf{n})\|$, *excepting possibly a finite set* $Q \subseteq N_1^k$. *If* $f \in LLBS$ *and* $f(N_1^k) \cap Q = \emptyset$ *then* $f^{\#} \in LLBS$.

Proof: We have to evaluate $M_{f\#}^{K\#}(\mathbf{n}, p)$. In view of the fact that $\|\mathbf{n}\| \leqq \alpha \|f(\mathbf{n})\|$ we have the following non-decreasing sequence of real numbers:

$$\|f^j(\mathbf{n})\| \leqq \alpha \|f^{j+1}(\mathbf{n})\| \leqq \alpha^2 \|f^{j+2}(\mathbf{n})\| \leqq \ldots \leqq \alpha^{p-j} f^p(\mathbf{n})\|,$$

for every j, $0 \leqq j \leqq p-1$, since $f(N_1^k) \cap Q = \emptyset$. Hence

$$\begin{aligned} M_{f\#}^{K\#}(\mathbf{n}, p) &= \sum_{j=0}^{p-1} M_f^K(\underbrace{f \circ f \circ \ldots \circ f}_{j}(\mathbf{n}))+p \\ &\leqq \sum_{j=0}^{p-1} c \,\|\underbrace{f \circ f \circ \ldots \circ f}_{j}(\mathbf{n})\|+p < \sum_{j=0}^{p-1} c\,\alpha^{p-j} \|f^p(\mathbf{n})\|+p \\ &= c\,\|f^p(\mathbf{n})\| \left(\sum_{k=1}^{p} \alpha^k\right)+p = c\,\|f^p(\mathbf{n})\| \frac{\alpha-\alpha^{k+1}}{1-\alpha}+p \\ &< \frac{\alpha c}{1-\alpha} \|f^p(\mathbf{n})\|+p. \end{aligned}$$

From $\|\mathbf{n}\| < \alpha \|f(\mathbf{n})\|$ it follows that $\|f(\mathbf{n})\| > A \|n\|$, where $A = 1/\alpha > 1$. Hence $\|f^p(\mathbf{n})\| > A^p \|\mathbf{n}\| > [1+p(A-1)] \|\mathbf{n}\| > p(A-1)$. Therefore $p < 1/(A-1) \|f^p(\mathbf{n})\|$ and the previous inequality is completed as follows:

$$\begin{aligned} M_{f\#}^{K\#}(\mathbf{n}, p) &< \left(\frac{\alpha c}{1-\alpha}+\frac{1}{A-1}\right)\|f^p(\mathbf{n})\| = \frac{c(1+\alpha)}{1-\alpha}\|f^{\#}(\mathbf{n}, p)\| \\ &= c_1 \|f^{\#}(\mathbf{n}, p)\|, \end{aligned}$$

hence $f^{\#} \in LLBS$.

THEOREM 5: *The class of functions which are computable by length-increasing grammars is closed under primitive recursion.*

Proof: Let us consider the cgs K_g and K_h which compute the functions $g : N_1^k \to N_1^l$ and $h : N_1^{k+1+l} \to N_1^l$, respectively

$$K_g = (I_{Ng}, I_{Tg}, \{z_1, \ldots, z_k\}, \{i'_1, \ldots, i'_l\}, b, \#, F_g),$$

$$K_h = (I_{Nh}, I_{Th}, \{s_1, \ldots, s_k, v, y_1, \ldots, y_l\}, \{i_1, \ldots, i_l\}, b, \#, F_f),$$

where $I_{Ng} \cap I_{Nh} = \emptyset$.

We shall construct the cg

$$K = (I_{Ng} \cup I_{N_h} \cup I_{N_S}, I_{Th}, \{x_1, \ldots, x_k, x_{k+1}\}, \{i_1, \ldots, i_l\}, b, \#, F'_g \cup F_f \cup F_S),$$

which computes the function $f : N_1^{k+1} \to N_1^l$ obtained by recursion from g and h. Here I_{NS} is the set of supplementary nonterminal smbols, F'_g is obtained from F_g by replacing each symbol i'_j by y_j, $1 \leqq j \leqq l$ and each occurence of "$\#$" with a new symbol v and F_S is the set of supplementary rules. The sets I_{NS} and F_S will be specified in the sequel.

Let us denote $f(\mathbf{n}, p) = (q_1^p, \ldots, q_k^p)$.

The activity of the cg K starts from the word $x_1^{n_1} \ldots x_k^{n_k} x_{k+1}^p$. We shall exhibit the rules which allow the derivation:

$$\# x_1^{n_1} \ldots x_k^{n_k} x_{k+1}^p \overset{\star}{\Rightarrow} \# s_1^{n_1} \ldots s_k^{n_k} v z_1^{n_1} \ldots z_k^{n_k} t_1^{n_1} \ldots t_k^{n_k} w x_{k+1}^{p-1}.$$

Namely, for the beginning we shall include in F_S the following sets of rules:

(i) $x_j\, x_{k+1} \to x_{k+1}\, s_j\, z_j\, t_j,\ 1 \leqq j \leqq k$;

(ii) $\#\, x_{k+1} \to \#\, \sigma$;

(iii) $\begin{cases} t_j\, s_h \to s_h\, t_j,\ 1 \leqq j, h \leqq k, j \leqq h; \\ t_j\, z_h \to z_h\, t_j,\ 1 \leqq j, h \leqq k, j \leqq h; \end{cases}$

(iv) $z_j\, s_h \to s_h\, z_j,\ 1 \leqq j, h \leqq k, j \leqq k$;

(v) $\sigma\, s_j \to s_j\, \sigma,\ 1 \leqq j \leqq k$;

(vi) $s_k\, \sigma\, z_1 \to s_k\, v\, \theta\, z_1$;

(vii) $\theta\, z_j \to z_j\, \theta,\ 1 \leqq j \leqq k$;

(viii) $z_k\, \theta\, t_1 \to z_k\, \tau\, t_1$;

(ix) $\tau\, t_j \to t_j\, \tau,\ 1 \leqq j \leqq k$;

(x) $t_k\, \tau\, x_{k+1} \to t_k\, w\, x_{k+1}$.

Indicating by subscripts the rules which were used we have the derivation:

$$\begin{aligned}
\# x_1^{n_1} \ldots x_k^{n_k} x_{k+1}^p &\underset{(\mathrm{i})}{\overset{\star}{\Rightarrow}} \# x_{k+1} (s_1 z_1 t_1)^{n_1} \ldots (s_k z_k t_k)^{n_k} x_{k+1}^{p-1} \\
&\underset{(\mathrm{ii})}{\Rightarrow} \# \sigma (s_1 z_1 t_1)^{n_1} \ldots (s_k z_k t_k)^{n_k} x_{k+1}^{p-1} \\
&\underset{(\mathrm{iii})}{\Rightarrow} \# \sigma (s_1 z_1)^{n_1} \ldots (s_k z_k)^{n_k} t_1^{n_1} \ldots t_k^{n_k} x_{k+1}^{p-1} \\
&\underset{(\mathrm{iv})}{\Rightarrow} \# \sigma s_1^{n_1} \ldots s_k^{n_k} z_1^{n_1} \ldots z_k^{n_k} t_1^{n_1} \ldots t_k^{n_k} x_{k+1}^{p-1}.
\end{aligned}$$

Before moving σ it is compulsory to arrange s_j, z_j, t_j in the manner which was indicated, in order to eliminate σ. This derivation can be continued as follows:

$$\begin{aligned}
&\# \sigma s_1^{n_1} \ldots s_k^{n_k} z_1^{n_1} \ldots z_k^{n_k} t_1^{n_1} \ldots t_k^{n_k} z_{k+1}^{p-1} \\
&\underset{(\mathrm{v})}{\overset{\star}{\Rightarrow}} \# s_1^{n_1} \ldots s_k^{n_k} \sigma z_1^{n_1} \ldots z_k^{n_k} t_1^{n_1} \ldots t_k^{n_k} x_{k+1}^{p-1} \\
&\underset{(\mathrm{vi})}{\Rightarrow} \# s_1^{n_1} \ldots s_k^{n_k} v \theta z_1^{n_1} \ldots z_k^{n_k} t_1^{n_1} \ldots t_k^{n_k} x_{k+1}^{p-1} \\
&\underset{(\mathrm{vii})}{\overset{\star}{\Rightarrow}} \# s_1^{n_1} \ldots s_k^{n_k} v z_1^{n_1} \ldots z_k^{n_k} \theta t_1^{n_1} \ldots t_k^{n_k} x_{k+1}^{p-1}
\end{aligned}$$

$$\underset{(\text{viii})}{\Rightarrow} \# s_1^{n_1} \dots s_k^{n_k} v z_1^{n_1} \dots z_k^{n_k} \tau t_1^{n_1} \dots t_k^{n_k} x_{k+1}^{p-1}$$

$$\underset{(\text{ix})}{\overset{\star}{\Rightarrow}} \# s_1^{n_1} \dots s_k^{n_k} v z_1^{n_1} \dots z_k^{n_k} t_1^{n_1} \dots t_k^{n_k} \tau x_{k+1}^{p-1}$$

$$\underset{(\text{x})}{\Rightarrow} \# s_1^{n_1} \dots s_k^{n_k} v z_1^{n_1} \dots z_k^{n_k} t_1^{n_1} \dots t_k^{n_k} w x_{k+1}^{p-1}$$

$$\underset{F'_g}{\overset{\star}{\Rightarrow}} \# s_1^{n_1} \dots s_k^{n_k} v y_1^{q_1} \dots y_k^{q_k} b \dots t_1^{n_1} \dots t_k^{n_k} w x_{k+1}^{p-1}.$$

Hence, we put in evidence the derivation

$$\# s_1^{n_1} \dots s_k^{n_k} x_{k+1}^{p} \overset{\star}{\Rightarrow} \# s_1^{n_1} \dots s_k^{n_k} v y_1^{q'_1} \dots y_k^{q'_k} b \dots t_1^{n_1} \dots t_k^{n_k} w x_{k+1}^{p-1}.$$

Now, our aim is to construct a derivation

$$\# s_1^{n_1} \dots s_k^{n_k} v^j y_1^{q_1^j} \dots y_l^{q_l^j} b \dots t_1^{n_1} \dots t_k^{n_k} w^j x_{k+1}^{p-j}$$

$$\overset{\star}{\Rightarrow} \# s_1^{n_1} \dots s_k^{n_k} v^{j+1} y_1^{q_1^{j+1}} \dots y_l^{q_l^{j+1}} b \dots t_1^{n_1} \dots t_k^{n_k} w^{j+1} x_{k+1}^{p-(j+1)},$$

$$1 \leqq j \leqq p-1.$$

Since in K_h have the derivation

$$\# s_1^{n_1} \dots s_k^{n_k} v^j y_1^{q_1^j} \dots y_l^{q_l^j} \overset{\star}{\Rightarrow} \# i_1^{q_1^{j+1}} \dots i_l^{q_l^{j+1}} b^{d_j}$$

it follows that in K it is possible to write

$$\# s_1^{n_1} \dots s_k^{n_k} v^j y_1^{q_1^j} \dots y_l^{q_l^j} b \dots t_1^{n_1} \dots t_k^{n_k} w^j x_{k+1}^{p-j}$$

$$\overset{\star}{\Rightarrow} \# i_1^{q_1^{j+1}} \dots i_l^{q_l^{j+1}} b \dots ^{+d_j} t_1^{n_1} \dots t_k^{n_k} w^j x_{k+1}^{p-j},$$

where $d_j = \mathrm{M}_h^{K_h}(\mathbf{n}, p, \mathbf{q}^j)$, $\mathbf{q}^j = (q_1^j, \dots, q_l^j)$.

At this stage we shall consider the rules:

(xi) $w\, x_{k+1} \to w\, \alpha\, uw;$

(xii) $w\, \alpha \to \alpha\, uw;$

(xiii) $t_j\, \alpha \to \alpha\, u_j\, v_j,\ 1 \leqq j \leqq k;$

(xiv) $b\, \alpha \to \alpha\, b;$

(xv) $i_c\, \alpha \to \alpha\, y_c,\ 1 \leqq c \leqq l;$

(xvi) $\#\, \alpha \to \#\, \xi;$

$$(\text{xvii}) \begin{cases} t_j\, u_h \to u_h\, t_j,\ 1 \leqq j \leqq h \leqq k; \\ b u_h \to u_h\, b; \\ y_c\, u_h \to u_h\, y_c,\ 1 \leqq c \leqq l; \end{cases}$$

and $1 \leqq h < k$.

$$\text{(xviii)} \begin{cases} t_j\, u \to ut_j,\ 1 \leqq h \leqq k; \\ bu \to ub; \\ i_c\, u \to ui_c,\ 1 \leqq c \leqq l; \end{cases}$$

$$\text{(xix)} \begin{cases} \xi\, u_h \to s_h\, \xi,\ 1 \leqq h \leqq k; \\ \xi\, u \to v\, \xi; \end{cases}$$

$$\text{(xx)}\ v\, \xi\, y_1 \to v\, \eta\, y_1;$$

$$\text{(xxi)}\ \eta\, y_c \to y_c\, \eta,\ 1 \leqq c \leqq l;$$

$$\text{(xxii)}\ y_l\, \eta\, b \to y_l\, bb;$$

Using the rules introduced here we have the continuation:

$$\# \, i_1^{q_1^{j+1}} \dots i_l^{q_l^{j+1}}\, b \dots t_k^{n_k}\, w^j\, x_{k+1}^{p-j}$$

$$\underset{\text{(xi)}}{\Rightarrow} \quad \# \, i_1^{q_1^{j+1}} \dots i_l^{q_l^{j+1}}\, b \dots t_1^{n_1} \dots t_k^{n_k}\, w^j\, \alpha\, uwx_{k+1}^{p-(j+1)}$$

$$\overset{\star}{\underset{\text{(xii)}}{\Rightarrow}} \quad \# \, i_1^{q_1^{j+1}} \dots i_l^{q_l^{j+1}}\, b \dots t_1^{n_1} \dots t_k^{n_k}\, \alpha\, (uw)^{j+1}\, x_{k+1}^{p-(j+1)}$$

$$\overset{\star}{\underset{\text{(xiii)}}{\Rightarrow}} \quad \# \, i_1^{q_1^{j+1}} \dots i_l^{q_l^{j+1}}\, b \dots \alpha\, (u_1\, t_1)^{n_1} \dots (u_k\, t_k)^{n_k}\, (uw)^{j+1}\, x_{k+1}^{p-(j+1)}$$

$$\overset{\star}{\underset{\text{(xiv)}}{\Rightarrow}} \quad \# \, i_1^{q_1^{j+1}} \dots i_l^{q_l^{j+1}}\, \alpha\, b \dots (u_1\, t_1)^{n_1} \dots (u_k\, t_k)^{n_k}\, (uw)^{j+1}\, x_{k+1}^{p-(j+1)}$$

$$\overset{\star}{\underset{\text{(xv)}}{\Rightarrow}} \quad \# \, \alpha\, y_1^{q_1^{j+1}} \dots y_l^{q_l^{j+1}}\, b \dots (u_1\, t_1)^{n_1} \dots (u_k\, t_k)^{n_k}\, (uw)^{j+1}\, x_{k+1}^{p-(j+1)}$$

$$\underset{\text{(xvi)}}{\Rightarrow} \quad \# \, \xi\, y_1^{q_1^{j+1}} \dots y_l^{q_l^{j+1}}\, b \dots (u_1\, t_1)^{n_1} \dots (u_k\, t_k)^{n_k}\, (uw)^{j+1}\, x_{k+1}^{p-(j+1)}$$

$$\overset{\star}{\underset{\text{(xvii)}}{\Rightarrow}} \quad \# \, \xi\, u_1^{n_1} \dots u_k^{n_k}\, u^{j+1}\, y_1^{q_1^{j+1}} \dots y_l^{q_l^{j+1}}\, b \dots t_1^{n_1} \dots t_k^{n_k}\, w^{j+1}\, x_{k+1}^{p-(j+1)}$$

$$\overset{\star}{\underset{\text{(xix)}}{\Rightarrow}} \quad \# \, s_1^{n_1} \dots s_k^{n_k}\, v^{j+1}\, \xi\, y_1^{q_1^{j+1}} \dots y_l^{q_l^{j+1}}\, b \dots t_1^{n_1} \dots t_k^{n_k}\, w^{j+1}\, x_{k+1}^{p-(j+1)}$$

$$\underset{\text{(xx)}}{\Rightarrow} \quad \# \, s_1^{n_1} \dots s_k^{n_k}\, v^{j+1}\, \eta\, y_1^{q_1^{j+1}} \dots y_l^{q_l^{j+1}}\, b \dots t_1^{n_1} \dots t_k^{n_k} w^{j+1} x_{k+1}^{p-(j+1)}$$

$$\overset{\star}{\underset{\text{(xxi)}}{\Rightarrow}} \quad s_1^{n_1} \dots s_k^{n_k}\, v^{j+1}\, y_1^{q_1^{j+1}} \dots y_l^{q_l^{j+1}}\, \eta\, b \dots t_1^{n_1} \dots t_k^{n_k}\, w^{j+1}\, x_{k+1}^{p-(j+1)}$$

$$\underset{\text{(xxii)}}{\Rightarrow} \quad \# \, s_1^{n_1} \dots s_k^{n_k}\, v^{j+1}\, y_1^{q_1^{j+1}} \dots y_l^{q_l^{j+1}}\, b \dots t_1^{n_1} \dots t_k^{n_k}\, w^{j+1}\, x_{k+1}^{p-(j+1)}.$$

Thus coupling the derivations we obtain in G the derivation

$$\# x_1^{n_1} \dots x_k^{n_k} x_{k+1}^p \overset{\star}{\Rightarrow} \# s_1^{n_1} \dots s_k^{n_k} v^j y_1^{q_1^j} \dots y_l^{q_l^j} b \dots t_1^{n_1} \dots t_k^{n_k} w^j x_{k+1}^{p-j}$$

for $1 \leqq j \leqq p-1$.

Taking $j = p-1$ it is possible to obtain in K:

$$\# x_1^{n_1} \dots x_k^{n_k} x_{k+1}^p$$

$$\overset{\star}{\Rightarrow} \# s_1^{n_1} \dots s_k^{n_k} v^{p-1} y_1^{q_1^{p-1}} \dots y_l^{q_l^{p-1}} b \dots t_1^{n_1} \dots t_k^{n_k} w^{p-1} x_{k+1}$$

$$\underset{F_h}{\overset{\star}{\Rightarrow}} \# i_1^{q_1^p} \dots i_l^{q_l^p} b \dots t_1^{n_1} \dots t_k^{n_k} w^{p-1} x_{k+1}.$$

The role of the next group of rules is to eliminate the last supernumerary symbols. Namely, we shall consider the rules:

(xxiii) $w\, x_{k+1} \rightarrow \delta\, b$;

(xxiv) $w\, \delta \rightarrow \delta\, b$;

(xxv) $t_j\, \delta \rightarrow \delta\, b$, $1 \leqq j \leqq k$;

(xxvi) $b\, \delta \rightarrow bb$.

Finally, we have obtained the derivation

$$\# x_1^{n_1} \dots x_k^{n_k} x_{k+1}^p \overset{\star}{\Rightarrow} \# i_1^{q_1^p} \dots i_l^{q_l^p} b \dots$$

The buffer size is given by

$$M_f^K(\mathbf{n}, p) = M_g^{K_g}(\mathbf{n}) + \sum_{j=1}^{p-1} M_h^{K_h}(\mathbf{n}, p, \mathbf{q}^j) + 2p + \|\mathbf{n}\| - 1.$$

COROLLARY 4: *Let $g : N_1^k \rightarrow N_1^l$ and $h : N_1^{k+1+l} \rightarrow N_1$ be two functions from LLBS where $M_g^{K_g}(\mathbf{n}) \leqq c_g \|\mathbf{n}\|$ for $\mathbf{n} \in N_1^k \setminus Q_g$ and*

$$M_h^{k_h}(\mathbf{n}, \mathbf{p}, \mathbf{q}) \leqq c_h \| h(\mathbf{n}, p, \mathbf{q}) \|$$

for $(\mathbf{n}, p, \mathbf{q}) \in N_1^{k+1+l} \setminus Q_h$, where Q_g and Q_h are finite sets. If there exist $\alpha, \beta \in (0, 1)$ so that $\|\mathbf{n}\| < \beta \| g(\mathbf{n}) \|$ and $\|\mathbf{q}\| < \alpha \| h(\mathbf{n}, p, \mathbf{q}) \|$ for all $(\mathbf{n}, p, \mathbf{q}) \in N_1^k \times N_1 \times N_1^k \setminus Q$, where $N_1^k \times N_1 \times h(N_1^k, N_1, N_1^l) \cap Q = \varnothing$, then the function f defined by recursion from g and h belongs to LLBS.

Proof: Using a similar approach as in corollary 3 we can write

$$\begin{aligned} M_f^K(\mathbf{n}, p) &= M_g^{K_g}(\mathbf{n})+\sum_{j=1}^{p-1} M_h^{K_h}(\mathbf{n}, p, \mathbf{q}^j)+2p+\|\mathbf{n}\|-1 \\ &\leqq c_g\|g(\mathbf{n})\|+\sum_{j=1}^{p-1} c_h\|h(\mathbf{n}, p, \mathbf{q}^j)\|+2p+\|\mathbf{n}\|-1 \\ &\leqq c_g\|g(\mathbf{n})\|+\sum_{j=1}^{p-1} c_h\|\mathbf{q}^{j+1}\|+2p+\|\mathbf{n}\|-1, \end{aligned}$$

if $\mathbf{n} \notin Q_g$ and $(\mathbf{n}, p, \mathbf{q}^j) \notin Q_h$, $1 \leqq j \leqq p-1$.

Since $\mathbf{q}^{j+1} = h(\mathbf{n}, j, \mathbf{q}^j)$ we have $\|\mathbf{q}^j\| < \alpha\|\mathbf{q}^{j+1}\|$, hence

$$\|\mathbf{q}^j\| < \alpha^{p-j}\|\mathbf{q}^p\|,$$

for every j, $1 \leqq j \leqq p$.

Afterwards, since $\mathbf{q}^1 = g(\mathbf{n})$ we have also $\|n\| < \beta\|\mathbf{q}^1\| < \beta\alpha^{p-1}\|\mathbf{q}^p\|$.

Putting together these evaluations we obtain

$$\begin{aligned} M_f^K(\mathbf{n}, p) &< c_g\|\mathbf{q}^1\|+c_h\|\mathbf{q}^p\|\sum_{j=2}^{p}\alpha^{p-j}+2p+\|\mathbf{n}\|-1 \\ &\leqq c\|\mathbf{q}^p\|\left(\sum_{j=1}^{p}\alpha^{p-j}\right)+2p+\beta\alpha^{p-1}\|\mathbf{q}^p\|-1 \\ &\leqq c\|\mathbf{q}^p\|\frac{1-\alpha^p}{1-\alpha}+2p+\|\mathbf{q}^p\| \leqq \|\mathbf{q}^p\|\left(\frac{c}{1-\alpha}+1\right)+2p \\ &= c_1\|\mathbf{q}^p\|+2p, \end{aligned}$$

where $c = \max(c_g, c_h)$ and $c_1 = [c/(1-\alpha)]+1$.

Taking into account that

$$\|\mathbf{q}^p\| > A^{p-1}\|\mathbf{q}^1\| = A^{p-1}\|g(\mathbf{n})\| > A^{p-1}B\|\mathbf{n}\|,$$

where $A = 1/\alpha > 1$ and $B = 1/\beta > 1$ it follows that

$$\|\mathbf{q}^p\| > [1+A(p-1)]\|\mathbf{n}\| > 1+A(p-1) > p.$$

Therefore we have

$$M_f^K(\mathbf{n}, p) \leqq (c_1+2)\|\mathbf{q}^p\| = (c_1+2)\|f(\mathbf{n}, p)\|,$$

hence f is in *LLBS*.

IV. SOME ELEMENTARY EXAMPLES

We shall present in the sequel some elementary examples of "basic" functions from *LLBS* from which we shall construct more complex functions using the closure properties of the *LLBS* class presented in corollary 1-4.

(i) The function $f : N_1 \to N_1$ given by $f(n) = an+c$, where $a \in N_1$, $c \in N$ is in *LLBS*. Indeed, let us take

$$K = (\{x_1\}, \{i_1, b, \#\}, \{x_1\}, \{i_1\}, b, \#,$$
$$\{\# x_1 \to \# i_1^c y x_1, y x_1 \to i_1^a y, y \to b\}).$$

In this grammar there exists the derivation

$$\# x_1^n \Rightarrow \# i_1^c y x_1^n \Rightarrow \# i_1^c i_1^a y x_1^{n-1} \overset{\star}{\Rightarrow} \# i_1^{na+c} y \Rightarrow \# i_1^{na+c} b,$$

and f is obviously from *LLBS* since $M_f^K(n) = 1$, $\forall n \in N_1$.

(ii) It is a very well known fact that the language $\{i_1^n \ldots i_k^n \mid n \geqq 1\}$ is context-sensitive. This fact can be recaptured here by proving that the function $h : N_1 \to N_1^k$ given by $h(n) = \underbrace{(n, n, \ldots, n)}_{k\,\text{times}}$ is from *LLBS*.

Let us consider the length-increasing cg:

$$K = (\{x_1\}, \{i_1, \ldots, i_k, b, \#\}, \{x_1\}, \{i_1, \ldots, i_k\}, b, \#,$$
$$\{x_1 \to y_1 \ldots y_k, y_j y_l \to y_l y_j, 1 \leqq l \leqq j \leqq k, y_j \to i_j, 1 \leqq j \leqq k\}).$$

If, in this grammar we have a derivation

$$\# x_1^n \overset{\star}{\Rightarrow} \# i_1^{n_1} \ldots i_k^{n_k} \tag{3}$$

we must have $n_1 = n_2 = \ldots = n_k$. Indeed, the derivation (3) has necessarily the form:

$$\# x_1^n \overset{\star}{\Rightarrow} \# (y_1 \ldots y_k)^n \overset{\star}{\Rightarrow} \# y_1^n y_2^n \ldots y_k^n \overset{\star}{\Rightarrow} \# i_1^{n_1} \ldots i_k^{n_k}$$

and the buffer size is zero.

(iii) The sum $f_S : N_1^2 \to N_1$ given by $f_S(n_1, n_2) = n_1 + n_2$ belongs to *LLBS*. This function is computed by the length-increasing cg:

$$K_S = (\{x_1, x_2\}, \{i_1, b, \#\}, \{x_1, x_2\}, \{i_1\}, b, \#, \{x_1 \to i_1, x_2 \to i_1\})$$

since we have the derivation $\# x_1^{n_1} x_2^{n_2} \overset{\star}{\Rightarrow} \# i_1^{n_1+n_2}$.

(iv) The product $f_P : N_1^2 \to N_1$, where $f_P(n_1, n_2) = n_1 n_2$ is a function from *LLBS*. To prove this let us consider the cg:

$$K_P = (\{x_1, x_2, y, z, v\}, \{i_1, b, \#\}, \{x_1, x_2\}, \{i_1\},$$

$$b,\ \#,\ \{x_1 x_2 \to i_1 b,\ x_1 x_1 x_2 \to x_1 v^2,\ x_1 x_2 x_2 \to v^2 x_2,$$
$$v^2 x_2 \to x_2 v^2,\ x_1 x_2 \to i_1 x_2 x_1, x_1 i_1 \to i_1 x_1,$$
$$x_2 i_1 \to i_1 x_2,\ x_1 v^2 \to x_1 yb, x_1 y \to yi_1,\ x_2 y \to yi_2,\ y \to i_1\}).$$

K_p computes f_p since in this cg it is possible to write:

$$\begin{aligned}
\# x_1^{n_1} x_2^{n_2} &\Rightarrow \# x_1^{n_1-1} v^2 x_2^{n_2-1} \overset{\star}{\Rightarrow} \# x_1^{n_1-1} x_2^{n_2-1} v^2\\
&\overset{\star}{\Rightarrow} \# i_1^{(n_1-1)(n_2-1)} x_2^{(n_2-1)} x_1^{(n_1-1)} v^2\\
&\Rightarrow \# i_1^{(n_1-1)(n_2-1)} x_2^{(n_2-1)} x_1^{(n_1-1)} yb\\
&\overset{\star}{\Rightarrow} \# i_1^{(n_1-1)(n_2-1)+(n_1-1)+(n_2-1)+1} b = \# i_1^{n_1 n_2} b.
\end{aligned}$$

We conclude also that $M_{f_p}^{K_p}(n_1, n_2) = 1$.

(v) In [3] S. Istrail exhibited a context-sensitive grammar for a language $L_P = \{ i^{P(n)} \mid n \in N \}$, where P is a polynomial having its coefficients in N_1 proving that L_p is a type-1 language.

This result can be retrived in our new approach as follows. Suppose that

$$\begin{aligned}
P(n) &= a_0 n^m + a_1 n^{m-1} + \ldots + a_{m-1} n + a_m\\
&= (\ldots((a_0 n + a_1) n + a_2) \ldots) n + a_m.
\end{aligned}$$

We shall act by induction on m.

Denoting by $\varphi_j : N_1^2 \to N_1$ the function $\varphi_j(m, n) = mn + a_j$, $2 \leqq j \leqq m$ and by $\varphi_1 : N_1 \to N_1$ the function $\varphi_1(n) = a_0 n + a_1$, $P(n)$ is given by

$$P(n) = \varphi_m(\ldots(\varphi_3(\varphi_2(\varphi_1(n), n), n), \ldots, n)).$$

We have proved at (i) that $\varphi_1 \in LLBS$, with $M_{\varphi_1}^K(n) = 1$, $\forall n \in N$ for a suitable cg. Since, for $\mathbf{n} = (m, p)$ we have $\|\mathbf{n}\| = m + p$ the condition $\|\mathbf{n}\| \leqq \|f(\mathbf{n})\|$ considered in theorem 3 becomes $m + n \leqq mn + a_j$ this condition is satisfied for all $(m, n) \in N_1^2$ since $a_j \geqq 1$.

Denoting by $\iota : N_1 \to N_1$ the identity map $\iota(n) = n$, the function $\varphi_1 \times \iota$ belongs to $LLBS$, hence $\varphi_2 \circ (\varphi_1 \times \iota) \in LLBS$, where

$$\varphi_2 \circ (\varphi_1 \times i)(n) = \varphi_2(\varphi_1(n), n) = (a_0 n + a_1) n + a_2.$$

Suppose that we have proved that each polynomial with degree less or equal to m is in $LLBS$ and let $H = a_0 n^{m+1} + a_1 n^m + \ldots + a_{m+1}$ be a polynomial whose degree is $m+1$.

Since $H = (a_0 n^m + a_1 n^{m-1} + \ldots + a_m) n + a_{m+1} = \varphi_{m+1}(P(n), n)$ it follows immediately that $H \in LLBS$.

(vi) Let us consider now the exponential function $g : N_1 \rightarrow N_1$ given by $g(n) = a^n$, where $a > 1$. This function is computed by the cg:

$$K = (\{x_1, y\}, \{i_1, b, \#\}, \{x_1\}, \{i_1\}, b, \#,$$
$$\{\# x_1 \rightarrow \# y i_1^{a-1}, i_1 x_1 \rightarrow x_1 i_1^a, y x_1 \rightarrow y i_1^{a-1}, y \rightarrow i_1\})$$

since it is possible to write

$$\# x_1^n \Rightarrow \# y i_1^{a-1} x_1^{n-1} \overset{\star}{\Rightarrow} \# y x_1 i_1^{a^2-a} x_1^{n-2}$$
$$\Rightarrow \# y i_1^{a^2-1} x_1^{n-2} \overset{\star}{\Rightarrow} \# y x_1 i_1^{a^3-3} x_1^{n-3}$$
$$\Rightarrow \# y i_1^{a^3-1} x_1^{n-4} \Rightarrow \dots \Rightarrow \# y i_1^{a^n-1} \Rightarrow \# i_1^{a^n}.$$

The last rule $y \rightarrow i_1$, can not be applied until the last step because x_1 can be eliminated only using the rule $y x_1 \rightarrow y i_1^{a-1}$.

Remark: By a slight modification of the proof of theorem 1 it is possible to prove the following assertion:

If the language $L_D = \{ j_1^{n_1} \dots j_k^{n_k} \mid (n_1, \dots, n_k) \in D \}$ is a context-sensitive one (where D is a suitable subset of N_1^k) and $f: N_1^k \rightarrow N_1^h$ is a function from *LLBS* then the language $\{ i_1^{m_1} \dots i_h^{m_h} \mid (m_1, \dots, m_h) \in f(D) \}$ is again context-sensitive (the proof is left for the reader).

For instance, starting from the fact that the language $\{ i^p \mid p \text{ is a prime} \}$ is context-sensitive (*see* [4]) it follows immediately, using the example (vi) that the language $\{ i^{2^p} \mid p \text{ is a prime} \}$ is again context-sensitive a. s. o.

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