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LIMITING RECURSION AND THE ARITHMETIC HIERARCHY

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Abstract. — *The Kleene hierarchy, with the exception of the first level is completely characterized by the iteration of the limit operation as defined by Gold* [1]. *Any arithmetic function can be obtained by applying a finite number of times the limit operator to a primitive recursive function.*

There seems to be a growing measure of interest in the notion of the limit of a recursive function [1] particularly in connection with the concepts of approximation, identification and learning of functions [2, 3, 4] and grammars [5] and of dialogical deductive systems [6].

Recently the notion of iteration of the limit procedure has been investigated by Schubert [7] in connection with the size-complexity of programs.

We give here a complete characterization of the Kleene hierarchy in terms of the iterated limit operation which fully answers an open problem indicated by Schubert and is interesting in its own right.

We will use only the following well known definition of limit of a number theoretic function.

Def. If g is a total function of $k + 1$ variables, then $\lim\limits_{n} g(x_1, ..., x_k, n) = a$ if there is an integer n_0 such that for all $n > n_0$ $g(x_1, ..., x_k, n) = a$.

« lim » is therefore a functional operator which associates to each total function g of $k + 1$ variables a partial function of k variables $f(x_1, ..., x_k)$ such that :

$$f(x_1, ..., x_k) = \begin{cases} \lim\limits_{n} g(x_1, ..., x_k, n) & \text{if such a limit exists} \\ \text{undefined} & \text{otherwise} \end{cases}$$

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We will write $f(x_1,..., x_k) = \lim_n g(x_1,..., x_k, n)$ from now on. Whenever the limit operation gives rise to a total function, the process may be iterated so that we have :

Def. $\lim^{(k)} g(x_1, ..., x_m, n_k, ..., n_1)$

$$= \lim_{n_k} \lim_{n_{k-1}} (... \lim_{n_1} g(x_1, ..., x_m, n_k, ..., n_1) ...)$$

under the condition that each lim operation is applied to a total function.

Def. If a (partial) function is expressible as the k-th limit of a total recursive function we say that it is a (partial) k-limiting recursive function.

Def. A set is k-limiting recursively enumerable if it is the domain of a partial k-limiting recursive function.

A set is k-limiting recursive if its characteristic function is k-limiting recursive.

The above definitions extend similar definitions of Gold [1] and practically coincide with those of Schubert [7]. Gold [1] and Putnam [8] proved that the Δ_2 sets are exactly the 1-limiting recursive sets. We have in general :

Theorem 1. For all k the Δ_{k+1} sets coincide with the k-limiting recursive sets.

This is an immediate consequence of the following lemma. Let G_f be the graph of the function f, then we have as a consequence of Schoenfield's Limit Lemma [9] and Post's theorem :

Lemma. Let $f(x)$ be a total function; then :

$$G_f \in \Delta_{k+1} \quad \text{iff} \quad f(x) = \lim^{(k)} g(x, n_k, ..., n_1)$$

for some recursive function $g(x, n_k, ..., n_1)$.

Proof. By Schoenfield's limit lemma we have :

i) $\deg f \leqslant_T \mathbf{0}^{(m+1)}$ iff there exists a function $g(x, n)$ such that

$$\deg g \leqslant_T \mathbf{0}^{(m)} \ \& \ f(x) = \lim_n g(x, n).$$

For any total function $h, \deg h = \deg G_h$ and by Post's theorem any set recursive in a relation Σ_k is in Δ_{k+1}, hence i) becomes :

ii) G_f is in Δ_{m+2} iff there exists a function $g(x, n)$ such that G_g is in Δ_{m+1} and $\lim_n g(x, n) = f(x)$.

Then the lemma easily follows by induction on k simply observing that the basis, $k = 1$, coincides with ii) when $m = 0$, that is with Gold's [1] and Putnam's [8] result, and that for the induction step again by ii) G_f is in Δ_{k+1} iff $f(x) = \lim_n g(x, n)$ for some function g such that $G_g \in \Delta_k$; by induction, g is $(k - 1)$-limiting recursive i.e.

$g(x, n) = \lim^{(k-1)} h(x, n, n_{k-1}, \ldots, n_1)$ where h is recursive, hence $f(x) = \lim_n \lim^{(k-1)} h(x, n, n_{k-1}, \ldots, n_1)$

Now one has more generally :

Theorem 2. The class of k-limiting recursively enumerable sets coincides with the Σ_{k+1} sets.

Proof. Let S be the domain of a partial k-limiting recursive function $f(x)$ then $f(x) = \lim_n g(x, n)$ and $S = \{ x \mid \exists n_0 \, \forall n > n_0 : g(x, n) = g(x, n_0) \}$ where $g(x, n)$ is total and $(k - 1)$-limiting recursive, hence by the lemma G is in Δ_k and therefore the relation $R(x, n, n_0)$ iff $g(x, n) = g(x, n_0)$ is in Δ_k and S is in Σ_{k+1}.

Let S be a Σ_{k+1} set. Then

$$x \in S \text{ iff } \exists y \, \forall z \, R(x, y, z)$$

where $R(x, y, z)$ is in Σ_{k-1}, hence a Δ_k relation. Let, following Gold [1], $f(x, n)$ and $g(x, n)$ be so defined :

$$f(x, 0) = 0$$

$$f(x, n+1) = \begin{cases} f(x, n) + 1 & \text{if} \quad R(x, f(x, n), g(x, n)) \text{ false} \\ f(x, n) & \text{otherwise} \end{cases}$$

$$g(x, 0) = 0$$

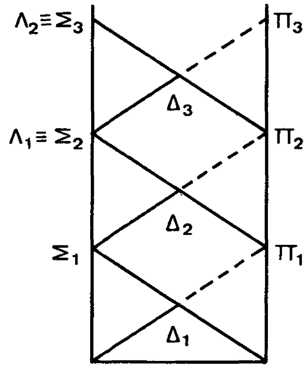
$$g(x, n+1) = \begin{cases} g(x, n) + 1 & \text{if} \quad R(x, f(x, n), g(x, n)) \text{ true} \\ 0 & \text{otherwise} \end{cases}$$

Now it is easy to convince oneself that $\lim_n f(x, n)$ is defined if and only if $\exists y \, \forall z \, R(x, y, z)$ that is iff $x \in S$ and because f is recursive in $R(x, y, z)$ and total by the lemma its graph is in Δ_k hence f is $(k - 1)$-limiting recursive and S is k-limiting r.e.

Therefore Schubert's conjecture, namely that not all Σ_{2k} sets, when $k > 1$ are k-limiting r.e., is correct and in fact such sets are in general $(2k - 1)$-limiting r.e.

Moreover, it is interesting to note that the Kleene Hierarchy, with the exception of the first level is completely characterized by the iteration of the limit operation. However, the Π sets are characterized only as being the complements of Λ sets as it is clear from the diagram where we named $\wedge_k$ the class of sets k-limiting r.e.

An interesting open problem is that of finding a characterization of the Π sets in terms of the iteration of some constructively significant operator.



Let us call arithmetic functions the functions partial recursive in some arithmetic predicates. We then introduce the following :

Def. The class of partial-limiting-recursive functions is the class of functions obtained by the application of a finite number of compositions, primitive recursions and limit operations applied to total functions starting with the zero, successor and the generalized identities functions.

(Note that « partial limiting-recursive » as used by Gold [1] translates as « partial 1-limiting recursive » in our and Schubert's [7] terminology.)

Theorem 3. A function f is partial k-limiting recursive iff it is partial recursive in some Δ_{k+1} predicates.

Proof. $\Rightarrow$ As we have already seen in the first part of the proof of Theorem 2 if f is partial k-limiting recursive then $f(x) = \lim\limits_n g(x, n)$ with G_g in Δ_k hence $f(x) = y$ iff $\exists n_0 \, \forall n : g(x, n) = y$, then G_f is a Σ_{k+1} set, hence f is partial recursive in a predicate Δ_{k+1}.

$\Leftarrow$ If f is partial recursive in some Δ_{k+1} predicates its graph G_f will be a Σ_{k+1} set and, by Theorem 2, a k-limiting r.e. set.

Thus $f(x) = y$ iff $(x, y) \in G_f$ iff

$\lim\limits_n s(x, y, n)$ is defined with s a $(k - 1)$-limiting recursive function.

We then define

$$u(x, 0) = 0$$

$$u(x, n + 1) = \begin{cases} u(x, n) & \text{if } s(x, u(x, n), n) = s(x, u(x, n), n + 1) \\ u(x, n) \dot{-} 1 & \text{if } s(x, u(x, n), n) \neq s(x, u(x, n), n + 1) \\ & \text{and } u(x, n) \neq 0 \\ n + 1 & \text{otherwise} \end{cases}$$

By the lemma u is a $(k - 1)$-limiting recursive function as well since it is recursive in s; hence $G_u \in \Delta_k$. It suffices to show that $f(x) \simeq \lim\limits_n u(x, n)$.

Now if $\lim_n u(x, n)$ is defined and is equal to a then also $\lim_n s(x, a, n)$ is defined so that $f(x) = a$. If on the other hand for a given x $\lim_n u(x, n)$ is undefined then it is easily seen that $u(x, n) = n$ for infinitely many n and u decreases in unit steps from one such n to the next one.

Therefore for each integer m the function u takes on the value m infinitely often and because m is not a limit value for $u(x, n)$, there exists an infinite sequence of integers $m_1, m_2 \ldots$ such that, for any i : $u(x, m_i) = m$ and $u(x, m_i + 1) \neq u(x, m_i)$.

This entails, by definition of u that for each i :

$$s(x, u(x, m_i), m_i) \neq s(x, u(x, m_i), m_i + 1)$$

i.e. that for all m and for infinitely many n $s(x, m, n) \neq s(x, m, n + 1)$. Therefore for each m, $\lim_n s(x, m, n)$ is not defined so that $f(x) \neq m$ for all m and $f(x)$ is not defined in x.

Theorem 4. The arithmetic functions coincide with the partial limiting recursive functions.

This is an immediate corollary of Theorem 3 and of the observation, due to Gold [1], that the limit of a recursive function can be replaced always by a limit of a primitive recursive function.

By comparing the minimalization and limit operators, one notices that, by applying just one operation of minimalization to functions total recursive in some predicates Δ_k, one obtains all the functions with graph in Σ_k, while in the case of the limit operation one gets all the functions in Σ_{k+1}. This happens because, while the minimalization operation, if it succeeds, does so in a signalled way (halting of the procedure), the eventual success of a limit operation is not similarly signalled. The effectiveness of the two operations is markedly different and the limit operation overshoots the bounds of Church' Thesis, although from the recent literature [2, 3, 4, 5, 6] one gets the feeling that its results are still effective enough to be of interest in various physicalistic interpretations of formalism : e.g. approximation, learning, identification, dialogic formal systems. What is really exploited in the latter processes is the result relative to $\Lambda_1 = \Sigma_2$ that is : one limit operation carries us from the primitive recursive domain to Σ_2 without trace of Σ_1 and Π_1.

It may be interesting to note that we can exhibit a normal form for Λ_1 functions analogous to Kleene's normal form for partial recursive functions. In a straightforward manner, using the Turing machines as computational basis, we define the primitive recursive function :

$$E(i, x, n) = \begin{cases} \langle \alpha_n \rangle & \text{if the Turing machine with index } i\text{, started on } q_1\bar{x}, \\ \varphi_i(x) & \text{has otherwise not stopped on the } n\text{-}th \text{ step} \end{cases}$$

An explicit definition of E can be obtained by use of Davis' functions and predicates [10]. It is clear that for all i

$$\lim_n E(i, x, n) \simeq \psi_i(x)$$

is a partial 1-limiting recursive function. (We use the symbol ψ_i to denote an (acceptable) enumeration of the partial 1-limiting recursive functions). It remains to show that every partial 1-limiting recursive function can be thus expressed. If $f(x)$ is partial 1-limiting recursive then :

$$f(x) = \lim_n \varphi_i(x, n)$$

where φ_i is total recursive. We now take a Turing machine Z such that

$$\Psi_Z(x, n) = \varphi_i(x, n)$$

and build a new Turing machine Z' such that :

i. converts the $1's$ of the input x into a symbol S_j not in the alphabet of Z and stores the result for subsequent use;

ii. simulates Z with S_j uniformly substituted for 1; in particular it starts computing $\varphi_i(x, 0)$;

iii. stores the result retranslated in $1's$ in an appropriate format;

iv. increases by one the second (limit) variable;

v. computes $\varphi_i(x, n)$ preserving untouched the previous result stored in $1's$;

vi. compares the new result, which will come out in $S_j's$, with the stored one without erasing or writing any $1's$;

vii. *a.* If different updates the stored result in $1's$ in the same format of iii.;

b. If equal does nothing;

viii. cycles indefinitely over iv. to vii.

Filling up the constructional details — the only care to be taken is that the # of $1's$ is never altered during the successive computations of $\varphi_i(x, n)$ — one easily convinces oneself that Z', with index i' is such that :

1. $$\forall n\, \exists t : E(i', x, t) = \varphi_i(x, n)$$

and

2. $$\lim_n \varphi_i(x, n) = a \Rightarrow \exists t_0\, \forall t\, [t_0 < t \Rightarrow E(i, x, t) = a].$$

We have thus proved the following :

Theorem 5. There exists a primitive recursive function $E(i, x, n)$ such that a function $f(x)$ is partial 1-limiting recursive if and only if for some i

$$f(x) \simeq \lim_n E(i, x, n).$$

This justifies the notation ψ_i for partial 1-limiting recursive functions.

Thus the function E substitutes with respect to the limit operator the rôle that U and T [10] played with respect to the minimalization operator.

Some interesting points arise which are perhaps worth mentioning as problems for further research.

For instance the use of time or step-count in the definition of E suggests a machine-independent formulation based on the axiomatic definition of computational complexity. We do not know if this is possible, but in working along these lines we came to feel that in limit procedures, which never stop, the step-count, or time, ceased to be perceived as complexity and that a different notion creeped in instead which is the intricacy of the path followed by the procedure in a space whose points would be all possible instantaneous descriptions. The reason may be that, possible as it is to express or simulate limit procedures with Turing machines or equivalent computational bases, the fundamental concept waiting to be formally introduced is that of a process as distinct from a procedure in that a procedure is a means to an end – the computation – while the process just happens, as physical processes do, even if some of them can obviously be used to perform computations. The main technical difference would reside of course in the different handling of the halting problem.

However, even if not yet formally introduced – although the identification procedures of [2] are very close – one has a unitary grasp of a process in the sense that each refers to a single piece of machinery whose assessment of effectivity is a matter of preference. Schubert [7] has tried to extend the concept to further levels but to do so he seems to propose as the counterpart of the intuitive notion of effective procedure the complex of computations carried on by an « expanding community of procedures » to the level Σ_3 or an « expanding community of expanding communities » to the level Σ_4 and so on. Now in a 1-limiting recursive procedure, i.e. a process, if some action depends on the limit value one can always take the last value as best guess and be certain to be wrong only finitely many times. In the 2-limiting or Σ_3 case, although one knows that only finitely many processes will be wrong other than finitely many times there is no effective way of getting at any time an ultimately correct guess of the correct value. This can be proved formally as a sort of inverse of Schoenfield's Modulus lemma [9], but one can convince oneself that if it were not so there would be a recursive pairing function which would reduce the 2-limiting procedures to the 1-limiting ones, which is impossible in view of our characterization in terms of Kleene hierarchy. The consequence is that it is markedly different on epistemological grounds to consider an individual procedure instead of a (potentially infinite) community of procedures in the limit processes because the dovetailing technique cannot be used. This is clearly not so in the plain recursive case and may furnish suggestions for a clear cut definition of what constitutes an individual system. As a concluding remark we might notice that this whole topic raises the interesting suggestion that important, apparentely rock-solid conceptions like Hilbert's finitist program and the related notion of constructivity might be more socially dependent than one's first guess, a more permissive society giving rise to more permissive standards of evidence.

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