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THE SYSTEM OF IDEMPOTENTS OF A REGULAR SEMIGROUP

by Alfred H. CLIFFORD

K. S. S. NAMBOORIPAD [5], [6] has characterized the system E_S of idempotents of idempotents of a regular semigroup S as a "biordered set". His main purpose in doing this was to generalize to regular semigroups W. D. Munn's fundamental representation of inverse semigroups [4]. However, we shall not be concerned with this aspect of the theory in the present account.

We may also regard E_S as a partial groupoid, with the product ef $(e , f \in E_S)$ undefined if $ef \notin E_S$. Such a partial groupoid is called a "regular partial band" by G. BAIRD [1], who proposed the interesting problem of characterizing a regular partial band axiomatically. The author [2], [3] developed the matter further, and the present talk is an exposition of this work.

In § 1, a (regular) warp is defined as a partial groupoid satisfying certain axioms and it is shown that E_S is a regular warp for any regular semigroup S. Further needed properties of warps are given in § 2. Nambooripad's axioms for a biordered set are stated in § 3, and it is shown that every regular warp determines a biordered set. In § 4 a method is given for constructing all regular warps determining a given biordered set. In § 5 a method is given for completing a regular warp to a regular partial band. § 6 deals with fundamental regular warps. In the final § 7, an example is given of a regular warp which is not a regular partial band.

Let S be a regular semigroup and $\overline{S} = S/\mu$, where μ is the greatest idempotent-separating congruence on S. Then, E_S and $E_{\overline{S}}$ are isomorphic as biordered sets, but not in general as partial groupoids. Thus, the partial groupoid approach gives a finer classification of regular semigroups than does the biordered set approach. In spite of § 7, the method of § 4 shows that the notion of regular warp is a quite natural one, and the results of § 3 and 5 show that it is an adequate approximation to that of regular partial band.

1. Axioms for a warp ; the warp of a semigroup.

By a warp, we mean a partial groupoid E satisfying axioms (W_1)-(W_5) below. If $e , f \in E$, then " $\exists\, ef$ " means that the product ef of e and f is defined in E. Except when emphasis is desired, a statement like " $\exists\, ef$ and $ef = g$ " will be abbreviated to " $ef = g$ ".

(W_1) Let e , f , g be elements of E such that $\exists\, ef$ and $\exists\, fg$. If either $(ef)g$ or $e(fg)$ is defined, then so is the other, and they are equal. (We then write efg for their common value in E).

(W_2) $ee = e$ *for all* e *in* E .

(W_3) *If* $ef = e$ *or* $ef = f$, *then* $\exists\, fe$.

(W_4) *If either*

(i) $ef = f$, $eg = g$, *and* $\exists\,(fe)(ge)$, *or*

(ii) $fe = f$, $ge = g$, *and* $\exists\,(ef)(eg)$,

then, $\exists\, fg$.

Definition 1.1 - For any pair of elements e , f of E , we define the *sandwich set* $\mathbb{S}(e\,,\,f)$ of e and f to be the set of all g in E such that

(i) $ge = g = fg$, *and*

(ii) $he = h = fh (h \in E) \Longrightarrow (eg)(eh) = eh$ *and* $(hf)(gf) = hf$.

(W_5) *Let* $g \in \mathbb{S}(e\,,\,f)$. *If* ef *and* $(eg)(gf)$ *are both defined, then they are equal*.

A warp E is called *regular*, if it satisfies (R_1) and (R_2). The empty set is denoted by $\square$.

(R_1) *For every pair of elements* e , f *of* E , $\mathbb{S}(e\,,\,f) \neq \square$.

(R_2) *If* $g \in \mathbb{S}(e\,,\,f)$ *and* $\exists\,(eg)(gf)$, *then* $\exists\, ef$.

If a and b are elements of a semigroup, we write $a \perp b$, if a and b are inverse to each other, that is, $aba = a$ and $bab = b$. If S is a semigroup, E_S denotes the set of idempotents of S . E_S becomes a partial groupoid, when we define the product of two elements e and f of E to be ef , if $ef \in E_S$, and otherwise undefined.

THEOREM 1.1. - *Let* S *be a semigroup such that* $E_S \neq \square$.

(i) E_S *is a warp*.

(ii) *For* e , f in E_S , *define*

$$\mathbb{S}_1(e\,,\,f) = \{g \in E_S : \ ge = g = fg \ \text{ and } \ egf = ef\} ,$$

$$\mathbb{S}_2(e\,,\,f) = \{g \in E_S : \ ge = g = fg \ \text{ and } \ g \perp ef\} .$$

Then $\mathbb{S}_1(e\,,\,f) = \mathbb{S}_2(e\,,\,f) \subseteq \mathbb{S}(e\,,\,f)$.

(iii) *If* e , $f \in E_S$, *and* ef *is a regular element of* S , *then* $\mathbb{S}(e\,,\,f) = \mathbb{S}_1(e\,,\,f) \neq \square$, *and* (R_2) *holds the pair* $(e\,,\,f)$.

(iv) *If* S *is regular, then* E_S *is a regular warp*.

Proof. - (i) Axioms (W_1) and (W_2) are immediate. As for (W_3), if $ef = e$ then $fefe = fee = fe$, so $\exists\, fe$; similarly if $ef = f$. To show that E_S satisfies (W_4), assume $ef = f$, $eg = g$, and $\exists\,(fe)(ge)$. Then

$$fge = fege = (fege)(fege) = fgfge .$$

Since $geg = gg = g$,

$$fg = fgeg = fgfgeg = fgfg .$$

Thus, $\exists\, fg$. The proof if $fe = f$, $ge = g$, and $\exists\,(ef)(eg)$, is dual. We defer the proof of (W_5) until we have proved (ii) and (iii).

(ii) Let e , f , g be elements of E such that $ge = g = fg$. Then

$$g(ef)g = (ge)(fg) = gg = g ,$$

$$(ef)g(ef) = e(fge)f = egf .$$

Hence $g \perp ef$ if, and only if, $egf = ef$, showing that $\mathbb{S}_1(e , f) = \mathbb{S}_2(e , f)$. Let $g \in \mathbb{S}_1(e , f)$, and let h be an element of E_S satisfying $he = h = fh$. Then

$$(eg)(eh) = egh = egfh = efh = eh ,$$

$$(hf)(gf) = hgf = hegf = hef = hf .$$

Hence $g \in \mathbb{S}(e , f)$, so $\mathbb{S}_1(e , f) \subseteq \mathbb{S}(e , f)$.

(iii) Since ef is regular, it has an inverse a in S : $aefa = a$ and $efaef = ef$. Let $h = fae$. Then $hh = f(aefa)e = fae = h$, so $h \in E_S$. Clearly $he = h = fh$. Since $ehf = efaef = ef$, it follows that $h \in \mathbb{S}_1(e , f)$, so $\mathbb{S}_1(e , f) \neq \Box$.

To show that $\mathbb{S}(e , f) \subseteq \mathbb{S}_1(e , f)$, let $g \in \mathbb{S}(e , f)$. From $he = h = fh$ and $g \in \mathbb{S}(e , f)$, and the definition of $\mathbb{S}(e , f)$, we conclude that $(eg)(eh) = eh$. Using this and $ehf = ef$, we have

$$egf = egef = egehf = ef .$$

Hence $g \in \mathbb{S}_1(e , f)$.

To show that (R_2) holds for the pair (e , f) , let $g \in \mathbb{S}(e , f)$, and assume $\exists\,(eg)(gf)$, i. e., $egf \in E_S$. Since $\mathbb{S}(e , f) = \mathbb{S}_1(e , f)$, $egf = ef$, and hence $ef \in E_S$.

Having concluded the proof of (ii) and (iii), we return to the proof of (W_5). Let $e , f \in E_S$ and $g \in \mathbb{S}(e , f)$. Assume that $\exists\, ef$ and $\exists\,(eg)(gf)$. But then $ef \in E_S$, and, in particular, ef is regular. By (iii), $g \in \mathbb{S}_1(e , f)$, and so $(eg)(gf) = egf = ef$. This concludes the proof of (i), and (iv) is immediate from (iii).

2. Some properties of warps.

Throughout this section, E denotes a warp, and the letters e , f , g , h , i , j denote arbitrary elements of E . Since the axioms for a warp are all left-right self-dual, the dual of any true proposition is also true, and in general will not be stated. The dual of proposition n will be called proposition n^* . Except in corollary 2.8, we use only axioms (W_1)-(W_4).

PROPOSITION 2.1. - $ef = f$ <u>and</u> $\exists\, fg \Longrightarrow e(fg) = fg$.

<u>Proof</u>. - The hypotheses imply that ef , fg , and $(ef)g$ are all defined. By (W_1), $e(fg) = (ef)g = fg$.

We define the relations ω^r and ω^ℓ on E as follows

$$(2.1)\qquad \begin{aligned} e\ \omega^r\ f &\Longleftrightarrow fe = e\ , \\ e\ \omega^\ell\ f &\Longleftrightarrow ef = e\ . \end{aligned}$$

Furthermore, we define $\omega = \omega^r \cap \omega^\ell$, $\mathcal{R} = \omega^r \cap (\omega^r)^{-1}$, and $\mathcal{L} = \omega^\ell \cap (\omega^\ell)^{-1}$.

We let $\omega^r(e) = \{f \in E : f\omega^r e\}$, and similarly for $\omega^\ell(e)$ and $\omega(e)$.

By proposition 2.1, ω^r and ω^ℓ are quasi-orders on E (reflexive, transitive relations), and thus $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations. It is immediate from (2.1) that

$$(2.2)\qquad e\ \omega^r\ f \ \text{ and } \ f\ \omega^\ell\ e \Longrightarrow e = f\ .$$

In particular, ω is anti-symmetric, hence a partial order on E . When $E = E_S$, $\mathcal{R}$ and $\mathcal{L}$ are just Green's relations restricted to E_S , and ω is the usual partial order $\leqslant$ on E_S . Denoting by $\mathcal{R}_e$ the $\mathcal{R}$-class containing e , and defining $R_e \leqslant R_f \Longleftrightarrow e\ \omega^r\ f$, then $\leqslant$ is the usual partial order on $\mathcal{R}$-classes.

The sandwich set $\mathcal{S}(e , f)$ of e and f is the set of all g in $\omega^\ell(e) \cap \omega^r(f)$ such that $eh\ \omega^r\ eg$ and $hf\ \omega^\ell\ gf$ for every h in $\omega^\ell(e) \cap \omega^r(f)$. The following is an immediate consequence.

PROPOSITION 2.2. - <u>If</u> $g \in \mathcal{S}(e , f)$, <u>then</u>

$$\mathcal{S}(e , f) = \{h \in \omega^\ell(e) \cap \omega^r(f) : eh\ \mathcal{R}\ eg \ \text{ and } \ hf\ \mathcal{L}\ gf\}\ .$$

A subset F of a partial groupoid E is called a <u>partial subgroupoid</u> of E if $e , f \in F$ and $\exists\, ef$ imply $ef \in F$. By a <u>subwarp</u> of a warp E , we mean a partial subgroupoid F of E such that if $e , f \in F$ then $\mathcal{S}_F(e , f) \subseteq \mathcal{S}(e , f)$, where $\mathcal{S}_F(e , f)$ denotes the sandwich set of e and f relative to F . Then (W_5) holds for F , and since (W_1)-(W_4) hold for any partial subgroupoid of a warp, it follows that a subwarp of a warp is also a warp.

PROPOSITION 2.3. - <u>For any</u> e <u>in</u> E , $\omega(e)$ <u>is a subwarp of</u> E .

<u>Proof</u>. - If $f , g \in \omega(e)$ and $\exists\, fg$, then $e(fg) = fg = (fg)e$ by proposition 2.1, so $fg \in \omega(e)$. Since

$$\omega^\ell(f) \cap \omega^r(f) \subseteq \omega^\ell(i) \cap \omega^r(i) = \omega(i)\ ,$$

it follows that

$$\mathcal{S}_{\omega(e)}(f , g) = \mathcal{S}(f , g)\ .$$

PROPOSITION 2.4. - $e\ \omega^r f \Longrightarrow ef\ \mathcal{R}\ e$ <u>and</u> $ef\ \omega\ f$.

<u>Proof</u>. – By (2.1), $fe = e$; and by (W_3), $\exists\ ef$. Also, $\exists\ e(fe)$. By (W_1), $(ef)e = e(fe) = ee = e$. Since $e(ef) = ef$ by proposition 2.1, we conclude that $ef \mathcal{R} e$. That $ef\ \omega\ f$ follows from proposition 2.1.

PROPOSITION 2.5. – <u>If</u> $e\ \omega^r\ f$ <u>and</u> $\exists\ ge$, gf , <u>then</u>, $ge\ \omega^r\ gf$. <u>Hence</u> $e \mathcal{R} f$ <u>and</u> $\exists\ ge$, gf <u>imply</u> $ge \mathcal{R} gf$.

<u>Proof</u>. – By (2.1), $fe = e$. By (W_1), $(gf)e = g(fe) = ge$. By proposition 2.1, $(gf)(ge) = (gf)[(gf)e] = (gf)e = ge$, that is $ge\ \omega^r\ gf$.

PROPOSITION 2.6. – <u>Let</u> $f , g \in \omega^r(e)$. <u>Then</u> $\exists\ fg$ <u>if, and only if,</u> $\exists\ (fe)(ge)$, <u>and if they both exist,</u> $(fe)(ge) = (fg)e$.

<u>Proof</u>. – Assume first that $\exists\ fg$. By (W_1), $fg = f(eg) = (fe)g$. By proposition 2.1, $e(fg) = fg$, so $\exists\ (fg)e$ by (W_3). By (W_1), $(fg)e = [(fe)g]e = (fe)(ge)$.

Conversely, if $\exists\ (fe)(ge)$ then $\exists\ fg$ by (W_4).

By an <u>E-square</u> we mean an array $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ of elements of E such that $e \mathcal{R} f$, $g \mathcal{R} h$, $e \mathcal{L} g$, and $f \mathcal{L} h$.

PROPOSITION 2.7. – <u>Let</u> $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ <u>be an E-square. If any one of the statements</u> $eh = f$, $fg = e$, $he = g$, $gf = h$ <u>is true, then they are all true, and the E-square is a rectangular band</u>.

<u>Proof</u>. – Of course, all horizontal and vertical products ($ef = f$, $ge = g$, etc.) hold by definition of $\mathcal{R}$ and $\mathcal{L}$. By cyclical symmetry, it suffices to show that $eh = f$ imples $fg = e$. But $e(hg) = eg = e$ and $eh = f$ imply, by (W_1), that $fg = (eh)g = e(hg) = e$.

PROPOSITION 2.8. – <u>If</u> $g \in \omega^\ell(e) \cap \omega^r(f)$ <u>and</u> $\exists\ ef$, <u>then</u> $\begin{pmatrix} g & gf \\ eg & egf \end{pmatrix}$ <u>is a rectangular band</u>.

<u>Proof</u>. – $\exists\ eg$ and $\exists\ gf$ by (W_3), and $gf \mathcal{R} g \mathcal{L} eg$ by proposition 2.1 and its dual. From $(ge)f = gf$ and $\exists\ ef$, we have $g(ef) = (ge)f = gf$. From $g\ \omega^\ell\ e$, $\exists\ gf$, ef and proposition (2.5)* we have $gf\ \omega^\ell\ ef$, and so $\exists\ (ef)(gf)$.

Since $f(gf) = gf$, $(ef)(gf) = e[f(gf)] = e(gf)$. Since $\exists\ eg$, we may write this egf . From $g \mathcal{R} gf$ and $\exists\ eg$, $e(gf)$, we have from proposition 2.5 that $eg \mathcal{R} egf$; dually, $gh \mathcal{L} egf$, so $\begin{pmatrix} g & gf \\ eg & egf \end{pmatrix}$ is an E-square. By (W_1),

$$g(egf) = (ge)(gf) = g(gf) = gf ,$$

and the square is a rectangular band, by proposition 2.7.

COROLLARY 2.9. – <u>A regular warp can be described as a partial groupoid satisfying axioms</u> (W_1)-(W_4), (R_1) <u>and</u> (R_2').

(R_2'). <u>If</u> $g \in S(e , f)$, <u>and one of</u> ef <u>and</u> $(eg)(gf)$ <u>exists, so does the other, and they are equal</u>.

Proof. - Clearly (R_2') implies (W_5) and (R_2). Conversely, (R_2') is a consequence of (W_5), (R_2), and proposition 2.7.

For each f in E we define $\tau^r(f) : \omega^r(f) \to E$ and $\tau^\ell(f) : \omega^\ell(f) \to E$ by

(2.3) $x\tau^r(f) = xf$ for all $x \in \omega^r(f)$, $x\tau^\ell(f) = fx$ for all $x \in \omega^\ell(f)$.

By proposition 2.1, $\tau^r(f)[\tau^\ell(f)]$ is a projection of $\omega^r(f)[\omega^\ell(f)]$ onto $\omega(f)$.

If $e \mathcal{R} f [e \mathcal{L} f]$, we define $\tau^r(e, f)[\tau^\ell(e, f)]$ to be the restriction of $\tau^r(f)[\tau^\ell(f)]$ to $\omega(e)$. Thus

$$\text{(2.4)}\qquad \begin{cases} x\tau^r(e, f) = xf \text{ for all } x \in \omega(e), \text{ where } e \mathcal{R} f, \\ x\tau^\ell(e, f) = fx \text{ for all } x \in \omega(e), \text{ where } e \mathcal{L} f. \end{cases}$$

If E and E' are warps, a bijection $\theta : E \to E'$ is called an isomorphism if, for all e, f in E, $\exists\, ef$ if, and only if, $\exists\, (e\varphi)(f\varphi)$, in which case $(e\varphi)(f\varphi) = (ef)\varphi$.

PROPOSITION 2.10.

(i) If $e \mathcal{R} f$ and $f \mathcal{R} g$, then $\tau^r(e, f)\tau^r(f, g) = \tau^r(e, g)$,

(ii) $\tau^r(e, e) = \varepsilon_e$, the identity transformation of $\omega(e)$,

(iii) $\tau^r(e, f)$ is an isomorphism of $\omega(e)$ onto $\omega(f)$, with inverse $\tau^r(f, e)$.

Proof.

(i) For every x in $\omega(e)$, $(xf)g = x(fg) = xg$, by (W_1),

(ii) Evident,

(iii) That $\tau^r(e, f)$ is a bijection of $\omega(e)$ onto $\omega(f)$, with inverse $\tau^r(f, e)$, is immediate from (i) and (ii). Let $x, y \in \omega(e)$. $e\, \omega^r\, f$ implies $x, y \in \omega^r(f)$. By proposition 2.6, $\exists\, xy$ if, and only if, $\exists\, (xf)(yf)$, in which case they are equal.

We call an E-square $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ τ-commutative, if the diagram

$$\text{(2.5)}\qquad \begin{array}{ccc} \omega(e) & \xrightarrow{\tau^r(e,\, f)} & \omega(f) \\ \downarrow{\scriptstyle \tau^\ell(e,\, g)} & & \downarrow{\scriptstyle \tau^\ell(f,\, h)} \\ \omega(g) & \xrightarrow[\tau^r(g,\, h)]{} & \omega(h) \end{array}$$

commutes. This notion is easily seen to be independent of which corner we begin in. As stated, it is equivalent to requiring that

(2.6) $h(xf) = (gx)h$, for all $x \in \omega(e)$.

PROPOSITION 2.11. - If an E-square is a rectangular band, it is τ-commutative.

Proof. - Assume $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ is a rectangular band, and let $x \in \omega(e)$. Then $x\, \omega^r\, f$, and $xf \in \omega(f)$ by proposition 2.4. Likewise $xf\, \omega^\ell\, f\, \omega^\ell\, h$, and $h(xf) \in \omega(h)$. From $f(xf) = xf$ we have

$$h(xf) = (gf)(xf) = g[f(xf)] = g(xf) = (gx)f = (gx)(eh) = [(gx)e]h = (gx)h \ .$$

3. The biordered set determined by a regular warp.

We begin with Nambooripad's definition [5] of a biordered set, making, however, slight changes in notation.

Let E be a set, and let ω^r and ω^ℓ be quasi-orders on E. Define

(3.1) $$\mathcal{R} = \omega^r \cap (\omega^r)^{-1} \ , \quad \mathcal{L} = \omega^\ell \cap (\omega^\ell)^{-1} \ , \quad \omega = \omega^r \cap \omega^\ell \ .$$

For each e on E, define $\omega^r(e) = \{f \in E : f \ \omega^r \ e\}$, and similarly for ω^ℓ and ω. For each e in E, let $\tau^r(e)$ and $\tau^\ell(e)$ be partial transformations of E, and let $\tau = \{\tau^r(e) : e \in E\}$ $\{\tau^\ell(e) : e \in E\}$. The system $(E, \omega^r, \omega^\ell, \tau)$ is called a <u>biordered set</u>, if axioms (B_1)-(B_5) below are satisfied, <u>together with their duals</u>. By the <u>dual</u> of a statement P involving $(E, \omega^r, \omega^\ell, \tau)$ we mean the statement P^* obtained from P by interchanging ω^r and ω^ℓ, and $\tau^r(e)$ and $\tau^\ell(e)$, for each e in E.

(B_1) For all e, f in E, $e \ \omega^r \ f$ and $f \ \omega^\ell \ e \Longrightarrow e = f$.

(B_2) For all e in E, $\tau^r(e)$ is an idempotent mapping (= projection) of $\omega^r(e)$ onto $\omega(e)$, such that

(a) $$f, g \in \tau^r(e) \text{ and } f \ \omega^\ell \ g \Longrightarrow f\tau^r(e) \ \omega^\ell \ g\tau^r(e) \ ,$$

(b) $$f \in \tau^r(e) \Longrightarrow f\tau^r(e) \ \mathcal{R} \ f \ .$$

Before stating the remaining axioms, we define the <u>basic partial binary operation on</u> E as follows. For e, f in E, the product ef is defined if, and only if, e and f are related by ω^r or ω^ℓ, and then

(3.2) $$ef = \begin{cases} e\tau^r(f) & \text{if } e \ \omega^r \ f \ , \\ e & \text{if } e \ \omega^\ell \ f \ , \\ f & \text{if } f \ \omega^r \ e \ , \\ f\tau^\ell(e) & \text{if } f \ \omega^\ell \ e \ . \end{cases}$$

We proceed to show that this definition is single-valued. From (B_2) we see that $\tau^r(e)$ induces the identity transformation on its image $\omega(e)$, so $f\tau^r(e) = f$ for all f in $\omega(e)$. In particular, $e\tau^r(e) = e$; and dually, $e\tau^\ell(e) = e$. Hence all four parts of (1.2) agree that $ee = e$.

Assume now that $e \neq f$, and that the pair (e, f) belongs to two or more of the relations ω^r, ω^ℓ, $(\omega^r)^{-1}$, $(\omega^\ell)^{-1}$. By (B_1) and the assumption $e \neq f$, the conjunctions $\omega^r \cap (\omega^\ell)^{-1}$ and $\omega^\ell \cap (\omega^r)^{-1}$ are impossible. Hence exactly one of the following must hold : $e \ \omega \ f$, $f \ \omega \ e$, $e \ \mathcal{R} \ f$, $e \ \mathcal{L} \ f$.
As remarked above, $e \ \omega \ f$ implies $e\tau^r(f) = e$, and the first two cases in (3.2) give the same value, namely $ef = e$. Dually, $f \ \omega \ e$ gives $fe = f$.
Assume $e \ \mathcal{R} \ f$, and let $g = e\tau^r(f)$. By (B_2), $g \in \omega(f)$ and also $g \ \mathcal{R} \ e$. From $g \ \mathcal{R} \ e$ and $e \ \mathcal{R} \ f$ we have $g \ \mathcal{R} \ f$. But, then $f \ \omega^r \ g$, and $g \ \omega^\ell \ f$, so $g = f$ by

(B_1). Hence the first and third cases of (3.2) give the consistent result $ef = f$. Dually, for $e \mathcal{L} f$, we find that the second and fourth cases of (1.2) give $ef = e$.

It is readily seen that the quasi-orders ω^r and ω^ℓ, and the partial transformations $\tau^r(e)$ and $\tau^\ell(e)$ can be expressed in terms of the basic product (3.2) as follows :

$$\text{(3.3)} \qquad \begin{aligned} e\ \omega^r\ f &\iff fe = e ,\\ e\ \omega^\ell\ f &\iff ef = e ,\end{aligned}$$

$$\text{(3.4)} \qquad \begin{aligned} e\tau^r(f) &= ef \quad \text{for all } e \text{ in } \omega^r(f) ,\\ e\tau^\ell(f) &= fe \quad \text{for all } e \text{ in } \omega^\ell(f) .\end{aligned}$$

In stating the remaining axioms, basic products will be used instead of the τ-mappings, but the relations ω^r and ω^ℓ will be retained. Moreover, we shall repeat (B_2), breaking it into its substatements (B_{21}), (B_{22}), (B_{23}), and similarly for the other axioms. The letters e, f, g denote arbitrary elements of E.

The sandwich set $\mathcal{S}(e, f)$ of a pair od elements e, f of E is defined to be the set of all g in $\omega^\ell(e) \cap \omega^r(f)$ such that $eh\ \omega^r\ eg$ and $hf\ \omega^\ell\ gf$ for all h in $\omega^\ell(e) \cap \omega^r(f)$.

(B_1) $e\ \omega^r\ f$ and $f\ \omega^\ell\ e \implies e = f$.

(B_{21}) $fe \in \omega(e)$ for all f in $\omega^r(e)$, and $ge = g$ for all g in $\omega(e)$.

(B_{22}) $f, g \in \omega^r(e)$ and $f\ \omega^\ell\ g \implies fe\ \omega^\ell\ ge$.

(B_{23}) $f \in \omega^r(e) \implies fe\ \mathcal{R}\ f$.

(B_{31}) $g\ \omega^r\ f\ \omega^r\ e \implies gf = (ge)f$.

(B_{32}) $f, g \in \omega^r(e)$ and $f\ \omega^\ell\ g \implies (ge)(fe) = (gf)e$.

(B_{41}) $\mathcal{S}(e, f) \neq \square$ (the empty set), for all e, f in E.

(B_{42}) $e, f \in \omega^r(g) \implies \mathcal{S}(e, f)g = \mathcal{S}(eg, fg)$.

We omit the final axiom (B_5) since NAMBOORIPAD has subsequently found that it is a consequence of the other axioms.

THEOREM 3.1. – Let E be a regular warp. Define ω^r and ω^ℓ by (2.1), and $\tau^r(f)$ and $\tau^\ell(f)$, for each f in E, by (2.3). Then $(E, \omega^r, \omega^\ell, \tau)$ is a biordered set.

Proof (with one omission). – (B_1) is immediate from (2.1). (B_{21}), (B_{22}), and (B_{23}) follow from propositions 2.1, (2.5)*, and 2.4, respectively. (B_{31}) follows from axioms (W_1) and (W_3). For $g\ \omega^r\ f\ \omega^r\ e$ implies $ef = f$ and $eg = g$, so $\exists\, ge$ and $gf = g(ef) = (ge)f$. (B_{32}) follows from proposition 2.6. (B_{41}) is the same as (R_1). we omit the rather long proof of (B_{42}), see ([3] proposition 2.10).

We call $(E, \omega^r, \omega^\ell, \tau)$ the biordered set determined by the regular warp E

4. Construction of all regular warps determining a given biordered set.

Most of the important concepts introduced for warps in § 2 are really biordered set concepts : the quasi-orders ω^r and ω^ℓ , the partial translations $\tau^r(f)$ and $\tau^\ell(f)$, and the sandwich sets $\mathbb{S}(e , f)$. The same holds for the restricted translations $\tau^r(e , f)$ and $\tau^\ell(e , f)$, both denoted by $\varepsilon(e , f)$ in [6], which play an important role in Nambooripad's construction. Proposition 2.10 and its dual hold for them ; the proof of part (i) is immediate from axiom (B_{31}). Consequently, the notion of a τ-commutative E-square is also biordered set-theoretical.

We saw in § 3 that every regular warp determines a biordered set. To every biordered set, there corresponds at least one regular warp (as we shall see), but in general more than one. For example, consider a completely simple semigroup S . The biordered set E_S is simply a rectangular array, with $\omega^r = \mathcal{R}$ and $\omega^\ell = \mathcal{L}$, and the basic products are all the horizontal and vertical products. Every E_S-square is r-commutative. Regarding E_S as a regular warp, the number of further products which exist can vary between the two extremes :

1° all of them, when, for example, S is a rectangular band,

2° none of them, when, for example, $S = \mathcal{M}(G ; I , \Lambda ; X)$, where $X = (x_{\lambda i})$, and G is the free group on the symbols $x_{\lambda i}$ $(\lambda \in \Lambda , i \in I)$.

In the present section, we begin with a biordered set E , and give a method for describing all possible (regular) warps $E(\cdot)$ which determine E . Clearly the partial binary operation $(\cdot)$ must include the basic products (3.2).

By a <u>row-singular</u> E-square we mean one of the form $\begin{pmatrix} e & f \\ ge & gf \end{pmatrix}$, where $e \mathcal{R} f$ and $e , f \in \omega^\ell(g)$. <u>Column-singular</u> is defined dually, and singular means either row- or column-singular. An E-square $\begin{pmatrix} e & f \\ e & f \end{pmatrix}$ is called <u>row-degenerate,</u> $\begin{pmatrix} e & e \\ f & f \end{pmatrix}$ is <u>column-degenerate,</u> and degenerate means either kind.

A set $\mathfrak{A}$ of τ-commutative E-square is called <u>effective</u> if it has the following three properties.

(Q_1) If $\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathfrak{A}$ and $\begin{pmatrix} g & h \\ i & j \end{pmatrix} \in \mathfrak{A}$, then $\begin{pmatrix} e & f \\ i & j \end{pmatrix} \in \mathfrak{A}$.
If $\begin{pmatrix} e & g \\ f & h \end{pmatrix} \in \mathfrak{A}$ and $\begin{pmatrix} g & i \\ h & j \end{pmatrix} \in \mathfrak{A}$, then $\begin{pmatrix} e & i \\ f & j \end{pmatrix} \in \mathfrak{A}$.

(Q_2) If $\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathfrak{A}$ and $x \in \omega(e)$, then $\begin{pmatrix} x & xf \\ gx & \bar{x} \end{pmatrix} \in \mathfrak{A}$, where $\bar{x} = h(xf) = (gx)h$. (Note (2.6))

(Q_3) $\mathfrak{A}$ contains all singular and all degenerate E-squares.

The partial binary operation $(.)$ on a biordered set E corresponding to an effective set $\mathfrak{A}$ of τ-commutative E-squares is defined as follows. Let $e , f \in E$. If, for some g in $\mathbb{S}(e , f)$ and some x in E , $\begin{pmatrix} g & gf \\ eg & x \end{pmatrix} \in \mathfrak{A}$, then we define $e . f = x$. The uniqueness of $e . f$ (if it exists) follows from proposition 2.2.

THEOREM 4.1. - <u>Let</u> E <u>be a biordered set, and let</u> $\mathfrak{A}$ <u>be an effective set of</u>

τ-commutative E-squares. Under the partial binary operation $(.)$ corresponding to $\mathfrak{A}$, $E(.)$ becomes a regular warp determining the biordered set E, and $\mathfrak{A}$ consists of those E-squares which are 2×2 rectangular bands in $E(.)$.

Conversely, if $E(.)$ is any regualr warp determining E, then the set $\mathfrak{A}$ of all E-squares which are 2×2 rectangular bands in $E(.)$ is an effective set, and $(\circ)$ coincides with the partial binary operation $(.)$ in E corresponding to $\mathfrak{A}$.

Proof of converse. – Let $E(\circ)$ be a regular warp determining E, and write ab for $a \circ b$. Let $\mathfrak{A}$ be the set of all E-squares which are 2×2 rectangular bands.

To show (Q_1), let $\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathfrak{A}$ and $\begin{pmatrix} g & h \\ i & j \end{pmatrix} \in \mathfrak{A}$.

Then, by (W_1), $ej = e(ih) = (ei)h = eh = f$, and $\begin{pmatrix} e & f \\ i & j \end{pmatrix} \in \mathfrak{A}$ by proposition 2.7. The second part of (Q_1) is proved dually.

To show (Q_2), let $\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathfrak{A}$ and $x \in \omega(e)$. Then $x \in \omega^{\ell}(g) \cap \omega^{r}(f)$. By proposition 2.8, $\begin{pmatrix} x & xf \\ gx & gxf \end{pmatrix} \in \mathfrak{A}$. From $(gx)e = gx$, and $eh = f$, we have

$$(gx)f = (gx)(eh) = [(gx)e]h = (gx)h = \overline{x},$$

so $\begin{pmatrix} x & xf \\ gx & \overline{x} \end{pmatrix} \in \mathfrak{A}$.

To show (Q), let $e \,\mathcal{R}\, f$ and $e, f \in \omega^{\ell}(g)$. Then $e(gf) = (eg)f = ef = f$, so $\begin{pmatrix} e & f \\ ge & gf \end{pmatrix} \in \mathfrak{A}$. Dually for column-singular E-squares. Trivially, $\mathfrak{A}$ contains all degenerate E-squares.

Let $e, f \in E$, and let $g \in \mathcal{S}(e, f)$. If $\exists\, ef$, then, by proposition 2.8 and $(R_2^!)$ in corollary 2.9, $\begin{pmatrix} g & gf \\ eg & ef \end{pmatrix} \in \mathfrak{A}$, and hence $ef = e \,.\, f$. Conversely, if $\exists\, e \,.\, f$, then $\begin{pmatrix} g & gf \\ eg & e \,.\, f \end{pmatrix}$ for some $g \in (e, f)$, by definition of $(.)$. Then $e \,.\, f = (eg)(gf) = ef$, by $(R_2^!)$.

For a proof of the direct part of the theorem, see ([3] p. 17-26).

5. The universal regular IG-semigroup on a regular warp.

By an IG-semigroup, we mean a semigroup which is generated by its idempotents.

Let E be a regular warp. Let $\mathfrak{F}_E$ be the free semigroup on the set E. If $a, b \in \mathfrak{F}_E$, write $a \sim b$, if we can pass from a to b by a finite sequence of elementary transitions of the following two kinds.

I. Replace two adjacent terms e, f in a word by the single term ef, if it exists, or the reverse.

II. Insert an element of $\mathcal{S}(e, f)$ between two adjacent terms e, f in a word, or the reverse.

Then $\sim$ is a congruence on $\mathfrak{F}_E$, and we define $B(E) = \mathfrak{F}_E/\sim$. It can be shown that the natural mapping of E into $B(E)$ is injective, and we shall regard E as a subset of $B(E)$.

If E and E' are biordered sets, a bijection $\theta : E \longrightarrow E'$ is called an isomorphism if it preserves ω^r , ω^ℓ , and τ (in the obvious sense) in both directions. In terms of basic products, this is equivalent to, for e , f in E , ef exists if and only if $(e\theta)(f\theta)$ exists, and then $(e\theta)(f\theta) = (ef)\theta$. If E and E' are warps, a mapping $\theta : E \longrightarrow E'$ is called a homomorphism if the existence of ef in E implies that of $(e\theta)(f\theta)$ in E' , and then $(e\theta)(f\theta) = (ef)\theta$.

THEOREM 5.1.

1° $B(E)$ <u>is a regular</u> IG-<u>semigroup with</u> $E_{B(E)} = E$ <u>as sets, and product in</u> $E_{B(E)}$ <u>extends that in</u> E .

2° <u>If</u> S <u>is any regular semigroup, and</u> θ <u>is a bijective homomorphism and biorder isomorphism of</u> E <u>onto</u> E_S , <u>then there is a unique semigroup homomorphism</u> $\tilde{\theta} : B(E) \longrightarrow S$ <u>extending</u> θ .

3° $E_{B(E)}$ <u>is the smallest partial regular band on the set</u> E <u>extending the partial binary operation on the warp</u> E .

We omit the proof, but remark that 3° is immediate from 2°, taking θ to be the inclusion of E in some regular semigroup S , identifying E with E_S . If e , f , g are elements of E such that $ef = g$ in $B(E)$, then

$$ef = (e\tilde{\theta})(f\tilde{\theta}) = (ef)\tilde{\theta} = g\tilde{\theta} = g \quad \text{in} \quad S ,$$

so that $ef = g$ in E_S . By theorem 5.1, we have a method for extending the partial product in a regular warp E in a minimal fashion to make it a partial regular band (namely, calculate $E_{B(E)}$).

An alternative construction of $B(E)$ has been given by NAMBOORIPAD in a paper not yet published.

6. Fundamental regular warp.

A regular semigroup S is called fundamental if the identity is the only congruence on S contained in Green's relation $\mathcal{H}$. A regular warp E is called <u>fundamental</u> if the converse of proposition 2.11 holds : <u>every</u> τ-<u>commutative</u> E-<u>square is a</u> rectangular band. It can be shown that a partial groupoid E is isomorphic with the warp E_S of some fundamental regular semigroup S if, and only if, it is a fundamental regular warp ([2] theorem 6.7).

If S is a regular semigroup, ρ a congruence on S contained in $\mathcal{H}$, and $\overline{S} = S/\rho$, then the mapping $e \longmapsto e\rho$ is a biorder isomorphism of the biordered set E_S onto the biordered set $E_{\overline{S}}$. It is also a bijective homomorphism of the warp E_S onto the warp $E_{\overline{S}}$. But it need not be an isomorphism. It may happen that $e, f \in E_S$, $ef \notin E_S$, but $(e\rho)(f\rho) \in E_{\overline{S}}$. For example, let S be completely simple, and take $\rho = \mathcal{H}$.

Let E be a biordered set, and let $\mathfrak{F}$ be the set of all τ-commutative E-squares (§ 4). It is easy to show that $\mathfrak{F}$ is effective. (Q_1) follows from transitivity of $\mathcal{R}$ and $\mathcal{L}$, and a standard commutative diagram argument. For (Q_2), the τ-commutativity of $\begin{pmatrix} x & xf \\ gx & \bar{x} \end{pmatrix}$ follows from the observation that if $x \;\omega\; e \;\mathcal{R}\; f$, then $\tau^r(x, xf)$ is the restriction of $\tau^r(e, f)$ to $\omega(x)$, and dually. As for (Q_3), the τ-commutativity of $\begin{pmatrix} e & f \\ ge & gf \end{pmatrix}$, where $e \;\mathcal{R}\; f$ and $e, f \in \omega^{\ell}(g)$, is equivalent to

$$(gf)(xf) = ((ge)x)(gf) \quad \text{for all} \quad x \in \omega(e) .$$

By (W_1), both sides are found to reduce to $(gx)f$.

Let $(*)$ denote the binary operation on E corresponding to $\mathfrak{F}$. From proposition 2.11 or theorem 4.1, we see that $(*)$ is an extension of every warp operation on E that corresponds to the given biorder structure on E; that is, $E(*)$ is the greatest (regular) warp determining E. Of all the warps determining E, $E(*)$ is the only one that is fundamental. Since no enlargement of $(*)$ can take place on passing from $E(*)$ to $E_{B(E)}$ (§ 5), it follows that $E(*)$ is a regular partial band.

7. A regular warp which is not a regular partial band.

Let $E = \{e_{i\lambda} \;;\; i \in I , \lambda \in \Lambda\}$ be an $I \times \Lambda$ rectangular band, with products defined by

$$e_{i\lambda} \cdot e_{j\mu} = e_{i\mu} \quad (\text{all } i , j \text{ in } I \;;\; \lambda , \mu \text{ in } \Lambda).$$

The biordered set $(E , \omega^r , \omega^{\ell} , \tau)$ determined by E can be described as follows

$$e_{i\lambda} \;\omega^r\; e_{j\mu} \iff i = j \quad (\text{so } \omega^r = \mathcal{R}) ,$$

$$e_{i\lambda} \;\omega^{\ell}\; e_{j\mu} \iff \lambda = \mu \quad (\text{so } \omega^{\ell} = \mathcal{L}) ,$$

$$e_{i\lambda} \;\tau^r(e_{i\mu}) = e_{i\mu} ,$$

$$e_{i\lambda} \;\tau^{\ell}(e_{j\lambda}) = e_{j\lambda} ,$$

$$\mathcal{S}(e_{i\lambda} , e_{j\mu}) = \{e_{j\lambda}\} .$$

The set E itself is an $I \times \Lambda$ E-array. The basic products are either horizontal $(e_{i\lambda}\, e_{i\mu} = e_{i\mu})$ or vertical $(e_{i\lambda}\, e_{j\lambda} = e_{i\lambda})$. Endowed with the basic partial binary operation, E is a regular warp which is isomorphic with the warp of idempotents E_S of the Rees matrix semigroup $S = \mathcal{M}(G ; I , \Lambda ; P)$, where G is the free group on $X = \{x_{\lambda i} \;;\; i \in I , \lambda \in \Lambda\}$, and $P = (P_{\lambda i})$ is defined by $P_{\lambda i} = x_{\lambda i}$.

Every E-square is τ-commutative, and there are no non-degenerate singular E-squares. A set $\mathfrak{A}$ of E-squares is effective if, and only if, it contains all degenerate E-squares and satisfies (Q_1). (Q_2) is trivially satisfied.

Now let, $I = \Lambda = \{1, 2, 3, 4\}$. Let $\mathfrak{A}$ consist of all degenerate E-squares and the following :

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}, \begin{pmatrix} e_{22} & e_{23} \\ e_{32} & e_{33} \end{pmatrix}, \begin{pmatrix} e_{33} & e_{34} \\ e_{43} & e_{44} \end{pmatrix}, \begin{pmatrix} e_{12} & e_{14} \\ e_{32} & e_{34} \end{pmatrix}, \begin{pmatrix} e_{21} & e_{23} \\ e_{41} & e_{43} \end{pmatrix}.$$

No two of them have a row or a column in common, so (Q_1) is vacuously satisfied, and so $\mathfrak{A}$ is an effective set of (τ-commutative) E-squares.

Let $(*)$ be the partial binary operation on E corresponding to $\mathfrak{A}$, and let $B(E)(\circ)$ be the universal regular IG-semigroup of $E(*)$. By proposition 2.8, each member of $\mathfrak{A}$ is a 2×2 rectangular band in $E(*)$. Calculating in $B(E)$, we have

$$\begin{aligned}
e_{14} \circ e_{41} &= e_{14} \circ e_{34} \circ e_{43} \circ e_{41} && \text{since } e_{14} \mathcal{L} e_{34} ,\ e_{43} \mathcal{R} e_{41} , \\
&= e_{14} \circ e_{33} \circ e_{41} && \text{since } e_{34} * e_{43} = e_{33} , \\
&= e_{14} \circ e_{32} \circ e_{23} \circ e_{41} && \text{since } e_{32} * e_{23} = e_{33} , \\
&= e_{12} \circ e_{21} && \text{since } e_{14} * e_{32} = e_{12} \\
& && \text{and } e_{23} * e_{41} = e_{21} , \\
&= e_{11} && \text{since } e_{12} * e_{21} = e_{11} .
\end{aligned}$$

But $\begin{pmatrix} e_{11} & e_{14} \\ e_{41} & e_{44} \end{pmatrix} \notin \mathfrak{A}$, so $e_{14} * e_{41}$ is undefined. Hence the bijection $\eta : E(*) \to E_B$ is not an isomorphism, and we conclude from theorem C, that $E(*)$ cannot be embedded in a regular semigroup ; i. e., $E(*)$ is not a regular partial band in the sense of Baird [1].

In the following diagram one sees the five non-degenerate members of $\mathfrak{A}$, and one sees also the missing square $\begin{pmatrix} e_{11} & e_{14} \\ e_{41} & e_{44} \end{pmatrix}$

$$\begin{tikzcd}
e_{11} \arrow[r, dash] \arrow[d, dash] & e_{12} \arrow[r, dash] & e_{13} \arrow[r, dash] & e_{14} \arrow[d, dash] \\
e_{21} \arrow[r, dash] \arrow[d, dash] & e_{22} \arrow[r, dash] \arrow[d, dash] & e_{23} \arrow[d, dash] & e_{24} \arrow[d, dash] \\
e_{31} \arrow[d, dash] & e_{32} \arrow[r, dash] & e_{33} \arrow[r, dash] \arrow[d, dash] & e_{34} \arrow[d, dash] \\
e_{41} \arrow[r, dash] & e_{42} \arrow[r, dash] & e_{43} \arrow[r, dash] & e_{44}
\end{tikzcd}$$

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