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Remarks on DOL growth sequences


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REMARKS ON DOL GROWTH SEQUENCES (*)

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Abstract. — Two theorems are given characterizing the position of DOL and PDOL growth sequences among N-rational sequences.

1. INTRODUCTION

A DOL system or a deterministic context-independent Lindenmayer system consists of an initial word $\omega$ and a set of productions $x \rightarrow \delta(x)$ which give for any letter $x$ and thus also for any word a unique successor. The growth sequence of a DOL system is the sequence formed by the lengths of the words $\omega, \delta(\omega), \delta^2(\omega), \ldots$ DOL sequences have been investigated e. g. in Paz and Salomaa [6], Salomaa [8], Vítanyi [10], Ruohonen [7] and Karhumäki [4].

A sequence $(r_n)$ is called $N$-rational if it can be represented in the form $r_n = PM^nQ$ where $P$ is a row vector, $M$ is a square matrix, $Q$ is a column vector and the entries of $P$, $M$ and $Q$ are natural numbers. (The name $N$-rational comes from the general theory of rational series founded by M. P. Schützenberger.) Now it is easy to see that a DOL sequence is $N$-rational; in fact it has a representation $PM^nQ$ where $Q$ consists merely of ones.

If a DOL sequence is not terminating, i. e. if $r_n \neq 0$ for every $n$, and if $L$ is the largest of the lengths of the words $\delta(x)$ then obviously $r_{n+1}/r_n \leq L$ for every $n$. If the system under consideration is such that $\delta(x)$ is always a non-empty word then this system is called a PDOL system and its growth sequence is called a PDOL sequence. Obviously a PDOL sequence is non-decreasing.

The goal of this paper is to illustrate the position of DOL and PDOL sequences among $N$-rational sequences. It will be seen that the satisfaction of an inequality $r_{n+1}/r_n \leq L$ is characteristic for DOL sequences. Further it will be seen that it is not the non-negativity but the $N$-rationality of the sequence $(r_{n+1} - r_n)$ that makes a DOL sequence to be a PDOL sequence.

2. PRELIMINARIES

A DOL system is at triple $G = (X, \delta, \omega)$ where $X = \{x_1, \ldots, x_k\}$ is an alphabet, $\delta$ is an endomorphism of the free monoid $X^*$ and $\omega \in X^*$. The mapping $\delta$ is usually given by writing the productions $x_i \rightarrow \delta(x_i)$ and the
word $\omega$ is called the axiom. If $\delta(x_i) \neq \lambda$ for each $i$ then $G$ is called a PDOL system. The function

$$f_G(n) = \lg(\delta^n(\omega)),$$

where $\lg$ means word length is called the growth function of $G$.

A pair $(X, \delta)$ where $X$ and $\delta$ are as above is called a DOL scheme. Introducing the axiom vector

$$P = (\lg_1(\omega), \ldots, \lg_k(\omega))$$

and the growth matrix

$$M = \begin{pmatrix}
\lg_1(\delta(x_1)) & \cdots & \lg_k(\delta(x_1)) \\
\vdots & \ddots & \vdots \\
\lg_1(\delta(x_k)) & \cdots & \lg_k(\delta(x_k))
\end{pmatrix}$$

of $G$ (here $\lg_j$ denotes the number of letters $x_j$) we obtain

$$(\lg_1(\delta^n(\omega)), \ldots, \lg_k(\delta^n(\omega))) = PM^n$$

and

$$f_G(n) = PM^n(1, \ldots, 1)^T.$$ 

A sequence $(r(n))$ is called $Z$-rational (resp. $N$-rational) if

$$r(n) = PM^n Q = (p_1, \ldots, p_k) \begin{pmatrix}
m_{11} & \cdots & m_{1k} \\
\vdots & \ddots & \vdots \\
m_{k1} & \cdots & m_{kk}
\end{pmatrix} \begin{pmatrix}
q_1 \\
\vdots \\
q_k
\end{pmatrix},$$

where all the entries are integers (resp. non-negative integers). If now $G$ is a DOL system (resp. a PDOL system) then by the above ($f_G(n)$) is a special $N$-rational sequence called a DOL sequence (resp. a PDOL sequence).

It is known (Schützenberger [9]) that a sequence $(r(n))$ is $Z$-rational ($N$-rational) if the series $\sum r(n) x^n$ is $Z$-rational ($N$-rational). If now $\sum r(n) x^n$ is a non-polynomial $N$-rational series then a theorem of Berstel [1] concerning its poles tells the following: there are a natural number $p$, algebraic numbers $A, A_1, \ldots, A_s (A > 0, |A_j| < A, s \geq 0)$ and polynomials $H_0, \ldots, H_{p-1}, h_1, \ldots, h_s$ such that

$$r(i + np) = H_i(i + np) A_i^{i+np} + \sum_{j=1}^s h_j(i + np) A_j^{i+np}$$

for large values of $n (i = 0, \ldots, p-1)$. In the case of a DOL sequence the polynomials $H_i$ must have a common degree $l$ because the quotients $r(n+1)/r(n)$ are bounded from above. We shall say that $(r(n))$ has the growth order $n^l A^n$.

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3. THE PDOL SEQUENCES

LEMMA 1: If \( (f(n)) = (P M^n Q) \) is an \( N \)-rational sequence and \( P \) has positive entries then \( (f(n)) \) is a DOL sequence.

Proof: Let \( G = (\{ x_1, \ldots, x_k \}, \delta, \omega) \) be a DOL system with axiom vector \( Q^T \) and growth matrix \( M^T \). Define \( G' = (\{ x_1, \ldots, x_k, x \}, \delta', \omega') \) where
\[
\delta'(x_i) = \delta(x_i) x^{\log_\delta(\delta(x_i))(p_i-1) + \cdots + \log_k(\delta(x_k))(p_k-1)},
\]
\[
\delta'(x) = \lambda,
\]
\[
\omega' = \omega x^{\log_\delta(\omega)(p_1-1) + \cdots + \log_k(\omega)(p_k-1)}.
\]
Then obviously \( f(n) = Q^T (M^n P^n) = f_{G'}(n) \).

THEOREM 1: Let \( (r(n)) \) be an \( N \)-rational sequence. Then we can find natural numbers \( m \) and \( p \) and DOL sequences \( (d_0(n)), \ldots, (d_{p-1}(n)) \) such that
\[
r(m+i+np) = d_i(n) \quad (i = 0, \ldots, p-1).
\]
Proof: Let \( r(n) = P M^n Q \) and let \( G = (X, \delta, \omega) \) be a DOL system with axiom vector \( P \) and growth matrix \( M \). Denote by \( X_n \) the set of letters occurring in \( \delta^n(\omega) \). Then we can find numbers \( m \) and \( p \) such that \( X_{m+i} = X_{m+i+np} \).
We may of course suppose that no \( y_n \) is empty.
Introduce now the DOL systems
\[
G_i = (X_{m+i}^n, \delta^m, \delta^{m+i}(\omega)) \quad (i = 0, \ldots, p-1)
\]
whose axiom vectors and growth matrices are denoted by \( P_i \) and \( M_i \). Then obviously
\[
r(m+i+np) = P_i M^n Q_i,
\]
where \( Q_i \) is composed of those entries of \( Q \) corresponding to letters of \( X_{m+i} \).
By lemma 1 we may define \( d_i(n) = P_i M^n Q_i \).

THEOREM 2: The following conditions are equivalent for a sequence \( (r(n)) \):
(i) \( (r(n)) \) is a PDOL sequence not identically zero;
(ii) \( r(0) \) is a positive integer and the sequence \( (s(n)) = (r(n+1) - r(n)) \) is \( N \)-rational.

Proof: Suppose (i) holds. If now \( (r(n)) \) corresponds to a PDOL system \( G = (\{ x_1, \ldots, x_k \}, \delta, \omega) \) then
\[
s(n) = \sum_{i=1}^{k} \log_i(\delta^m(\omega))(\log(\delta(x_i))-1)
\]
and each of the sequences \( (\log_i(\delta^m(\omega))) \) \( (i = 1, \ldots, k) \) is \( N \)-rational.

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Suppose then that (ii) holds. Write according to theorem 1
\[ s(m+i+np) = d_i(n) \quad (i = 0, \ldots, p-1) \]
where \((d_i(n))\) corresponds to a system \(G_i = (X_i, \delta_i, \omega_i)\). Assuming that the alphabets \(X_i\) are mutually disjoint we construct the PDOL system \(G = (X, \delta, \omega)\) where
\[
X = \left( \bigcup_{i=0}^{p-1} \bigcup_{j=0}^{p-1} X_i^{(j)} \right) \cup \{ y \},
\]
\[
\omega = \omega_0^{(p-1)} \omega_1^{(p-2)} \ldots \omega_{p-2}^{(1)} \omega_{p-1}^{(0)}
\]
and
\[
x^{(j)} \rightarrow x^{(j+1)} \quad \text{when} \quad x^{(j)} \in X_i^{(j)} , \ j < p-1,
\]
\[
x^{(p-1)} \rightarrow \delta_i(x)^{(0)} y \quad \text{when} \quad x^{(p-1)} \in X_i^{(p-1)},
\]
\[
y \rightarrow y.
\]
Disregarding the non-commutativity of letters we may write
\[
\omega \rightarrow \delta_0 (\omega_0)^{(0)} y^{lg(\omega_0)} \omega_1^{(p-1)} \ldots \omega_{p-2}^{(2)} \omega_{p-1}^{(1)}
\]
\[
\rightarrow \delta_0 (\omega_0)^{(1)} y^{lg(\omega_0)} \delta_1 (\omega_1)^{(0)} y^{lg(\omega_1)} \ldots \omega_{p-1}^{(2)}
\]
\[
\rightarrow \ldots
\]
Thus
\[
r(n) = (r(0) + s(0) + \ldots + s(m-1))
\]
\[
+ (s(m) + \ldots + s(m + p - 1))
\]
\[
+ s(m + p) + \ldots + s(n-1)
\]
\[
= (r(0) + s(0) + \ldots + s(m - 1)) + f_G(n - m - p),
\]
when \(n \geq m + p\). It is now easy to extend \(G\) to a PDOL system \(G'\) for which
\[
f_{G'}(n) = r(n).
\]

**LEMMA 2:** Let \((r(n))\) be a \(Z\)-rational sequence. Then for any large natural number \(R\) the sequence defined by
\[
d(0) = d(1) = 1,
\]
\[
d(2n) = R^{2n} - r(2n),
\]
\[
d(2n+1) = R^{2n} - r(2n+1) \quad (n > 0)
\]
is a DOL sequence.

**Proof:** Let at first \((r(n))\) be a DOL sequence corresponding to the system \(G = (X, \delta, \omega)\). Construct the system \(H = (X \cup \overline{X} \cup \overline{X} \cup \{ a, b \}, \delta', \Omega)\) where
\[
\Omega = a^{R^2 - 2r(2) - r(3)} \delta^2(\omega) \delta^3(\omega)
\]
and
\[ x \rightarrow b \bar{x}, \quad \bar{x} \rightarrow \lambda, \quad a \rightarrow b, \quad b \rightarrow a^R, \quad \bar{x} \rightarrow a^{R^2 (1 + \log (\delta(x))) - 2 \log (\delta^2 (x)) - \log (\delta^3 (x))} \delta^2 (x) \delta^3 (x). \]

It is immediately seen that \( f_H (n) = d (n+2). \)

This implies our lemma because of the following. It is known that every \( \mathbb{Z} \)-rational sequence is the difference of two \( \mathbb{N} \)-rational sequences (see [9] remark 2 or [2] p. 218). Furthermore, every \( \mathbb{N} \)-rational sequence is the difference of two DOL sequences for
\[ PM^n Q = (P + (1, \ldots, 1)) M^n Q - (1, \ldots, 1) M^n Q. \]

Hence every \( \mathbb{Z} \)-rational sequence can be written as the difference of two DOL sequences.

**Theorem 3:** Not every increasing DOL sequence is a PDOL sequence.

**Proof:** Using lemma 2 we see that when \( R \) is a large natural number then the sequence \( (d(n)) \) where
\[ d(0) = d(1) = 1, \quad d(2n) = R^{2n}, \quad d(2n+1) = R^{2n} + (\Re (3 + 4i)^{2n+1})^2 \quad (n > 0) \]
is an increasing DOL sequence. Now
\[ \Re (3 + 4i)^n = \cos 2\pi n \alpha \cdot 5^n, \]
where \( \alpha \) is irrational because for every positive \( n \) \( \Im (3 + 4i)^n < 4 \mod 5 \).

The theorem of Berstel [1] then implies that the sequence \( (d(2n+1) - d(2n)) \) cannot be \( \mathbb{N} \)-rational. Therefore \( (d(n)) \) is not a PDOL sequence.

**Note:** Let \( (d(n)) = (PM^n Q) \) be a DOL sequence. By lemma 4 below we have \( M^{m+p} \geq M^m \) for some integers \( m \) and \( p (p > 0) \). But then each of the sequences
\[ (d(m+i+(n+1)p) - d(m+i+np)) \quad (i = 0, \ldots, p-1) \]
is a DOL sequence and so the sequences
\[ (d(m+i+np)) \quad (i = 0, \ldots, p-1) \]
are PDOL sequences. This result also appears in [5] (proof of th. 4.12).

**4. THE DOL SEQUENCES**

Let \( G = \{ x_1, \ldots, x_k \}, \delta \) be a DOL scheme such that for any letter \( x_i \):
\[ \log (\delta^n (x_i)) \sim g_i A^n \quad \text{as} \quad n \to \infty \quad (g_i > 0, A \geq 1). \]

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If \( w \in \{ x_1, \ldots, x_k \}^+ \) then the number
\[
g(w) = \log_1(w) g_1 + \ldots + \log_k(w) g_k
\]
is called the growth coefficient of \( w \).

Suppose we have \( p \) DOL schemes \( G_i = (X_i, \delta_i) \) \((i = 0, \ldots, p-1)\) satisfying the condition of the above definition with a common number \( A \). Introduce the infinite alphabet
\[
X = \{(W_0, \ldots, W_{p-1}) \mid W_i \in X_i^+ \}
\]
ad define
\[
\delta(W_0, \ldots, W_{p-1}) = (\delta_0 W_0, \ldots, \delta_{p-1} W_{p-1})
\]
where \( \delta \) means Parikh-vector. Take then a fixed element \((\omega_0, \ldots, \omega_{p-1})\) of \( X \) and denote
\[
Y = \left\{ (W_0, \ldots, W_{p-1}) \mid \frac{g(W_0)}{g(\omega_0)} = \ldots = \frac{g(W_{p-1})}{g(\omega_{p-1})} \right\}.
\]
Obviously \((W_0, \ldots, W_{p-1}) \in Y\) implies that also \( \delta(W_0, \ldots, W_{p-1}) \in Y \).

**Lemma 3:** There are vectors \( V_1, \ldots, V_j \) of \( \pi(Y) \) such that any vector in \( \pi(Y) \) is a sum of these.

**Proof:** Let \( \pi(Y) \subseteq N^k \). By the definition of \( Y \) there are algebraic numbers \( a_{ij} (i = 1, \ldots, p-1; j = 1, \ldots, k) \) such that \( \bar{v} = (n_1, \ldots, n_k) \in Z^k \) is in \( \pi(Y) \) iff
\[
\bar{v} \neq 0, \quad \bar{v} \geq 0,
\]
and
\[
a_{i1} n_1 + \ldots + a_{ik} n_k = 0 \quad (i = 1, \ldots, p-1).
\]
Let \( F \) be the additive subgroup of \( Z^k \) defined by the above linear system.

We shall need the following simple lemma (see [3]):

**Lemma 4:** Any subset of \( N^k \) contains only a finite number of minimal vectors (with respect to the natural componentwise ordering).

Let now \( \bar{V}_1, \ldots, \bar{V}_j \) be the minimal vectors of \( \pi(Y) \). If \( \bar{V} \in \pi(Y) \) then it has a representation \( \bar{V} = \bar{V}_i + \bar{U} \) where \( \bar{U} \in N^k \). But if \( \bar{U} \neq \bar{0} \) it is in \( \pi(Y) \) because it belongs to \( F \). Repeating this process we obtain \( \bar{V} \) as a sum of the minimal vectors.

**Theorem 4:** Let \( G_i = (X_i, \delta_i, \omega_i) \) \((i = 0, \ldots, p-1)\) be DOL systems such that if \( x_j \in X_i \) then
\[
\log(\delta_i^n(x_j)) \sim g_{ij} A^n \quad \text{as} \quad n \to \infty \quad (g_{ij} > 0, A \geq 1).
\]
Then the sequence defined by
\[ d(np + i) = \log(\delta_i^n(\omega_j)) \quad (n = 0, 1, \ldots, i = 0, \ldots, p - 1) \]
is a DOL sequence.

Proof: Take \( p \) copies \( X_0^{(0)}, \ldots, X_{p-1}^{(p-1)} \) of each \( X_i \) and define
\[
Y = \bigcup_{j=0}^{p-1} \left\{ \left( W_0^{(j)}, \ldots, W_{p-1}^{(j)} \right) \mid W_i^{(j)} \in X_i^{(j)+}, \quad \frac{g(W_0)}{g(\omega_0)} = \ldots = \frac{g(W_{p-1})}{g(\omega_{p-1})} \right\}.
\]
Let \( V_0^{(0)}, \ldots, V_p^{(0)}, \ldots, V_0^{(p-1)}, \ldots, V_p^{(p-1)} \) be elements of \( Y \) corresponding to the vectors given by lemma 3. We may say that any element of \( Y \) is a commutative product of these elements.

Introduce the DOL system
\[
G = \left( \{ V_0^{(0)}, \ldots, V_p^{(p-1)} \} \cup \{ y \}, \delta, \omega \right) = (Z \cup \{ y \}, \delta, \omega),
\]
where \( \delta \) and \( \omega \) are as follows:
- \( \omega \) consists of \( (\omega_0^{(0)}, \ldots, \omega_p^{(p-1)}) \) written commutatively in the alphabet \( Z \) and of so many \( y \)'s that \( \log(\omega) \) becomes equal to \( \log(\omega_0) \);
- \( y \) produces \( \lambda \):
  - when \( j < p-1 \) \( (W_0^{(j)}, \ldots, W_{p-1}^{(j)}) \) produces \( (W_0^{(j+1)}, \ldots, W_{p-1}^{(j+1)}) \) and so many \( y \)'s that the length of the produced word will be \( \log(W_{i+1}) \);
  - \( (W_0^{(p-1)}, \ldots, W_{p-1}^{(p-1)}) \) produces \( (\delta_0(W_0)^{(0)}, \ldots, \delta_{p-1}(W_{p-1})^{(0)}) \) written commutatively in the alphabet \( Z \) and so many \( y \)'s that the produced word will have length \( \log(\delta_0(W_0)) \).

Heuristically, derivations in the systems \( G_i \) are simulated in the components of the letters of \( Z \). With the aid of the \( p \) copies taken of the alphabets the simulation is delayed to happen only at intervals of \( p \) steps. By using the letter \( y \) the length of the word \( \delta_i^n(\omega_i) \) is adjusted to be equal to that of \( \delta_i^n(\omega_j) \). This is possible because the components of the letters of \( Z \) are non-empty words.

It should be clear now that \( (d(n)) \) is the growth sequence of \( G \).

Let \( G = (X, \delta) \) be a DOL scheme which gives a growth of order \( n! A^n \quad (A \geq 1, l \leq 0) \) but does not give a growth of higher order. We divide \( X \) into classes \( \Sigma, \Sigma_0, \ldots, \Sigma_l \) as follows: the letters of \( \Sigma \) generate a growth having smaller order than \( A^n \) and the letters of \( \Sigma_i \) generate a growth of order \( n! A^n \).

It is clear that a letter of \( \Sigma_i \) cannot produce letters of \( \Sigma_{i+1} \cup \ldots \cup \Sigma_l \), it must produce a letter of \( \Sigma_i \) and it may produce letters of \( \Sigma \cup \Sigma_0 \cup \ldots \cup \Sigma_{i-1} \); a letter of \( \Sigma \) may produce only letters of \( \Sigma \) or \( \lambda \).

**Lemma 5:** Any letter of \( \Sigma_i \) \((l > 0) \) generates letters of \( \Sigma_{i-1} \). If all letters of \( \Sigma \cup \Sigma_0 \cup \ldots \cup \Sigma_{l-1} \) are deleted then the resulting scheme \( H \) is such that all letters generate a growth of order \( A^n \).

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Proof: Suppose \( x \) generates in \( H \) a growth whose order is at least \( n \ A^n \). Then \( x \) generates in \( G \) in \( 2^n \) steps a word whose length is at least of the order
\[
n A^n n^l A^n = \left(\frac{1}{2}\right)^{l+1} (2n)^{l+1} A^{2^n}.
\]
This shows that the growth of \( x \) in \( H \) has the order \( A^n \) at most.

Assume that \( x \in \Sigma_1 (l > 0) \) never generates a letter of \( \Sigma_{l-1} \). By the above \( \delta^n (x) \) contains \( O (A^n) \) letters of \( \Sigma \cup \Sigma_0 \cup \ldots \cup \Sigma_{l-2} \) directly produced by letters of \( \Sigma_l\). But
\[
\sum_{n=0}^{N} A^n (N-n)^{l-2} A^{N-n} = A^n \sum_{n=0}^{N} n^{l-2} = o (N^l A^N)
\]
and so \( x \) cannot generate a growth of order \( n^l A^n \). Hence \( x \) must generate letters of \( \Sigma_{l-1} \).

Assume further that the growth of \( x \) in \( H \) is majorized by \( a^n (a < A) \). Because
\[
\sum_{n=0}^{N} a^n (N-n)^{l-1} A^{N-n} = a^n \sum_{n=0}^{N} n^{l-1} (A/a)^n = O (N^{l-1} A^N)
\]
we have a contradiction as above. Thus the growth of \( x \) in \( H \) has the order \( A^n \) when \( l > 0 \).

Suppose \( x \in \Sigma_0 \) and the growth of \( x \) in \( H \) as well as the growth of any letter of \( \Sigma \) in \( G \) is majorized by \( a^n (a < A) \). Because
\[
\sum_{a=0}^{N} a^n A^{N-n} = o (A^N)
\]
we see that the above result is true also when \( l = 0 \).

**Theorem 5:** Let \( (r (n)) \) be an \( N \)-rational sequence such that \( r (n) \neq 0 \) for every \( n \) and the quotient \( r (n+1)/r (n) \) remains bounded. Then \( (r (n)) \) is a DOL sequence.

**Proof:** We know that there are numbers \( m \) and \( p \) and DOL sequences \( (d_0 (n)), \ldots, (d_{p-1} (n)) \) such that
\[
r (m+i+np) = d_i (n).
\]
By our assumption all these sequences have the same order of growth. We may suppose that it is of the form \( n^l A^n (A > 1) \) for the polynomial case is covered by a theorem of Ruohonen [7].

Let \( G_i = (X_i, \delta_i, \omega_i) \) be a DOL system corresponding to the sequence \( (d_i (n)) \). Write \( X_i = \Sigma_i \cup \Sigma_{i0} \cup \ldots \cup \Sigma_{i_l} \) as before and denote by \( H_{ij} \) the DOL schema obtained from \( (X_i, \delta_i) \) by deleting all letters except those of \( \Sigma_{ij} \).
Suppose \( W \in \Sigma_1^+ \) and \( w \in \Sigma^* \). If \( W \) generates \( V \) \( (V \in \Sigma_{10}^+, v \in \Sigma_i^*) \) and \( w \) generates \( u \) in \( k \) steps then there are constants \( N, M \) and \( L \) (independent of \( k \) and \( i \)) such that

\[
\log(V) \geq NA^k \log(W),
\]
\[
\log(v) \leq MA^k \log(W),
\]
\[
\log(u) \leq La^k \log(w) \quad (a < A).
\]

We now see that if \( \log(w)/\log(W) \leq \alpha M/N (\alpha \geq 2, \alpha M/N \text{ integer}) \) and if \( k \) is so large that \( (L/N) (a/A)^k \leq 1/2 \) then

\[
\log(u)/\log(V) \leq M/N + (L/N)(a/A)^k(\log(w)/\log(W)) \\
\leq M/N + (1 - 1/\alpha) \alpha M/N = \alpha M/N,
\]
too.

By taking a multiple of \( p \), if necessary, we may suppose that the following three conditions hold:

(i) any growth in \( H_{ij} \) \( (i = 0, \ldots, p-1; \ j = 0, \ldots, l) \) is asymptotically equal to constant times \( A^n \);

(ii) any letter of \( \Sigma_{ij} \) \( (i = 0, \ldots, p-1; \ j = 0, \ldots, l) \) produces in a step letters of all the alphabets \( \Sigma_{i,j-1}, \ldots, \Sigma_{i,0} \);

(iii) the equation (1) holds with \( k = 1 \).

Moreover, by increasing \( m \) we obtain the following situation:

(iv) the axiom of \( G_i \) contains letters of all the alphabets \( \Sigma_{i,0}, \ldots, \Sigma_{i,l} \).

We are now ready to give an induction proof showing that the sequence \( (s(n)) = (r(n-m)) \) is a DOL sequence. This immediately implies our theorem.

If \( l = 0 \) we at first neglect all letters of the \( \Sigma_i \)'s and construct a system just as in the proof of theorem 4. Then we take \( p \) copies \( x^{(0)}, \ldots, x^{(p-1)} \) of each neglected letter \( x \) and join these copies to the components of the letters of \( Z \) so that the original systems \( G_i \) become simulated. Condition (iii) assures that this can be done. At the same time we add \( y \)'s so that the right lengths are obtained.

When taking the induction step we at first delete all letters of the alphabets \( \Sigma_i \cup \Sigma_{i,0} \cup \ldots \cup \Sigma_{i,l-1} \) and construct a system according to theorem 4. By conditions (ii) and (iv) the letters of this system as well as the \( p \)-tuple \( (\omega_0, \ldots, \omega_{p-1}) \) give axioms for systems whose existence is guaranteed by the induction hypothesis.

**Example:** Let

\[
G_0 = (\{ A, B, C \}, \delta_0, AB),
\]

where

\[
A \to A^4 B, \quad B \to B^4 b, \quad b \to b
\]

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and

\[ G_1 = (\{ \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F} \}, \delta_1, \mathcal{C} \mathcal{D}) \]

where

\[
\begin{align*}
\mathcal{C} & \rightarrow \mathcal{C}^4, \\
\mathcal{D} & \rightarrow \mathcal{D}^2 \mathcal{E}^3, \\
\mathcal{E} & \rightarrow \mathcal{E}^2 \mathcal{F}^4, \\
\mathcal{F} & \rightarrow \mathcal{D} \mathcal{F}.
\end{align*}
\]

The common order of growth is \( nA^n \) and

\[ \Sigma_0 = \{ b \}, \quad \Sigma_{00} = \{ B \}, \quad \Sigma_{01} = \{ A \}, \]

\[ \Sigma_1 = \emptyset, \quad \Sigma_{10} = \{ D, E, F \}, \quad \Sigma_{11} = \{ C \}. \]

The procedures described in the preceding proof yield e. g. the following system: The axiom is

\[ (A^{(0)}, C^{(0)})(B^{(0)}, D^{(0)}) \]

and the productions are

\[
\begin{align*}
(A^{(0)}, C^{(0)}) & \rightarrow (A^{(1)}, C^{(1)}), \\
(A^{(1)}, C^{(1)}) & \rightarrow (A^{(0)}, C^{(0)})^4 (B^{(0)}, D^{(0)}), \\
(B^{(0)}, D^{(0)}) & \rightarrow (B^{(1)}, D^{(1)}), \\
(B^{(1)}, D^{(1)}) & \rightarrow (B^{(0)} b^{(0)}, D^{(0)})(B^{(0)}, D^{(0)})(B^{(0)} D^{(0)}2, E^{(0)3}) y^2, \\
(B^{(0)} b^{(0)}, E^{(0)}3) & \rightarrow (B^{(1)} b^{(1)}, E^{(1)3}) y^2, \\
(B^{(1)} b^{(1)}, E^{(1)3}) & \rightarrow (B^{(0)} b^{(0)}, E^{(0)3})^2 (B^{(0)}, F^{(0)3})^4 y^4, \\
(B^{(0)}, F^{(0)3}) & \rightarrow (B^{(1)}, F^{(1)3}) y^2, \\
(B^{(1)}, F^{(1)3}) & \rightarrow (B^{(0)} b^{(0)}, F^{(0)3})(B^{(0)}, D^{(0)})^3 y, \\
(B^{(0)} b^{(0)}, D^{(0)}) & \rightarrow (B^{(1)} b^{(1)}, D^{(1)}), \\
(B^{(1)} b^{(1)}, D^{(1)}) & \rightarrow (B^{(0)} b^{(0)}, D^{(0)})^2 (B^{(0)} D^{(0)}2, E^{(0)3})^3 y^3, \\
(B^{(0)} D^{(0)}2, E^{(0)3}) & \rightarrow (B^{(1)} D^{(1)} b^{(1)}, E^{(1)3}) y^2, \\
(B^{(1)} D^{(1)} b^{(1)}, E^{(1)3}) & \rightarrow (B^{(0)} D^{(0)}2 b^{(0)}, E^{(0)3})^2 (B^{(0)} b^{(0)}, F^{(0)3})(B^{(0)}, F^{(0)3})^3 y^5, \\
(B^{(0)} b^{(0)}, F^{(0)3}) & \rightarrow (B^{(1)} b^{(1)}, F^{(1)3}) y^2, \\
(B^{(1)} b^{(1)}, F^{(1)3}) & \rightarrow (B^{(0)} b^{(0)}, F^{(0)3})(B^{(0)} b^{(0)}, D^{(0)})(B^{(0)}, D^{(0)})^2 y^2, \\
y & \rightarrow \lambda.
\end{align*}
\]

Note: Let \( (d(n)) \) be a DOL sequence such that the rational function \( \sum d(n) x^n \) is not a polynomial. Then it is easy to see that the growth order of \( (d(n)) \) is \( n A^n \) where \( 1/A \) is the smallest positive pole of \( \sum d(n) x^n \) and \( l+1 \)
Remarks on DOL growth sequences

is its multiplicity. This enables us to effectively compare the growth orders of two DOL sequences; we describe the method briefly in general form.

Given integer polynomials \( q_1(x), \ldots, q_p(x) \) we can (using symmetric polynomials) construct a polynomial \( Q(x) \) such that any difference of two zeros of \( q(x) = q_1(x) \ldots q_p(x) \) is a zero of \( Q(x) \). Thus we can give a positive number \( \gamma \) such that if \( z_1 \) and \( z_2 \) are zeros of \( q(x) \) then either \( z_1 = z_2 \) or \( |z_1 - z_2| > \gamma \).

The polynomial

\[
Q_i(x) = \frac{q_i(x)}{\gcd(q_i(x), q'_i(x))}
\]

has simple zeros which are the same as those of \( q_i(x) \). Therefore we can compare the real roots of the polynomials \( q_i(x) \) by examining the sign changes of the polynomials \( Q_i(x) \).

Because \( \alpha \) is a \( k \)-fold zero of \( q_i(x) \) iff it is a zero of \( q_i(x), q'_i(x), \ldots, q^{(k-1)}(x) \) but not a zero of \( q^{(k)}(x) \) it is possible to determine the multiplicity of any real zero of \( q_i(x) \).

An \( N \)-rational sequence \( (r(n)) \) is by theorem 5 a DOL sequence iff one of the following conditions holds:

(i) there is a natural number \( L \) such that \( r(n) > 0 \) when \( n \leq L \) and \( r(n) = 0 \) when \( n > L \);

(ii) every \( r(n) \) is positive and the DOL sequences given by theorem 1 have the same growth order.

Hence it is possible to decide whether or not a given \( N \)-rational sequence is a DOL sequence.

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References

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