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Groupe d'étude d'algèbre, tome 1 (1975-1976), exp. nº 6, p. 1-8
<http://www.numdam.org/item?id=GEA_1975-1976__1__A6_0>

Groupe d'étude d'ALGÈBRE
(Marie-Paule MALLIAVIN)
1re année, 1975/76, n° 6, 8 p.

3 mai 1976

PARTITION IDEALS OF ORDER 1, THE ROGERS-RAMANUJAN IDENTITIES AND COMPUTERS

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1. Introduction.

The Rogers-Ramanujan identities are two of the most surprising results in the theory of partitions. They may be stated as follows (see [5], p. 175-176).

THEOREM 1. - Let $A_{2,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, i, -i \pmod 5$. Let $B_{2,i}(n)$ denote the number of partitions of n into parts where 1 appears at most $i-1$ times and the difference between any two parts is at least 2. Then for $i = 2$ or 1, $A_{2,i}(n) = B_{2,i}(n)$.

For example, when $n = 12$, $i = 1$, $B_{2,1}(12)$ enumerates the six partitions : 12, $10 + 2$, $9 + 3$, $8 + 4$, $7 + 5$, $6 + 4 + 2$, while $A_{2,1}(12)$ enumerates the six partitions 12, $8 + 2 + 2$, $7 + 3 + 2$, $3 + 3 + 3 + 3$, $3 + 3 + 2 + 2 + 2$, $2 + 2 + 2 + 2 + 2 + 2$.

Recently, the theory of such identities has been studied in a lattice-theoretic framework [6], and partition functions such as $A_{2,i}(n)$ have been associated with "partition ideals of order 1".

In section 2, we shall present a short survey of the theory of partition ideals. In section 3, we shall associate with each partition ideal C a certain sequence $a_n(C)$ that will be of great use to us in section 4, where we describe computer searches for identities of the Rogers-Ramanujan type.

In section 5, we present two partition theorems, that we have discovered from such computer searches.

2. Partition ideals.

We require a number of definitions which we shall always associate with the related intuitive concept about partitions.

Definition 1. - We let S denote the set of all sequences

$$\{f_1 , f_2 , f_3 , \ldots\} = \{f_i\}_{i=1}^{\infty} = \{f_i\}$$

of nonnegative integers only finitely many of which are nonzero.

Explanation 1. - The elements of S correspond to the intuitive concept of partitions by the stipulation that f_i denotes the number of times i appears as a

(*) Sponsored by United States Army under Contract N° DAAG29-75-C-0024.

part. Thus, $\{1, 2, 1, 0, 3, 0, 0, 0, 0, \ldots\}$ corresponds to 1+2+2+3+5+5+5.

It is a straight-forward exercise to show that S is a distributive lattice under the partial ordering $\{f_i\} \leq \{g_i\}$ which means $f_i \leq g_i$ for all $i \geq 1$. Furthermore, the mapping $\sigma : S \to \underset{\sim}{N}$ given by $\sigma(\{f_i\}) = \sum_{i\geq 1} f_i \cdot i$ is a positive valuation on S ([9], p. 230).

Explanation 2. - The positive valuation σ maps each element of S onto the number that is being partitioned. For example,

$$\begin{aligned}\sigma(\{1, 2, 0, 1, 0, 0, 0, \ldots\}) &= 1.1 + 2.2 + 1.4 \\ &= 1 + 2 + 2 + 4 \\ &= 9 .\end{aligned}$$

Definition 2. - The semi-ideals of S are called partition ideals.

Recall [9], p. 56 that a semi-ideal C of a lattice L is a subset of L such that, if $x \in C$ and $y \leq x$, then $y \in C$.

Definition 3. - If C is a partition ideal, we define $p(C, n)$ to be the cardinality of $\{\pi ; \pi \in C, \sigma(\pi) = n\}$.

We call $p(C, n)$ the C-partition function.

Explanation 3. - $p(C, n)$ is the number of partitions of n whose associated $\{f_i\}$ sequence is in C.

Definition 4. - We say that two partition ideals C_1 and C_2 are equivalent, and we write $C_1 \sim C_2$ if $p(C_1, n) = p(C_2, n)$ for all $n \geq 0$.

Example 1. - Let $\mathcal{B}_{2,i} = \{\{f_j\} ; f_1 \leq i - 1$ and $f_j + f_{j+1} \leq 1$ for all $j\}$ and let $\mathcal{A}_{2,i} = \{\{f_j\} ; f_j > 0$ implies $j \not\equiv 0, \pm i \pmod 5\}$. Then theorem 1 asserts

$$\mathcal{B}_{2,2} \sim \mathcal{A}_{2,2} \quad \text{and} \quad \mathcal{B}_{2,1} \sim \mathcal{A}_{2,1} .$$

We note that there is a certain "local" property of some partition ideals ; intuitively, we may determine whether $\{f_i\}$ is in $\mathcal{A}_{2,2}$ or not by examining each f_i alone for $i \geq 1$ and asking whether $f_i = 0$ whenever $i \equiv 0, \pm 2 \pmod 5$. On the other hand, we must examine pairs f_i, f_{i+1} to check whether $\{f_i\} \in \mathcal{B}_{2,2}$. This difference is empressed in the following definition of the order of a partition ideal.

Definition 5. - If C is a partition ideal, then C has order k provided k is the least integer such that whenever $\{f_i\} \notin C$, then there exists m such that $\{f'_i\} \notin C$ where

$$f'_i = \begin{cases} f_i, & i = m, m+1, \ldots, m+k-1, \\ 0, & \text{otherwise.} \end{cases}$$

Given the above setting, we may ask several fundamental questions about the equivalence classes of partition ideals produced by the equivalence in definition 4 (see [6] § 2). The one of concern to us here is the following [6] p. 20 :

Second fundamental problem. - Fully characterize those equivalence classes of partition ideals modulo $\sim$ that contain a partition ideal of order 1.

3. Partition ideals of order 1.

The following theorem are proved in detail in [6] p. 20-23. They form the central core for our computer search described in section 4.

THEOREM 2. - *Let* C *be a partition ideal of order 1. Let* $\gamma_\ell = \sup_{\{f_i\}\in C} f_\ell$ (γ_ℓ *is thus a nonzero integer or* $+\infty$). *Then for* $|q| < 1$

$$\sum_{n=0} p(C\ ,\ n)q^n = \left(\prod_{\ell=1,d_\ell<\infty}^{\infty} (1 - q^{\ell(d_\ell+1)})\right)/\left(\prod_{\ell=1}^{\infty} (1 - q^\ell)\right) = \prod_{n=1}^{\infty} (1 - q^n)^{a_n(C)-1} ,$$

where $a_n(C)$ is the number of solutions j of the equation $j(d_j + 1) = n$.

THEOREM 3. - *The ideals in* S *(semi-ideals closed under union) are the partition ideals of order* 1.

THEOREM 4. - *Suppose* C *and* C' *are two partition ideals of order* 1, *then* $C \sim C'$ *if, and only if,* $a_n(C) = a_n(C')$ *for all* n .

The reader may question the significance of theorem 4. At first glance one assumes that checking $p(C\ ,\ n) = p(C'\ ,\ n)$ for all n should be no more or less difficult than checking $a_n(C) = a_n(C')$ for all n .
The following example should point up the difference.

Example 2. - Let $\mathcal{Q} = \{\{f_i\} : f_i \leqslant 1 \text{ for all } i\}$, $\mathcal{O} = \{\{f_i\} : f_i = 0$ if i is even$\}$. A famous elementary theorem of Euler [5] p. 154, asserts that $\mathcal{O} \sim \mathcal{Q}$. Full elevation of $p(\mathcal{Q}\ ,\ n)$ and $p(\mathcal{O}\ ,\ n)$ is tedious even for small n since $p(\mathcal{O}\ ,\ 3) = p(\mathcal{Q}\ ,\ 3) = 2$, $p(\mathcal{O}\ ,\ 50) = p(\mathcal{Q}\ ,\ 50) = 3658$, $p(\mathcal{O}\ ,\ 100) = p(\mathcal{Q}\ ,\ 100) = 444793$. On the other hand, by the definition $a_n(\mathcal{O})$ we see immediately that $a_n(\mathcal{O}) = 1$ if n is even, and 0 if n is odd, while $a_n(\mathcal{Q})$ also equals 1 if n is even, and 0 if n is odd ; hence Euler's theorem is immediate from theorem 4.

4. Computer searches for equivalent partition ideals.

To attack the second fundamental problem described in section 2, we begin with a simple observation that has its origins in the work of Euler, but has been explicitly stated only recently by BENDER and KNUTH [8] p. 41. Namely, for any sequence of integers, b_1 , b_2 , ... , there corresponds a sequence, a_1 , a_2 , ... , such that

$$1 + \sum_{n=1}^{\infty} b_n q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-a_n} .$$

The existence and uniqueness of the a_n is clear : $a_1 = b_1$, and given a_n for $n < k$, we determine a_k as that integer that is the coefficient of q^k in $(1 + \sum_{n=1}^{\infty} b_n q^n)\prod_{n<k} (1 - q^n)^{a_n}$. Using the logarithmic derivative, we see immediately that

$$nb_n = \sum_{j=1}^{n} b_{n-j} D_j \tag{4.1}$$

where

$$D_j = \sum_{d|j} da_d . \tag{4.2}$$

Now, we may utilize these observations by noting first that <u>every</u> nonempty partition ideal C has associated with it a unique sequence of integers $a_n(C)$ such that

$$1 + \sum_{n=1}^{\infty} p(C , n)q^n = \prod_{n=1}^{\infty} (1 - q^n)^{a_n(C)-1} . \tag{4.3}$$

Furthermore theorem 4 may be extended to <u>all</u> nonempty partition ideals. Also since a computer can generally easily compute $p(C , n)$ for small n (say $1 \leqslant n \leqslant 30$) we may use (4.1) and (4.2) to compute $a_n(C)$ for small n for arbitrary partition ideals. Now, when C is of order 1, the $a_n(C)$ must satisfy the rather restrictive conditions set forth in the following theorem. Consequently for many C' with order > 1 , we can show that $C' \not\sim C$ (C of order 1) by a simple examination of a few values of $a_n(C')$. More important, the first few $a_n(C')$ will often suggest an equivalent partition ideal C of order 1 if such exists.

THEOREM 5. - <u>If</u> C <u>is a partition ideal of order</u> 1, <u>then for all</u> $n > 0$

(i) $a_n(C) \geqslant 0$,

(ii) $a_n(C) \leqslant d(n)$ <u>the number of divisors of</u> n , <u>and</u>

(iii) $\sum_{1\leqslant j\leqslant n} a_j(C) \leqslant n$.

<u>Proof</u>. - All of these facts are obvious from the definition that $a_n(C)$ is the number of solutions of the equation $j(d_j + 1) = n$ where $d_j = \sup_{\{f_i\}\in C} f_j$.

Now let us examine some results from the computer search. We first considered the possibility of generalizing work of B. GORDON [11].

THEOREM 6 (Gordon's theorem). - <u>If</u>

$$\mathcal{A}_{k,a} = \{\{f_i\} ; \text{ for all } i \geqslant 1 , f_i = 0 \text{ if } i \equiv 0 , \pm a \pmod{2k + 1}\} ,$$

and

$$\mathcal{B}_{k,a} = \{\{f_i\} ; f_1 \leqslant a - 1 , \text{ and for all } i \geqslant 1 , f_i + f_{i+1} \leqslant k - 1\} ,$$

then (for $1 \leqslant a \leqslant k$)

$$\mathcal{A}_{k,a} \sim \mathcal{B}_{k,a} .$$

Let us look at partition ideals (provided $h_0 \geqslant h_1 \geqslant h_2 \geqslant \dots \geqslant h_{k-1}$)

$\mathcal{B}_k(h_0 , h_1 , \dots , h_{k-1})$

$= \{\{f_i\} ;$ for all $i \geqslant 1$, $f_i \leqslant k - 1$, and if $f_i = j$ then $f_{i+1} \leqslant h_j\}$.

Note that $\mathcal{B}_k(k - 1 , k - 2 , \dots , 1 , 0) = \mathcal{B}_{k,k}$. Technically the $\mathcal{B}_k(h_0 , h_1 , \dots , h_{k-1})$ constitute all those linked partition ideals with all spans and modulus equal to 1 [7], § 4. The following table lists several small values for k and $(h_0 , \dots , h_{k-1})$ and the associated first sixteen

$$a_n(\mathcal{B}_k(h_0 , k_1 , \dots , h_{k-1})) = a_n .$$

k	$h_0,h_1,\dots,h_{k-1}$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
3	2 0 0	0	0	2	0	0	1	0	0	2	0	0	1	0	0	2	0
3	2 1 0	0	0	1	1	0	0	1	0	0	1	1	0	0	1	0	0
3	2 1 1	0	0	1	0	1	1	−2	2	2	−2	1	−1	0	8	−4	−6
3	2 2 0	0	0	1	1	−1	1	2	−2	−1	5	0	−4	3	2	−6	4
3	2 2 1	0	0	1	0	0	2	−1	−1	3	0	−1	2	−3	0	8	−2
3	2 2 2	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
4	3 0 0 0	0	0	1	1	1	−2	2	2	−2	0	1	4	−1	−6	9	−2
4	3 1 0 0	0	0	0	2	1	−2	1	2	1	−5	0	10	2	−13	−5	22
4	3 1 1 0	0	0	0	1	2	−1	−1	2	2	−1	−5	2	10	1	−11	−10
4	3 1 1 1	0	0	0	1	1	0	0	1	0	0	0	1	1	0	0	−1
4	3 2 0 0	0	0	0	2	0	−1	2	1	−2	−2	5	5	−7	−7	13	7
4	3 2 1 0	0	0	0	1	1	0	0	0	1	0	0	0	1	1	0	0
4	3 2 1 1	0	0	0	1	0	1	1	−1	−1	2	2	−1	−2	1	1	1
4	3 2 2 0	0	0	0	1	1	−1	1	1	0	−2	0	5	1	−5	−1	4
4	3 2 2 1	0	0	0	1	0	0	2	0	−2	0	3	2	−3	−1	3	−3
4	3 2 2 2	0	0	0	1	0	0	1	1	−1	−1	2	2	−1	−2	3	0
4	3 3 0 0	0	0	0	2	0	−1	1	2	−1	−2	1	6	−1	−8	4	10
4	3 3 1 0	0	0	0	1	1	0	−1	1	2	0	−3	−1	5	5	−4	−7
4	3 3 1 1	0	0	0	1	0	1	0	0	0	1	0	0	0	1	0	1
4	3 3 2 0	0	0	0	1	1	−1	0	2	1	−2	−3	4	6	−3	−6	0
4	3 3 2 1	0	0	0	1	0	0	1	1	−1	−1	1	3	0	−3	1	1
4	3 3 2 2	0	0	0	1	0	0	0	2	0	−2	0	3	1	−2	0	3
4	3 3 3 0	0	0	0	1	1	−1	0	1	2	−1	−3	2	4	1	−5	−2
4	3 3 3 1	0	0	0	1	0	0	1	0	0	0	0	2	0	−1	−1	2
4	3 3 3 2	0	0	0	1	0	0	0	1	1	−1	−1	2	1	−1	0	2
4	3 3 3 3	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1
6	5 5 1 1 1 1	0	0	0	0	0	2	0	0	0	0	0	1	0	0	0	0

The above table provides only a few of the many cases treated by the computer. The cases of obvious interest are the first, second, sixth, twelfth, nineteenth, twenty sixth and twenty seventh lines. All other lines have negative entries for some a_n , and so by theorem 5 the related partition ideal in not equivalent to any partition ideal of order 1.

The tenth line, however, is intriguing especially when we remark that the $a_{16} , \dots , a_{30}$ are 2, 1, −1, 2, −2, 1, 2, −1, 2, −2, 0, 2, −2, 4. This suggests that there may be some relationship of interest between two partition ideals one "approximately" $\mathcal{B}_4(3 , 1 , 1 , 1)$ and one "approximately"

$$\{\{f_i\} ;\ f_i > 0 \text{ implies } i \equiv 0 , 4 , 5 \pmod 8\} .$$

Line one corresponds to a valid partition identity found in [4]. Lines two and twelve are from theorem 6 in the cases $k = a = 3$, 4 . Lines six and twenty six are from an elementary partition identity due to J. W. L. GLAISHER [10]. Line nineteen corresponds to a result proved in [1]. The first time that our search produced an unknown theorem was the case $k = 6$ $h_0 = h_1 = 5$, $h_2 = h_3 = h_4 = h_5 = 1$. We continue our discussion in section 5, where we prove a general partition identity which includes this result from line twenty seven as essentially a special case.

5. A new partition identity.

To illustrate some of the methods available to us in proving partition theorems, we prove a general family of partition identities related to line twenty seven of the table in section 4.

THEOREM 7. - Let $0 \leqslant a < k$, $\lambda \geqslant 1$ all be integers. Let ${}_\lambda A_{k,a}(n)$ denote the number of partitions of n into parts $\not\equiv 0$, $\pm \lambda(2a+1)$ $(\mathrm{mod}\ \lambda(4k+2))$. Let ${}_\lambda B_{k,a}(n)$ denote the number of partitions of n considered as $\{f_i\} \in S$ such that $f_1 \leqslant 2a\lambda + \lambda - 1$, and for each i , if $f_i = 2\lambda h - \lambda + r$, with $0 \leqslant r < 2\lambda$, then $f_{i+1} \leqslant 2\lambda(k-h) + \lambda - 1$. Then ${}_\lambda A_{k,a}(n) = {}_\lambda B_{k,a}(n)$ for all n .

Remarks. - When $\lambda = 1$, theorem 7 reduces to the main theorem in [2] ; the similarity of the general proof to the case $\lambda = 1$ allows us to be somewhat sketchy in the following. Actually line twenty seven in our table is not covered by theorem 7 ; however, the result

$$1 + \sum_{n\geqslant 1} p(\mathcal{B}_6(5,5,1,1,1,1)\ ,\ n)q^n = \prod_{n=1}^{\infty} ((1 - q^{\ell 2n-6})^2(1 - q^{\ell 2n}))/(1 - q^n) \ ,$$

may be easily deduced from the following proof.

Proof. - We begin by studying ${}_\lambda b_{k,a}(m\ ,\ n)$, the number of partitions enumerated by ${}_\lambda B_{k,a}(n)$ with exactly m parts.

First we recall that the empty partition of zero is the only partition of a nonpositive integer and is also the only partition with a nonpositive number of parts. Therefore for $0 \leqslant a \leqslant k$

$$(5.1)\qquad {}_\lambda b_{k,a}(m\ ,\ n) = \begin{cases} 1 & \text{if } m = n = 0\ , \\ 0 & \text{if } m \leqslant 0\ ,\ n \leqslant 0\ , \text{ but } (m\ ,\ n) \neq (0\ ,\ 0)\ . \end{cases}$$

Next we observe that ${}_\lambda b_{k,0}(m\ ,\ n)$ counts those partitions in question that at most $\lambda - 1$ ones (i. e. $f_1 \leqslant \lambda - 1$). Thus, in such partitions $f_2 \leqslant 2\lambda k + \lambda - 1$ times. If we transform the partitions enumerated by ${}_\lambda b_{k,a}(m\ ,\ n)$ by deleting all the ones that appear and subtracting 1 from each of the remaining parts, we see that such a transformation establishes a bijection between certain finite sets of partitions, and the cardinalities of the sets in question show that

$$(5.2)\qquad {}_\lambda b_{k,0}(m\ ,\ n) = {}_\lambda b_{k,k}(m\ ,\ n-m) + {}_\lambda b_{k,k}(m-1\ ,\ n-m) + \dots + {}_\lambda b_{k,k}(m-\lambda+1\ ,\ n-m)\ .$$

Following the reasoning of the preceding paragraph, we may observe that for $0 < a \leqslant \lambda$, ${}_\lambda p_{k,a}(m , n) - {}_\lambda p_{k,a-1}(m , n)$ counts those partitions for which $(2a - 1)\lambda \leqslant f_1 < (2a + 1)\lambda$, in which case $f_2 \leqslant 2\lambda(k - a) + \lambda - 1$. The same bijection used above shows that

$$(5.3)\quad {}_\lambda b_{k,a}(m , n) - {}_\lambda b_{k,a-1}(m , n)$$
$$= {}_\lambda b_{k,k-a}(m - (2a - 1)\lambda , n - m) + {}_\lambda b_{k,k-a}(m - (2a - 1)\lambda - 1 , n - m)$$
$$+ \dots + {}_\lambda b_{k,k-a}(m - (2a - 1)\lambda - 2\lambda + 1 , n - m) .$$

By a double mathematical induction (first on n, then on a) we see that (5.1), (5.2) and (5.3) uniquely determine the ${}_\lambda b_{k,a}(m , n)$.

Now let us define

$$(5.4)\quad {}_\lambda R_{k,a}(x;q)$$
$$= \prod_{m=1}^{\infty} (1-xq^m)^{-1} \sum_{r=0}^{\infty} (-1)^r x^{2\lambda kr} q^{\lambda(2k+1)r(r+1)-(2a+1)r\lambda}(1-x^{\lambda(2a+1)} q^{\lambda(2r+1)(2a+1)})$$
$$\times \frac{(1-x^{2\lambda}q^{2\lambda})(1-x^{2\lambda}q^{4\lambda})\dots(1-x^{2\lambda}q^{2r\lambda})}{(1-q^{2\lambda})(1-q^{4\lambda})\dots(1-q^{2r\lambda})} .$$

Then straightforward techniques developed in [2] show that for $0 \leqslant a \leqslant k$

$$(5.5)\quad {}_\lambda R_{k,a}(0 ; q) = {}_\lambda R_{k,a}(x ; 0) = 1$$

$$(5.6)\quad {}_\lambda R_{k,a}(x ; q) = (1 + xq + x^2 q^2 + \dots + x^{\lambda-1} q^{\lambda-1}) {}_\lambda R_{k,k}(x , q)$$

$$(5.7)\quad {}_\lambda R_{k,a}(x ; q) - {}_\lambda R_{k,a-1}(x ; q)$$
$$= x^{(2a-1)\lambda} q^{(2a-1)\lambda} (1 + xq + x^2 q^2 + \dots + x^{2\lambda-1} q^{2\lambda-1})$$
$$\times {}_\lambda R_{k,k-a}(xq ; q) .$$

We now deduce from (5.5), (5.6) and (5.7) the identities satisfied by the coefficients of $x^m q^n$, and we find that these identities are just (5.1), (5.2) and (5.3) ; hence by the uniqueness of the solutions of (5.1), (5.2) and (5.3), it follows that

$$(5.8)\quad {}_\lambda R_{k,a}(x ; q) = \sum_{m,n\geqslant 0} {}_\lambda b_{k,a}(m , n)x^m q^n .$$

Finally for $0 \leqslant a < k$

$$(5.9)\quad \sum_{n\geqslant 0} {}_\lambda B_{k,a}(n)q^n = \sum_{n,m\geqslant 0} {}_\lambda b_{k,a}(m , n)q^n = {}_\lambda R_{k,a}(1 ; q)$$
$$= \Big(\sum_{r=0}^{\infty} (- 1)^r q^{\lambda(2k+1)r(r+1)-(2a+1)r\lambda}(1 - q^{\lambda(2r+1)(2a+1)})\Big)\Big/\prod_{m=1}^{\infty} (1 - q^m)$$
$$= \prod_{n=1,n\not\equiv 0,\pm\lambda(2a+1)(\mathrm{mod}\ \lambda(4k+2)}^{\infty} (1 - q^n)^{-1} = \sum_{n\geqslant 0} {}_\lambda A_{k,a}(n)q^n ,$$

(by [5] p. 169-170) and by comparing coefficients of q^n in the extremes of (5.9) we establish theorem 7.

Another theorem arising from our computer search is as follows :

THEOREM 8. - Let $k \geq 2$. Let $\mathfrak{a}_k(n)$ denote the number of partitions of n into parts $\equiv 0, -1 \pmod{k}$ but $\not\equiv k(k-1) \pmod{k^2}$. Let $\mathfrak{B}_k(n)$ denote the number of partitions of n of the form $n = b_1 + \ldots + b_s$, where $b_i - b_{i+1} \geq j(k-1)$ (assume $b_{s+1} = 0$) with j the least nonnegative residue of $-b_i$ modulo k. Then $\mathfrak{a}_k(n) = \mathfrak{B}_k(n)$ for all n.

The methods of [3] suffice to prove this result.

6. Conclusion

In this short survey we have tried to show how some of the lattice theoretic development of partition identities has been implemented through computers to search for identities of this type. We hope in the future to study this method further so that we may discover new partition identities that require genuinely new methods for their establishment.

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