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# A NOTE ON GRAPH COLORING

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Abstract. — *A result concerning edge colorings in graphs is extended to the case of vertex colorings. Let* $S_1, \ldots S_k$ *be a coloring of the vertices of G and let* $s_i$ *be the cardinality of* $S_i$. *It is shown that there always exists a k-coloring with*

$$|s_j - s_i| \leqslant (l - 2) \min (s_i, s_j) + 1 \quad \textit{for any} \quad i, j.$$

*where l is such that no vertex belongs to more than l maximal cliques.*

A multigraph consists of a finite nonempty set $X$ of vertices and a set $U$ of edges. A *k-edge-coloring* is a partition of $U$ into subsets $H_1$, $H_2$, ..., $H_k$ such that no two edges in the same $H_k$ are adjacent. Let $h_i$ be the cardinality of $H_i (i = 1, \ldots, k)$, We will say that the sequence $(h_1, h_2, \ldots, h_k)$ where $h_1 \geqslant h_2 \geqslant \ldots \geqslant h_k$ is *color-feasible* in $G$.

The following proposition appears in [1] and [2] :

**Proposition 1 :** If $(h_1, h_2, \ldots, h_k)$ is color-feasible in $G$, then any sequence $(h_1, h_2, \ldots, h_k)$ with :

a) $$h'_1 \geqslant \ldots \geqslant h'_k$$

b) $$\sum_{i=1}^{l} h'_i \leqslant \sum_{i=1}^{l} h_i \qquad l = 1, \ldots, k - 1$$

c) $$\sum_{i=1}^{k} h'_i = \sum_{i=1}^{k} h_i$$

is color-feasible in $G$.

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Let the *chromatic index* $q(G)$ of $G$ be the smallest $k$ for which $G$ has a $k$-edge-coloring. As a consequence of proposition 1 we have :

**Proposition 2** : For any $k \geqslant q(G)$, $G$ has a $k$-edge coloring

$$H_1, \ldots, H_k \text{ with } |h_i - h_j| \leqslant 1, \qquad i, j = 1, \ldots, k$$

In this note we will extend these results to the more general case of vertex colorings.

A *k-coloring* of $G$ is a partition of its vertices into subsets $S_1, S_2, \ldots, S_k$ of nonadjacent vertices.

(Note that whenever we are dealing with vertex colorings, we just have to consider *simple* graphs, i.e. graphs without multiple edges.)

The *chromatic number* $\gamma(G)$ of $G$ is the smallest $k$ for which $G$ has a $k$-coloring.

A *clique* $K$ in $G$ is a subset of vertices such that any two vertices in $K$ are adjacent in $G$. A clique $K$ is *maximal* if there is no clique $K'$ in $G$ which strictly contains $K$.

Given a subset $A$ of $X$, $\langle A \rangle$ will denote the subgraph spanned by $A$ : its edges are those edges of $G$ with both endpoints in $A$. The *degree* of a vertex $x$ in $G$ is the number of edges in $G$ which are adjacent to $x$.

Let $S_1, S_2, \ldots, S_k$ define a $k$-coloring of $G$ ; $s_i$ will denote the cardinality of $S_i$. We assume that no vertex in $G$ belongs to more than $l$ maximal cliques ($l \geqslant 2$). If $l = 1$, each connected component $G'$ of $G$ is a clique.

**Proposition 3** : The degrees in the subgraph $\langle S_i \cup S_j \rangle$ are at most $l$ for any $i, j$.

*Proof* : Assume a vertex $x$ in $S_i$ is adjacent to $p > l$ vertices $x_1, x_2, \ldots, x_p$ in $S_j$ ; any two of these vertices are nonadjacent, so they cannot belong to the same clique. Hence the maximal cliques $K_i$ containing $x$ and $x_i$ are distinct ($i = 1, \ldots, p$) which is a contradiction.

**Proposition 4** : Let $S'_i \subset S_i$, $S'_j \subset S_j$ define a connected component $G' = \langle S'_i \cup S'_j \rangle$ of $\langle S_i \cup S_j \rangle$ ; then $|s'_j - s'_i| \leqslant (l - 2) \min (s'_i, s'_j) + 1$.

*Proof* : Suppose $s'_i = p$ and $s'_j > (l - 1)p + 1$; since $G'$ is bipartite, it has at most $l \cdot p$ edges (no degree exceeds $l$) ; however $G'$ has more than $p + (l - 1)p + 1 = l \cdot p + 1$ vertices, hence it cannot be connected, so $s'_j \leqslant (l - 1)p + 1$ and the proposition follows.

**Proposition 5 :** Given a $k$-coloring $S_1, S_2, ..., S_k$ of a graph $G$ where no vertex belongs to more than $l$ maximal cliques, any two subsets $S_i, S_j$ with $s_j > (l-1)s_i + 1$ may be replaced by two subsets $\bar{S}_i, \bar{S}_j$ satisfying

$$|\bar{s}_j - \bar{s}_i| \leqslant (l-2) \min(\bar{s}_i, \bar{s}_j) + 1.$$

*Proof* : Let $s_j = s_i + K$ with $K > (l-2)s_i + 1$ ; then $G' = \langle S_i \cup S_j \rangle$ is not connected and there is a connected component $\langle S_i' \cup S'_j \rangle$ of $G'$ with $s_j' = s_i' + K'$ where $0 < K' \leqslant (l-2)s_i' + 1 \leqslant (l-2)s_i + 1 < K$.

By interchanging the vertices of $S_i'$ and $S_j'$ we obtain two subsets $\bar{S}_i$ and $\bar{S}_j$ of nonadjacent vertices.

They satisfy :

$$s_i = s_j - K < \bar{s}_j = s_j - K' < s_j$$

$$s_i < s_i + K' = \bar{s}_i < s_i + K = s_j$$

Hence $|\bar{s}_j - \bar{s}_i| < K = |s_j - s_i|$.

If $|\bar{s}_j - \bar{s}_i| > (l-2) \min(\bar{s}_i, \bar{s}_j) + 1 > K$ the interchange procedure may be reiterated and finally we will obtain two subsets $\bar{S}_i, \bar{S}_j$ satisfying

$$|\bar{s}_j - \bar{s}_i| \leqslant (l-2) \min(\bar{s}_i, \bar{s}_j) + 1.$$

Quite similarly to the case of edge-colorings, we say that a sequence $(s_1, s_2, ..., s_k)$ with $s_1 \geqslant s_2 \geqslant ... \geqslant s_k$ is color-feasible in $G$ if there exists a $k$-coloring $S_1, S_2, ..., S_k$ of $G$ where $S_i$ has cardinality $s_i (i = 1, ..., k)$.

According to Proposition 5, let $S = (s_1, s_2, ..., s_k)$ be color-feasible in $G$ ; if $S' = (s_1', s_2', ..., s_k')$ is any sequence obtained from $S$ by interchanges between subsets $S_i, S_j$ with $|s_j - s_i| > (l-2) \min(s_i, s_j) + 1$, then $S'$ is also color-feasible.

In particular, by making successive interchanges, we obtain :

**Proposition 6 :** Let $G$ be a graph with chromatic number $\gamma(G)$ and where no vertex belongs to more than $l$ maximal cliques ; then for any $k \geqslant \gamma(G)$, there exists a color-feasible sequence $(s_1, s_2, ..., s_k)$ with $s_1 \leqslant (l-1)s_k + 1$.

We conclude this note with a few remarks :

REMARK 1 : Proposition 6 should be related to a theorem of Hajnal and Szemerédi [3] : For any graph $G$ with maximum degree $h$, there exists a color-feasible sequence $(s_1, s_2, ..., s_{h+1})$, with $s_1 \leqslant s_{h+1} + 1$.

In other words, if $h + 1$ colors are to be used for the vertices of $G$, then it is always possible to find an $(h + 1)$ — coloring where all cardinalities of the $S_i'$s are within 1.

However if less than $h + 1$ colors may be used, then it is not always possible to do so. As an exemple consider graph $G_1$ with 4 vertices $u, v, w, x$ and 3 edges $(u, v)$, $(u, w)$, $(u, x)$ ; the only way of coloring its vertices with $2 < h + 1 = 4$ colors is $S_1 = \{ v, w, x \}$., $S_2 = \{ u \}$ and so we have $s_1 - s_2 = 3 - 1 = (l - 2)s_2 + 1 = 1 + 1 > 1$ since $u$ belongs to $l = 3$ maximal cliques.

REMARK 2 : It is well known that an edge coloring problem in $G$ may be reduced to a vertex coloring problem in a graph $G'$ whose vertices are the edges of $G$ : any two adjacent edges in $G'$ are represented by adjacent vertices in $G'$ and there exists in $G'$ a family $F$ of cliques such that :

*a*) each pair of adjacent vertices belongs to exactly one clique of $F$ ;

*b*) each vertex belongs to at most 2 cliques of $F$ ($F$ contains all maximal cliques of $G'$ which are not normal triangles (4, p. 390]).

Thus any subset $S$ of vertices in $G'$ with $|S \cap K| \leqslant 1$ for any clique $K$ of $F$ represents a subset of nonadjacent edges in $G$.

It is thus possible to consider that the only « maximal » cliques of $G'$ are those in $F$ ; so $l = 2$ and it follows from Proposition 5 that interchanges can be made between $S_i$ and $S_j$ whenever $|s_j - s_i| > 1$. This means of course that Propositions 1 and 2 are valid.

REMARK 3 : One could think of deducing the result of Hajnal and Szemerédi from Proposition 6 in the following way : if for any graph $G$ with maximum degree $h$ it is possible to introduce some edges in such a way that

*a*) the maximum degree is still $h$

*b*) each vertex belongs to at most 2 maximum cliques of the new graph, then obviously (since $\gamma(G) \leqslant h + 1$) it is possible to find an $(h + 1)$-coloring $S_1, ..., S_k$ with $s_1 - s_k \leqslant 1$.

Unfortunately, this is not true as is shown by considering graph $G_2$ with vertices $x_1, x_2, x_3, y_1, y_2, y_3$ and edges $(x_i, y_j) i, j = 1, 2, 3$ ; each vertex belongs to 3 maximal cliques and the introduction of any supplementary edge increases the maximum degree.

REMARK 4 : Finally Proposition 6 may be formulated in terms of hypergraphs (notions which are not defined here can be found in [4]) ; we want to color the edges of a hypergraph $H$ in such a way that no 2 edges $E_i$, $E_j$ with $E_i \cap E_j \neq \varnothing$ are of the same color. Now $l$ is the rank of $H$ i.e.

$$r(H) = \max_i |E_i| = l$$

and let $q(H)$ be the minimum number of colors required to color the edges of $H$ ; then for any $k \geqslant q(H)$ there exists a $k$-edge-coloring $S_1, ..., S_k$ of $H$ with $|s_i - s_j| \leqslant (r(H) - 2) \min (s_i, s_j) + 1$.

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