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Groupe d'Etude d'Analyse Ultramétrique
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DIMENSION OF THE SPACE OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $y' = \omega y$

by

Alain ESCASSUT*

Let K be an algebraically closed field of characteirstic 0 provided with an ultrametric absolue value $|.|$. For all set D in K we will denote by R(D) the K-algebra of the rational function $h(x) \in K(x)$ with no pole in D. When D is closed and bounded, the algebra R(D) is provided with the norm of uniform convergence on D denoted by $\|.\|_D$[E_1] that makes it a normed K-algebra. Its completion for that norm is then a K-Banach algebra denoted by H(D), the elements of which are called *the analytic elements on* D[K_5,A,E_1,E_2].

A set D is said to be *infraconnected* if for all $a \in D$, the adherance of the set $\{|x-a| \mid a \in D\}$ in $\mathbb{R}$ is an interval. We know that *a bounded closed set* D *is infraconnected if and only if* H(D) *does not have non trivial idempotent* [E_2].

In Chapter I we will prove the analytic elements with null derivative on a clopen infraconnected set is a constant and more generally when the derivative of an analytic element is an analytic element, we obtain the Mittag-Leffler series of the derivative in deriving the Mittag-Leffler series of the considered analytic element.

In chapter II we will study the dimension of the space of solutions of the differential equation $(\mathcal{E})y' = fy$ with $f,y \in H(D)$, D a clopen bounded infraconnected set. We will prove a solution is either invertible in H(D) or strictly annulled by a T-filter on D [E_3]. If $\mathcal{S}$ contains a solution invertible in H(D) then $\mathcal{S}$ has dimension 1. If H(D) has no divisors of zero, then $\mathcal{S}$ has dimension o or 1.

In Chapter III, we will suppose the residue characteristic p is different from zero and we will construct clopen bound infraconnected sets D with elements $f \in H(D)$ such that $\mathcal{S}$ has dimension n ($n \in \mathbb{N}$) or infinite dimension.

* Chapter II and III were made in common with Marie-Claude Sarmant. The questions taken up here were pointed out to my attention in talking to Labib Haddad at the Clermont Ferrand Analysis Seminary.

I. DERIVATIVE OF ANALYTIC ELEMENTS ON INFRACONNECTED CLOPEN SETS

Asserting the theorems requires to introduce a lot of definitions and notations.

For all $a \in K$, $r \in K_+$, $d(a,r)$ denotes the disk $\{x \in K \mid |x-a| \leq r\}$ $d^-(a,r)$ is the disk $\{x \in K \mid |x-a| < r\}$, and $C(a,r)$ is the circle $\{x \mid |x-a| = r\}$.

Let D be a bounded closed set of diameter R and let $\tilde{D}$ be the disk $d(a,r)$ with $a \in D$. Then $\tilde{D} \setminus D$ admits a partition by a unique family $(T_i)_{i \in I}$ where each T_i is a disk $d^-(a_i,r_i)$ and r_i is maximal. The T_i are called the *holes* of D.

Let $T = d^-(a,r)$ be a hole of D, and let $\check{R}(T)$ be the algebra of the $h(x) \in R(K \setminus T)$ such that $\lim_{|x| \to \infty} h(x) = o$. $\check{R}(T)$ is then provided with the norm of the uniform convergence on $K \setminus T$ and we will denote by $\check{H}(T)$ the Banach algebra completed of $\check{R}(T)$ for that norm.

We know that $\check{H}(T)$ is the algebra of the Laurent series $\sum_{n=1}^{\infty} \frac{\lambda_n}{(x-a)^n}$ such that $\lim_{n \to \infty} \left(\frac{|\lambda_n|}{r^n} \right) = o[A,K_2,R]$.

Now assume D is infraconnected closed and bounded. We have the Mittag-Leffler theorem [K,R] for an $f \in H(D)$. *There exists a unique sequence of holes* $(T_n)_{n \in \mathbb{N}^*}$ of D *and a unique set of analytic elements* $(f_n)_{n \in \mathbb{N}^*}$ *with* $f_n \in \check{H}(T_n)$ *and* $f_n \neq o$ *and a unique* $f_o \in H(\tilde{D})$ *such that the series* $\sum_{n=o}^{\infty} f_n$ *converges to f in* H(D) *and satisfies* $\|f\|_D = \max(\|f_o\|_{\tilde{D}}, \sup_{n \in \mathbb{N}^*} \|f_n\|_{K \setminus T_n})$.

Here we will call a f-hole of any one of the holes T_n $(n \in \mathbb{N})$.

The classical Theorem 1 is well known and it will be helpful :

<u>Theorem I.1</u>. *Let* Δ *be a bounded closed infraconnected set in* K *and let* $g \in H(\Delta)$. *Let* $(T_n)_{n \in \mathbb{N}}$ *be the sequence of the g-holes of g on the infraconnected set* Δ *and let* $\Delta' = \tilde{\Delta} \setminus (\bigcup_{n=1}^{\infty} T_n)$ *Then g extends into an element* $h \in H(\Delta')$ *such that* $\|h\|_{\Delta'} = \|g\|_{\Delta}$.

It is well known and easily seen that an analytic element on an open set D has a derivative at each point of D [R]. Now we will look when f' also belongs to H(D).

<u>In all of the following theorems and corollairies we denote by</u> D <u>a clopen bounded infraconnected set of diameter</u> R <u>in</u> K , <u>by</u> S <u>the set consisting of the diameter of the holes of</u> D <u>and the diameter</u> R <u>or</u> D <u>and by</u> λ <u>the lower bound of</u> S. <u>We denote by</u> f <u>an element of</u> H(D), <u>by</u>

$Q(f)$ the set consisting of $\mathbb{R}$ and of the set of the diameters of the f-holes of D, and by $\rho(f)$ the lower bound of $Q(f)$.

Let $f_o + \sum_{n=1}^{\infty} f_n$ be the Mittag-Leffler series of f on D with $f_o \in H(\tilde{D})$, $f_n \in \check{H}(T_n)$ (T_n a f-hole of D) $n \in \mathbb{N}^*$.

Theorem I.2. *Assume* $\rho(f) > o$. *Then* f' *belongs to* H(D) *and satisfies* $\|f'\|_D \leq \frac{\|f\|_D}{\rho(f)}$, *and the series* $\sum_{n=o}^{\infty} f'_n$ *converges to* f' *in* H(D).

Corollary . *Assume* $\lambda > o$. *For all* $f \in H(D)$, f' *belongs to* H(D) *and*

$$\|f'\|_D \leq \frac{\|f\|_D}{\lambda} .$$

The main problem we have got to study is whether the infraconnectedness characterizes the implication "$f'(x) = o$ whenever $x \in D \Rightarrow f$ = constant in D".

An answer is "yes" on the clopen set D. But it is not the same on a set Δ that is only open but not closed.

Actually, we find solutions to those problems in answering a more general question : *if* f' *belongs to* H(D), *does the series* $\sum_{n=o}^{\infty} f'_n$ *converge to* f' *in* H(D) ?

Theorem I.3 . *The three following assertions are equivalent :*

a) f' *belongs to* H(D)

b) the series $\sum_{n=o}^{\infty} f'_n$ *converges to* f' *in* H(D)

c) the series $\sum_{n=o}^{\infty} f'_n$ *converges in* H(D).

On the first hand, Theorem I.3 helps us characterize the infraconnected clopen bounded sets D such that all the elements $g \in H(D)$ have derivative in H(D).

Theorem I.4. λ *is different from zero if and only if for every* $g \in H(D)$, g' *belongs to* H(D).

On the second hand, Theorem I.3. gives us the implication : if $f'(x)=o$ for all $x \in \Delta$, then f is a constant, whenever Δ be a clopen infraconnected set and this is characteristic of the infraconnected sets in the class of

the clopen bounded sets. However, that characterization does not hold any more when Δ is not supposed to be closed. Theorem I.5. *A clopen bounded set* E *is infraconnected if and only if for every* $g \in H(E)$ *such that* $g'(x) = o$ *for all* $x \in E$, *g is a constant in* E.

Remark.

The derivation clearly is not continuous in R(D). Like in the proof of Theorem I.5, consider a clopen bounded infraconnected set D with a sequence of holes $T_n = d^-(b_n, r_n)$ with $\lim_{n\to\infty} r_n = o$ and take a sequence (λ_n) in K such that $\lim_{n\to o} \frac{|\lambda_n|}{r_n} = o$ and $\lim_{n\to o} \frac{|\lambda_n|}{r_n^2} = +\infty$.

Clearly, the sequence $g_n = \frac{\lambda_n}{x-b_n}$ converges to 0 although the sequence g'_n does not converge. As a consequence the series $\sum_{n=1}^{\infty} g_n$ has a derivative that does not belong to H(D).

However the following Theorem I.6 will be helpfull in further articles.

Theorem I.6 . *Suppose* f'*belongs to* H(D). *For every* $\varepsilon > o$, *there exists* $h \in R(D)$ *such that* $\|f-h\|_D \le \varepsilon$ *together with* $\|f'-h'\|_D \le \varepsilon$.

Recall briefly the proof of Theorem I.1.

Let $g = \sum_{n=o}^{\infty} g_n$ be the Mittag-Lefflerian series of g on the infraconnected set Δ with $g_o \in H(\tilde{\Delta})$ and $g_n \in \check{H}(T_n)$. For each $n \in \mathbb{N}$ let $h_n = \sum_{j=o}^{n} g_j$; then h_n clearly belongs to $H(\Delta')$. Now $g_{n+1} \in \check{H}(T_{n+1}) \subset H(\Delta')$and by the Mittag-Leffler Theorem $\|g_{n+1}\|_{K\setminus T_{n+1}} = \|g_{n+1}\|_\Delta$ hence $\|h_{n+1}-h_n\|_{\Delta'} = \|g_{n+1}\|_\Delta$ and then the sequence h_n does converge in $H(\Delta')$ to an element h which extends to g. Finally, $\|h\|_{\Delta'} = \|g\| = \max(\|g_o\|_{\tilde{\Delta}}; \sup_{n\in\mathbb{N}} \|g_n\|_{K\setminus T_n})$.

Proof of Theorem I.2.

Since D is open, it is well known and easily seen that f has a derivative at each point of D (because in every disk $d(a,r) \subset D$, f(x) is equal to a series $\sum_{n=o}^{\infty} \lambda_n (x-a)^n$ which converges for $|x-a| \le r$) [R].

Now let (T_n) be the sequence of the f-holes of f in D and let $D' = \tilde{D}\setminus(\bigcup_{n=1}^{\infty} T_n)$. By Theorem I.1, $f \in H(D')$ and $\|f\|_{D'} = \|f\|_D$. Thus we can

assume every hole of D is a f-hole.

Let $a \in D$ and let $r(a)$ be the distance from a to $K \setminus D$. In $d^-(a,r(a)),f(x)$ is equal to a series $\sum_{n=o}^{\infty} \lambda_n (x-a)^n$ convergent for $|x-a|<r(a)$ is equal to a series $\sum_{n=o}^{\infty} \lambda_n (x-a)^n$ convergent for $|x-a| < r(a)$ and then $f'(x) = \sum_{n=1}^{\infty} n\lambda_n (x-a)^{n-1}$ does satisfy $|f'(x)| \leq \frac{1}{r(a)} \|f\|_{d(a,r(a))} \leq \frac{\|f\|_D}{r(a)}$

On the other hand, by definition of $r(a)$, we can see that for all $\varepsilon > o$ there exists a f-hole T included in the disk $d(a,r(a)+ \varepsilon)$. Then $\mathrm{diam}(T) \leq r(a) + \varepsilon$ hence $\rho(f) \leq r(a) + \varepsilon$. As that is true for all $\varepsilon > 0$ clearly $\rho(f) \leq r(a)$ hence $|f'(x)| \leq \frac{\|f\|_D}{\rho(f)}$. Finally, this inequality is true for all $x \in D$ hence we obtain the inequality (1) $\|f'\|_D \leq \frac{\|f\|_D}{\rho(f)}$.

Now let $\sum_{n=o}^{\infty} f_n$ be the Mittag-Leffler series of f in H(D) with $f_o \in H(\tilde{D})$ and $f_n \in \check{H}(T_n)$, $f_n \neq o$. It is well known that $f'_o \in H(\tilde{D})$ and in the same way $f'_n \in \check{H}(T_n)$ for each $n \in \mathbb{N}^*$. Then by (1) the series $\sum_{n=o}^{\infty} f_n$ clearly converges in H(D) to a limit ℓ. We just prove $\ell = f'$. Indeed, let ε be a positive number, and let $N(\varepsilon) \in \mathbb{N}$ be such that $\|f_q\|_D \leq \varepsilon\, \rho(f)$ for $q \geq N(\varepsilon)$. Then $\|f' - \sum_{n=o}^{q} f'_n\|_D \leq \varepsilon$ hence $\|f' - \ell\|_D \leq \varepsilon$, finally $f' = \ell$; that ends the proof.

<u>Definition</u>. Let D be an infraconnected closed bounded set and let $f \in H(D)$. Let T be a hole of D. By the Mittag-Leffler Theorem and by Theorem I.1. there is a unique $h_T \in \check{H}(T)$ such that $f-h_T$ extends to an element of $H(D \cup T)$; h_T will be called the *f-singular element associated to* T *with respect to* D.

The following proposition is then a direct consequence of the Mittag-Leffler Theorem.

<u>Proposition</u>. *Let* D *and* D' *be two infraconnected closed bounded sets such that* $D' \subset D$. *Assume there is a hole* T *of* D *which is also a hole of* D'. *For every* $f \in H(D)$, *the f-singular element associated to* T *with respect to* D *has a restriction on* D' *that is the f-singular element associated to* T *with respect to* D'.

Notations. For all $a \in K$, r', $r'' \in \mathbb{R}_+^*$ with $r' < r''$, we will denote by $\Gamma(a,r',r'')$ the annulus $\{x \in K \mid r' < |x-a| < r''\}$, and by $\Lambda(a,r',r'')$ the annulus $\{x \in K \mid r' \leq |x-a| \leq r''\}$.

Lemme I.A. *assume the series* $\sum_{n=o}^{\infty} f'_n$ *converges to a limit* h *in* H(D). *Then* $h(x) = f'(x)$ *for all* $x \in H(D)$.

Proof. Let α be a point in D and let $r > o$ be such that $d(\alpha,r) \subset D$. For every $n \in \mathbb{N}$ let $g_n = \sum_{j=o}^{n} f'_j$, and let g_n be the restriction of g_n to $d(\alpha,r)$. By theorem I.2, the sequence g_n converges to the restrictionh of f' to $d(\alpha,r)$ in $H(d(\alpha,r))$ hence $h(x) = f'(x)$ for all $x \in d(\alpha,r)$. This is true for all $\alpha \in D$ and that ends the proof.

Proof of Theorem I.3. First b) trivially implies a) and c). After, by Lemma I.A, it is easily seen that c) implies b), and then it only remains to us to prove for example a) implies b).

Let us assume a) is true and prove b). For every hole T of D, let f_T(resp g_T) be the f-singular element (resp. the f'-singular element) associated to T with respect to D. Let $\mathcal{G}$ be the set of the f-holes T such that $(f_T)' \neq g_T$ and let $\mathfrak{J}$ be the set of the f-holes such that $(f_T)' = g_T$. If we can show that $\mathcal{G} = \emptyset$, Theorem I.3 is clearly proven. Suppose then $\mathcal{G} \neq \emptyset$. All of the g_T are null except maybe a countable family of them. The series $\sum_{t \in \mathfrak{J}} g_T$ and $\sum_{T \in \mathcal{G}} g_T$ obviously converge in H(D), and then

$$f' = \sum_{T \in \mathfrak{J}} (f_T)' + \sum_{T \in \mathcal{G}} g_T$$

By Lemma I.A, the series $\sum_{T \in \mathfrak{J}} (f_T)'$ is clearly equal to the derivative of $\sum_{T \in \mathfrak{J}} f_T$. Let $h = \sum_{T \in \mathcal{G}} f_T = f - \sum_{T \in \mathcal{G}} f_T$; then $h' = f' - \sum_{T \in \mathfrak{J}} (f_T)' = \sum_{T \in \mathcal{G}} g_T$.

Let $\mathfrak{D}$ be the family of the diameters of the holes T that belong to $\mathcal{G}$, and let δ be its lower bound. Suppose $\delta > 0$. By Theorem I.2, the series $\sum_{T \in \mathcal{G}} f'_T$ converges to h', hence $\sum_{T \in \mathcal{G}} f'_T$ is the Mittag-Leffer series of h' on D, hence $f'_T = g_T$ for all $T \in \mathcal{G}$ and that contradicts the definition of $\mathcal{G}$. Hence $\delta = 0$.

Now, we will prove there exists a hole $\mathfrak{T} = d^-(a,r) \in \mathcal{G}$ with an annulus $\Lambda(a,r,s)$ such that the set $\mathcal{S}$ of the diameters ρ of the f-holes included in $\Lambda(a,r,s)$ has a strictly positive lower bound. Indeed suppose such a hole $\mathfrak{T}$

does not exist. Then we can easily construct a sequence of f-holes $\tau_n = d^-(a_n,r_n)$ with (1) $r_n \leq \frac{1}{n}$ and (2) $|a_{n+1}| \leq \frac{2}{n}$. For example, assume the sequence has just been constructed up to the range q, satisfying (1) and (2) for $n \leq q$. Since $\mathcal{S}$ does not exist, then in $\Lambda(a_q,r_q,\frac{2}{q})$ we can find a f-hole $T_{q+1} = d^-(a_{q+1},r_{q+1})$ with $r_{q+1} < \frac{2}{q+1}$ and then the sequence is clearly constructed by induction in taking first any f-hole $T_1 = d^-(a_1,r_1)$

The sequence (T_n) clearly converges to a point $\omega \in D$ and that contradicts the hypothesis "D is clopen". Hence we have now proven the existence of the f-hole τ with an annulus $\Lambda(a,r,s)$ and a number $\delta > 0$ such that every f-hole $T \subset \Lambda(a,r,s)$ satisfies (3) $\mathrm{diam}(T) \geq \delta$

Let $\mathcal{L}$ be that family of the f-holes included in $\Lambda(a,r,s)$. Let $\ell = \sum_{T\in\mathcal{L}} f_T$; by Theorem I.2, the series $\sum_{T\in\mathcal{L}} (f_T)'$ converges to ℓ' in H(D).

Now let $\varphi = h-\ell-f_\tau$. Clearly φ belongs to H(D) and no hole T (of D). included in d(a,s) is a φ-hole. Hence φ extends to an element of H((D $\cup$ d(a,s)). In d(a,s),$\varphi(x)$ is equal to a Taylor series $\Phi(x)\in H(d\ (a,s))$, hence $\Phi'\in H(d(a,s))$.

Thus in D $\cap$ d(a,s),$\varphi'(x)$ is equal to the series $\Phi'(x)$ and then for every hole T of D included in d(a,s) the φ'-singular element associated to T with respect to D is null.

On the other hand, $\varphi' = h'-\ell\ (f_\tau)' = \sum_{T\in\mathcal{G}} g_T - \sum_{T\in\mathcal{L}} (f_T)' - (f_\tau)'$ and then the φ'-singular element associated to τ with respect to D is $g_{\mathcal{G}}-(f_\tau)' \neq 0$ so that we have a contradiction with $\varphi'\in H(d(a,s))$, and then Theorem I.3 is finally proven.

<u>Proof of Theorem</u> I.4.

By the Corollary, if $\lambda > 0$, every $g\in H(D)$ has its derivative g' in H(D). Now let us assume that $\lambda = 0$ and let us find a $g\in H(D)$ such that g' does not belong to H(D). Let $(T_n)_{n\in\mathbb{N}^*} = (d^-(a_n,r_n))_{n\in\mathbb{N}^*}$ be a sequence of holes such that $\lim_{n\to\infty} r_n = 0$.

Now let $(\lambda_n)_{n\in\mathbb{N}^*}$ be a sequence in K such that $\lim_{n\to\infty} \frac{|\lambda_n|}{r_n} = 0$ and $\lim_{n\to\infty} \frac{|\lambda_n|}{r_n^2} = +\infty$. We know that $|\frac{\lambda_n}{x-a_n}| = \frac{|\lambda_n|}{r_n}$ and then the series $\sum_{n=1}^{\infty} \frac{\lambda_n}{x-a_n}$ converges to a limit g in H(D) and for each n, the g-singular element of T_n with respect to D is obviously $\frac{\lambda_n}{x-a_n}$, hence the Mittag-Leffler series

of g on D on $\sum_{n=1}^{\infty} \frac{\lambda_n}{x-a_n}$.

Now suppose g' belongs to H(D) . By theorem I.3, the series $\sum_{n=1}^{\infty} \frac{\lambda_n}{(x-a_n)^2}$ must converge to g' in H(D). But $\left\| \frac{\lambda_n}{(x-a_n)^2} \right\|_D = \frac{|\lambda_n|}{r_n^2}$ and that clearly shows the series $-\sum_{n=1}^{\infty} \frac{\lambda_n}{(x-a_n)^2}$ does not converge in H(D). Theorem I.4 is then proven.

<u>Proof of Theorem</u> I.5. It is well known that if E is not infraconnected, H(E) has an idempotent $u \neq 0$ and 1 and then $u(x) = 0$ in a subset E_1 of E, while $u(x) = 1$ in $E_2 = E \setminus E_1$, hence $u'(x) = 0$ for all $x \in E$ (though u is not constant in E).

Now suppose E is infraconnected, let $g \in H(E)$ be such that $g'(x) = 0$ whenever $x \in E$ and let us prove that g is a constant.

Let $\sum_{n=0}^{\infty} g_n$ be the Mittag-Leffler series of g on E. Since g' is the null analytic element, the series $\sum_{n=0}^{\infty} (g_n)'$ converges to zero in H(E) hence it is easily seen that $g'_n = 0$ whenever $n \geq 1$ and g_0 is a constant in D, and that ends the proof of Theorem I.5.

<u>Proof of Theorem</u> I.6.

Let $d(a,r) = \tilde{D}$ and let $g(x) = \sum_{m=0}^{\infty} \lambda_m (x-a)^m \in H(\tilde{D})$. For each $q \in \mathbb{N}$, let $(g)_q$ be the polynomial $\sum_{m=0}^{q} \lambda_m (x-a)^m$. Now let $T = \bar{d}\,(b,r)$ be a hole of D, and let $t(x) = \sum_{m=1}^{\infty} \frac{\mu_m}{(x-b)^m}$; for each $q \in \mathbb{N}^*$, let $(t)_q = \sum_{m=1}^{q} \frac{\mu_m}{(x-b)^m}$.

Now let ε be > 0 and let us find the $h \in R(D)$ satisfying Theorem I.4.

By Theorem I.3, we have an integer $N(\varepsilon)$ such that (1) $\left\| \sum_{n=0}^{N(\varepsilon)} f_n - f \right\|_D \leq \varepsilon$ and (2) $\left\| \sum_{n=0}^{N(\varepsilon)} f'_n - f' \right\| \leq \varepsilon$.

On the other hand we obviously have an integer $Q(\varepsilon)$ such that $\left\| f_n - (f_n)_{Q(\varepsilon)} \right\|_D \leq \varepsilon$ whenever $n = 0,\ldots,N(\varepsilon)$ and then by (1) and (2) it is easily seen we have $\left\| \sum_{n=0}^{N(\varepsilon)} (f_n)_{Q(\varepsilon)} - f \right\|_D \leq \varepsilon$ and $\left\| \sum_{n=0}^{\infty} (f'_n)_{Q(\varepsilon)} - f' \right\|_D \leq \varepsilon$

Putting $h = \sum_{n=0}^{N(\varepsilon} (f_n)_{Q(\varepsilon)}$ we obtain the $h \in R(D)$ we have been looking for.

II THE DIFFERENTIAL EQUATION $y' = fy$ IN THE ALGEBRAS $H(D)$

Here we take a clopen bounded infraconnected set D, a f in H(D), we consider the differential equation $(\mathcal{E})$ $y' = fy$ with $y \in H(D)$, and we denote by $\mathcal{S}$ the space of the solutions $g \in H(D)$ of $(\mathcal{E})$.

By classical results, we know that $\mathcal{S}$ may be reduced to $\{0\}$. (For example, if D is the disk $|x| \leq 1$, it is easily seen the equation $y' = y$ has no solution in H(D)). Here we will give sufficient conditions on the algebra H(D) to have dimension 1 or 0 for $\mathcal{S}$.

In the three Theorems that follow, D *is a clopen bounded infraconnected set,* f *belongs to* H(D), $(\mathcal{E})$ *denotes the differential equation* $y' = fy$ *and* $\mathcal{S}$ *is the linear space of the solutions of* $(\mathcal{E})$ *in* H(D).

The notions of T-filter and strictly annulled element involved in Théorem II.2 will be recalled below.

Theorem II.1. *If* $(\mathcal{E})$ *has at least one solution* g *inversible in* H(D) *then* $\mathcal{S}$ *has dimension* 1.

Theorem II.2. *We assume* $(\mathcal{E})$ *has at least one solution* g *non identically null. Then* g *has no zero isolated in* D. *Besides*

either g *is invertible in* H(D)

or g *is strictly annulled by a T-filter on* D.

Theorem II.3. *If* H(D) *has no divisor of zero, then* $\mathcal{S}$ *has dimension* 0 *or* 1.

The proof of Theorem 1 is easily obtained.

Proof of Theorem II.1. Let g be a solution of $(\mathcal{E})$ invertible in H(D), and let h be another solution. We verify $\frac{h}{g}$ is a constant in H(D). Indeed, by hypothesis, $\frac{h}{g}$ does belong to H(D). Then $(\frac{h}{g})' = \frac{h'g - hg'}{g^2} = \frac{fgh - hgh}{g^2} = 0$ and then by Theorem I.5 we know that $\frac{h}{g}$ is a constant in D.

Now we have to recall the definitions linked to the Monotonous Filters.

Technical definitions and proof of Theorem II.2.

The technique used in the proofs of the Theorems requires a lot of classical definitions previously given in $[G, E_2, E_3, E_4, E_5]$.

We will denote by "log" a real logarithm function of base $\omega > 1$ and by v the valuation defined on K by $v(x) = = -\log|x|$.

Now we have to define the monotonous filters. Henceforth, D *will denote*

a closed bounded infraconnected set and we will specify when it is supposed to be open; f will denote an element of H(D); ($\mathcal{E}$) is the equa+ion $y' = fy$ with $y \in H(D)$.

We call an *increasing filter* (resp. a *decreasing filterk)* of *center* $a \in \tilde{D}$ *of diameter* r the filter on D that admits for base the family of sets $\Gamma(a,r,s) \cap D$ with $0 < s < r$ (resp. $\Gamma(a,r,s) \cap D$ with $r < s$).

We call a *decreasing filter with no center on* D a filter that admits for base a sequence D_n in the form $D_n = d(a_n,r_n) \cap D$ with :

$$d(a_{n+1},r_{n+1}) \subset d(a_n,r_n),\ \lim_{n\to\infty} r_n > 0,\ \bigcap_{n=1}^{\infty} d(a_n,r_n) = \phi$$

and the limit of (r_n) is called the *diameter* of the filter.

We call a *monotonous filter* a filter that is either increasing or decreassing.

We know that if $\mathcal{F}$ is a monotonous filter on D and $f \in H(D)$, then the function defined in D by $|f(x)|$ has a limit along the filter $\mathcal{F}$ and the mapping $f \to \lim_{\mathcal{F}} |f(x)|$ is a multiplicative semi-norm on H(D) continuous with respect to the norm $\|.\|_D[G,E_3]$.

If $\mathcal{F}$ is a monotonous filter of center a, of diameter r, we also have

$$\lim_{\mathcal{F}} |f(x)| = \lim_{\substack{|x-a| \to r \\ |x-a| \neq r \\ |x \in D}} |f(x)|.$$

For convenience we introduce the valuation function $v_a(f,\mu)$ defined by

$$v_a(f,-\log r) = \lim_{\substack{|x-a| \to r \\ |x-a| \neq r \\ x \in D}} v(f(x) \quad \text{if} \quad \lim_{\substack{|x-a| \to r \\ |x-a| \neq r \\ x \in D}} |f(x)| \neq 0$$

$$v_a(f,-\log r) = +\infty \quad \text{if} \quad \lim_{\substack{|x-a| \to r \\ |x-a| \neq r \\ x \in D}} f(x) = 0$$

Let R be the diameter of D. Then for all $a \in \tilde{D}$, the function $\mu \to v_a(f,\mu)$ is continuous and piecewise linear in its interval of definition I. If a does not belong to a hole of D, I is $[-\log R, +\infty[$. If a belongs to a hole $T = d^-(a,\rho)$, then $I = [-\log R, -\log \rho]$.

When $a = 0$ we will only write $v(f,\mu)$ for $v_o(f,\mu)$.

<u>For</u> $\mu < v(a-b)$ <u>we have</u> $v_a(f,\mu) = v_b(f,\mu)$ <u>for all</u> $h \in H(D)$. $[E_1,E_3]$..

By the definition of $v_a(f,\mu)$ it is easily seen that $-\log\|f\|_D \leq v_a(f,\mu)$ for all $a\in D$, and $\mu \geq -\log R$. In particular, *if* f *and* $g\in H(D)$ *are such that* $-\log\|f-g\|_D < v_a(f,\mu)$ then $v_a(f,\mu) = v_a(g,\mu)$.

Lef f belong to H(D). f is said to be *strictly annulled by a increasing filter (resp. a decreasing filter) of center* a, *of diameter* r, if there exists $\lambda < -\log r$ (resp. $\lambda > -\log r$) such that $v_a(f,\mu) < +\infty$ whenever $\mu \in]-\log r,\lambda]$ (resp. whenever $\mu\in[\lambda,-\log r[$) and if $\lim_{\mathcal{F}} f(x) = 0$.

f is said to be *strictly annulled by a decreasing filter* $\mathcal{F}$ *with no center, of diameter* r, *of base* (D_n) *with* $D_n = d(a_n, r_n)\cap D$, if there exists $\lambda > -\log r$ such that $v_{a_n}(f,\mu) < +\infty$ whenever $\mu\in[\lambda,-\log r_n]$, whenever $n\in\mathbb{N}$, and if $\lim_{\mathcal{F}} f(x) = 0$.

Now recall a monotonous filter is called a T-filter if the holes of the elements of its bases form a sequence that satisfies a condition given in $[E_4]$ (we won't explicitly need it in the present work). Then we know that *given a monotonous filter* $\mathcal{F}$, *there exist elements* $f\in H(D)$ *strictly annuled par* $\mathcal{F}$ *if and only if* $\mathcal{F}$ *is a* T-*filter* $[E_4]$

An element $f\in H(D)$ is said to be *quasi-invertible* if it factorizes in the form $P(x)g(x)$ with P a polynomial the zeros of which are in the interior of D, and g an invertible element in H(D).

Then if D *is a clopen bounded infraconnected set, an element* $f\in H(D)$ *is not quasi-inversible if and only if it is annuled by a* T-*filter on* $D[E_4]$.

<u>Proof of Theorem</u> II.2. Let us assume g has an isolated zero a in D. Since D is open we know g factorizes in the form $(x-a)^q h(x)$ with $h\in H(D)$ and $h(a) \neq 0$ $[E_1,E_2]$ hence $g' = (x-a)^{q-1}(qh + (x-a)h')$ hence $qh = (x-a)(f-h')$ which contradicts the hypothesis $h(a) \neq 0$, (since $q \neq 0$). Thus g has no isolated zero in D.

Now suppose g is not invertible; since it has no isolated zero it is not quasi-invertible, and since D is open, that implies g is strictly annulled by a T-filter on D $[E_3;E_4]$.

<u>Beaches, integrity and proof of Theorem</u> II.3.

Let $\mathcal{F}$ be an increasing (resp.a decreasing) filter of center a, of diameter $r>0$. Let set of the $x\in D$ such that $|x-a| \geq r$ (resp. $|x-a| \leq r$) is called *the beach* of $\mathcal{F}$, denoted by $\mathcal{P}(\mathcal{F})$. The beach $\mathcal{P}(\mathcal{F})$ of a decreasing filter $\mathcal{F}$ with no center is the empty set ϕ . We denote by $\mathcal{E}(\mathcal{F})$ the set

$D \setminus \mathcal{P}(\mathcal{F})$, by $\mathcal{I}(\mathcal{F})$ the ideal of the $f \in H(D)$ such that $\lim_{\mathcal{F}} f(x) = 0$ and by $\mathcal{I}_o(\mathcal{F})$ the ideal of the $f \in \mathcal{I}(\mathcal{F})$ such that $f(x) = 0$ whenever $x \in \mathcal{P}(\mathcal{F})$. *Then* $\mathcal{I}(\mathcal{F})$ *and* $\mathcal{I}_o(\mathcal{F})$ *are closed prime ideals* [E_3,E_4,E_5].

Two monotonous filters $\mathcal{F}$ and $\mathcal{G}$ on D are said to be <u>complementary</u> if $\mathcal{P}(\mathcal{F}) \cup \mathcal{P}(\mathcal{G}) = D$.

The Banach algebra H(D) *has no divisors of zero if and only if* D *is infraconnected with no couple of complementary* T-*filters* [E_5].

In all of the following lemmas, D *will denote a closed bounded infraconnected set* and we will specify when it is open.

<u>Lemma II.A</u> *Let* $a \in D$ *and let* $r \in \mathbb{R}_+$. *Assume* $f(x) = 0$ *whenever* $x \in d(a,r) \cap D$. *Assume there exists* $b \in D$ *such that* $f(b) \neq 0$. *Then there exists a* T-filter $\mathcal{F}$ *on* D *such that* $b \in \mathcal{C}(\mathcal{F})$ *and* $d(a,r) \subset \mathcal{P}(\mathcal{F})$ [E_5].

<u>Lemma II.B</u> *Let* $\mathcal{F}$ *be a* T-*filter on* D *with no complementary* T-*filter. Then* $\mathcal{I}(\mathcal{F}) = \mathcal{I}_o(\mathcal{F})$.

<u>Proof of Lemma II.B.</u> The equality $\mathcal{I}(\mathcal{F}) = \mathcal{I}_o(\mathcal{F})$ is trivial when $\mathcal{P}(\mathcal{F}) = \phi$ hence we will assume that $\mathcal{F}$ has center a. Let r be its diameter and let $\theta = -\log r$. Let $f \in \mathcal{I}(\mathcal{F})$ and let us show $f \in \mathcal{I}_o(\mathcal{F})$. For this, let us assume $f \notin \mathcal{I}_o(\mathcal{F})$ and let $b \in \mathcal{P}(\mathcal{F})$ be such that $f(b) \neq 0$.

Let $\lambda = v(a-b)$.

1) *Assume* $\mathcal{F}$ *is increasing.*

1)α) *Assume first* $v_a(f,\lambda) < +\infty$.

By hypothesis since $f \in \mathcal{I}(\mathcal{F})$, we know $v_a(f,\theta) = +\infty$. Hence there exists $\gamma \in [\theta,\lambda]$ such that $v_a(f,\gamma) = +\infty$ and $v_a(f,\mu) < +\infty$ whenever $\mu \in [\gamma,\lambda]$. Then f is strictly annulled by the decreasing filter $\mathcal{G}$ of center a, of diameter $s = \omega^{-\gamma}$. The filter $\mathcal{G}$ is then a T-filter complementary to $(\mathcal{F})$ which contradicts the hypothesis.

1)β) Assume now $v_a(f,\lambda) = +\infty$. We know $v_b(f,\lambda) = v_a(f,\lambda)$ since $\lambda = v(a-b)$ and therefore $v_b(f,\lambda) = +\infty$, while $v_b(f,\mu) < +\infty$ when μ approaches $+\infty$ because $f(b) \neq 0$.

It then exists $\gamma \geq \lambda$ such that $v_b(f,\mu) < +\infty$ whenever $\mu > \gamma$ and $v_b(f,\gamma) = +\infty$. Hence f is strictly annulled by the increasing filter of center b, of diameter $s = \omega^{-\gamma}$. This filter is then a T-filter $\mathcal{G}$. Since

$\max(r,s) \leq |a-b|$, $\mathcal{G}$ is complementary to $\mathcal{F}$ which contradicts the hypothesis.

2) Now, let us assume $\mathcal{F}$ is decreasing. Then a and b belong to $\mathcal{P}(\mathcal{F}) = d(a,r) \cap D$ therefore $|a-b| \leq r$ hence $v_b(f,\theta) = +\infty$ .It then exists $\gamma > \theta$ such that $v_b(f,\gamma) = +\infty$ for all $\mu > \gamma$, hence the increasing filter of center b, of diameter $s = \omega^{-\gamma} < r$ is a T-filter complementary to $\mathcal{F}$ which ends the proof of Lemma II.B.

<u>Corollary II.C.</u> *If* H(D) *has no divisor of zero then for every* T-*filter* $\mathcal{F}$ *on* D, $\mathcal{J}(\mathcal{F}) = \mathcal{J}_o(\mathcal{F})$.

<u>Definition.</u> Let $g \in H(D)$. We call *support* of g the set $\sum$ of the $x \in D$ such that $g(x) \neq 0$, and $\sum$ will be said to be *reinforced* if for every $a,b \in \sum$, the function $\mu \to v_a(f,\mu)$ is bounded in the interval $[v(a,b),+\infty[$.

<u>Proposition II.E.</u> *Assume* H(D) *has no divisor of zero. Then every* $f \in H(D) \setminus \{0\}$ *has a reinforced support.*

<u>Proof</u>. Let $f \in H(D)$, let $\sum$ be the support of f, let $a,b \in \sum$. Let us show $v_a(f,\mu)$ is bounded when $\mu \in [v(a-b),+\infty[$. Indeed assume it is not. Since $a \in \sum$ $f(a) \neq 0$, hence there exists $\gamma \in \mathbb{R}$ such that $v_a(f,\mu) = v(f(a))$ whenever $\mu \geq \gamma$. Since $v_a(f,.)$ is a continuous function, if it is not bounded in $[v(a-b),+\infty[$, there exists $\lambda \geq v(a-b)$ such that $v_a(f,\mu) < +\infty$ whenever $\mu > \lambda$ and $v_a(f,\lambda) = +\infty$, so that D has an increasing T-filter $\mathcal{F}$ of center a, of diameter $r = \omega^{-\lambda}$.

Assume first $v_a(f,v(a-b)) < \infty$. Then there exists $\alpha \in]v(a-b),\lambda]$ such that $v_a(f,\mu) < \infty$ whenever $\mu \in]v(a-b),\alpha[$ and $v_a(f,\alpha) = +\infty$ which means that D has a decreasing T-filter $\mathcal{G}$ of center a, of diameter $\omega^{-\alpha} > r$. Then $\mathcal{G}$ is complementary to $\mathcal{F}$, which contradicts the hypothesis "H(D) has no divisor of zero". Thus $v_a(f,\mu)$ is finally bounded in $[v(a-b),+\infty[$ and that ends the proof of Proposition II.E.

<u>Lemma II.F</u>. *Let* A *and* B *be infraconnected closed bounded sets such that* $\tilde{A} = \tilde{B}$. *Then* $A \cup B$ *is infraconnected.*

<u>Proof</u> . Let $d(\alpha,R) = \tilde{A} = \tilde{B}$. Let $a \in A$. Since $A \cup B = d(\alpha,R) = d(a,R)$ the set

$I(a) = \{|x-a| \,|\, x\in A\cup B\}$ is included in $[0,R]$. Since A is infraconnected, of diameter R, the set $I(a) = \{|x-a| \,|\, x\in A\}$ is dense in $[0,R]$. In the same way $a\in B$ $I(a)$ is still dense in $[0,R]$ and that finishes proving Lemma II.F.

<u>Proposition II.G</u>. *Assume* D *is open. Let* $f\in H(D)$ *and assume the support* $\sum$ *of* f *is reinforced. Then for every couple* $(a,b)\in \sum \times \sum$ *, there exists a clopen bounded infraconnected set* $\Omega_a^b \subset \sum$ *and a number* $\delta > 0$ such that $|f(x)| \geq \delta$ *whenever* $x\in\Omega_a^b$.

<u>Proof</u>. Let $r = |a-b|$. By hypothesis there exists $M\in\mathbb{R}_+$ such that $v_a(f,\mu) \leq M$ and $v_b(f,\mu) \leq M$ for all $\mu \geq v(a-b)$. Then the equality

$$v(f(x)) = v_a(f,v(x-a)) \text{ (resp.} v(f'(x)) = v_b(f,v(x-b)))$$

is true in all $D \cap d(a,r)$ (resp. $D \cap d(b,r)$) except may be in a finite number of circles of center a (resp. b) of radii $\rho \leq r[E_3]$.
Let $C(a,\rho_i)_{1\leq i\leq m}$ (resp. $C(b,\sigma_j)_{1\leq i\leq n}$) be the circles of center a (resp. b) that contain points $x\in D$ such that $v(f(x)) \neq v_a(f,v(x-a))$ (resp. $v(f(x)) \neq v_b(f,v(x-b))$) and let

$$\Delta_a^b = (d(a,r) \cap D)\setminus (\bigcup_{i=1}^{m} C(a,\rho_i)).$$

(resp. $\Delta_b^a = d(b,r) \cap D) \setminus (\bigcup_{j=1} C(b,\sigma_j))$). Then Δ_b^a (resp. Δ_b^a) is clearly infraconnected and clopen.

Moreover by hypothesis we have $v(f(x)) = v_a(f,v(x-a)) \leq M$ in all Δ_a^b. Let us put $\Omega_a^b = \Delta_a^b \cup \Delta_b$. Then $v(f(x)) \leq M$ whenever $x\in\Omega_a^b$ hence we can take $\delta = \omega^{-M}$ to obtain the relation $|f(x))| \geq \delta$ in Ω_a^b .
Now Ω_a^b is clearly clopen. At last by Lemma II.E, Ω_a^b is infraconnected because Δ_a^b and Δ_b^a are infraconnected sets such that $\tilde{\Delta}_a^b = \tilde{\Delta}_b^a = d(a,r)$.
Proposition II.F is then proven.

<u>Proposition</u> II.H. Let D be clopen, let $f\in H(D)$ and let $(\mathcal{E})$ be the differential equation $y' = fy$. *We assume* $(\mathcal{E})$ *has a solution* g *whose support is reinforced. Let* h *be another solution of* $(\mathcal{E})$ *. Then there exists* $\lambda\in K$ *such that* $h(x) = \lambda g(x)$ *whenever* $x\in\sum$.

<u>Proof</u>. Since D is open, $\sum$ is clearly open in K, hence for every $a\in\sum$ there exists a disk $\Delta(a)$ included in $\sum$. Let $(\mathcal{E}_a)$ be the equation $y' = f(x)y$ for $x\in\Delta(a)$; then $(\mathcal{E}_a)$ has non null solutions (like the restriction of g to $\Delta(a)$) hence the space of the solutions has dimension one by classical

results (and by Theorem II.1). It only remains to show $\lambda(a)$ is constant when a runs in $\sum$.

Let us fix a and b in $\sum$. By Proposition II.G, there exists a clopen bounded infraconnected set $\Omega_a^b \subset \sum$, with $a,b \in \Omega_a^b$, and $\delta > 0$ such that $|g(x)| \geq \delta$ whenever $x \in \Omega_a^b$. The restriction $\tilde{g}$ of g to Ω_a^b is then invertible in $H(\Omega_a^b)$. Hence the restriction (h/g) of h/g to Ω_a^b is a locally constant element of $H(\Omega_a^b)$. As Ω_a^b is clopen and infraconnected, by Theorem I.5. we know that h/g is a constant in $H(\Omega_a^b)$, hence $h/g(b) = h/g(a)$ and then Proposition II.H is proved.

<u>Proof of Theorem II.3</u>. Assume ($\mathcal{E}$) has a non identically null solution g. By Proposition E, the support $\sum$ of g is reinforced. Let h be another non identifically null solution. Since H(D) has no divisor of zero, the support $\sum'$ of h does have common points with $\sum$. By Proposition H there exists $\lambda \in K$ such that $h(x) = \lambda g(x)$ whenever $x \in \sum$. Since $\sum \cap \sum' \neq \phi$, λ can't be zero. Hence $h(x) \neq 0$ whenever $x \in \sum$, therefore $\sum' \supset \sum$. By the same reasoning we just have $\sum' \subset \sum$ hence $\sum = \sum'$. The relation $h(x) = \lambda g(x)$ is then true in $\sum$ and it is trivially true in $D \setminus \sum$ where $h(x) = g(x) = 0$. Theorem II.3 is then proved.

III. ZERO RESIDUE CHARACTERISTIC

In the chapter we will suppose the characteristic p of the residue class field k to be equal to zero.

D will still denote a clopen bounded infraconnected set, f an element of H(D) and $\mathcal{S}$ the space of the solutions of the equation $\mathcal{E}$ $y' = fy$ with $y \in H(D)$.

<u>Theorem III</u>. *If $\mathcal{S}$ is not reduced to {0}, is has dimension one and every non identically null solution is invertible in* H(D).

Before proving the Theorem, we have to establish the Lemmas and Propositions A, B, C, D, E mainly dedicated to the behaviour of the valuation function $\mu \to v(f,\mu)$ when the residue characteristic is zero.

<u>Lemma III.A</u>. *Let* r *and* $R \in \mathbb{R}_+$ *with* $0<r<R$ *and let* D *be* $\Gamma(o,r,R)$. *Let* μ *belong to* $]-\text{Log } R, -\text{Log } r[$ *and let* f *be a Laurent series* $\sum_{-\infty}^{\infty} a_n x^n \in H(D)$ *such that* $v(f,\mu) = v(a_q) + q\mu$ *with* $q \neq 0$. *Then* $v(f,\mu) = v(f',\mu) + \mu$.

Proof. $f'(x) = \sum_{-\infty}^{+\infty} n_a x^{n-1}$ hence $v(f',\mu) = \inf_{n\in\mathbb{Z}^*} v(na_n)+(n-1)\mu$. Since the residue characteristic of K is zero, $v(na_n) = v(a_n)$ whenever $n\neq 0$ hence

$$\inf_{n\in\mathbb{Z}^*} v(na_n)+(n+1)\mu = v(qa_q)+(q-1)\mu = v(f,\mu) - \mu.$$

Lemma III.B. *Let* r', r" *be numbers such that* $0 < r' < r''$ *and let* h(x) *be a rational function in* K(x) *such that* $v(h,\mu)$ *is not constant in any interval included in* [r',r"]. *Then* $v(h',\mu) = v(h,\mu) - \mu$ *whenever* $\mu\in[-\text{Log } r'', -\text{Log } r']$.

Since the function $\mu \to v(h,\mu)$ is continuous in μ, it is enough to prove the relation in $]-\text{Log } r'', -\text{Log } r'[$. Let $\sigma\in]-\text{Log } r'', -\text{Log } r'[$ and let $s = \omega^{-\sigma}$. We will prove the relation at σ in considering $t\in]s,r''[$ such that h has no pole in $\Gamma(o,s,t)$. Then h(x) is equal to a Laurent series $\sum_{-\infty}^{+\infty} a_n x_n$ and we can apply Lemma III.A that shows the relation is true whenever $\mu\in]-\text{Log } r,\sigma[$. By continuity the relation then is true at σ.

Proposition III.C. *Let* D *be a clopen bounded infraconnected set, of diameter* R, *such that* 0 *belongs to* $\tilde{D}$. *Let* r *be the distance from* 0 *to* D *and let* r', $r''\in\mathbb{R}_+^*$ *be such that* $0 < r' < r'' \leq R$ *and* $r \leq r'$. *Let* $f\in H(D)$. *We* H(D) *assume the function* $\mu\to v(f,\mu)$ *is bounded in the interval* $I = [-\text{Log } r'', -\text{Log } r']$ *and it is not constant in any interval* $\mathcal{J} \subset I$. *Then* $v(f,\mu) = v(f',\mu) + \mu$ *whenever* $\mu\in I$.

Proof. Let M be the upper bound of $v(f,\mu)$ in I and let $\delta = \omega^{-M} \min(1,\frac{1}{R})$. By Theorem I.6 there exists $h\in R(D)$ satisfying (1) $\|f-h\|_D < \delta$ together with $\|f'-h'\|_D < \delta$. Relation (1) also implies $v(f-h,\mu) > M \geq v(f,\mu)$ hence (3) $v(f,\mu) = v(h,\mu)$ whenever $\mu\in I$.

By then the function $\mu\to v(h,\mu)$ is not constant in any interval included in I, hence by Lemma III.B, we have (4) $v(h',\mu) = v(h,\mu)-\mu$ whenever $\mu\in I$. Then relations (3) and (4) do show that $v(f',\mu) = v(f,\mu) - \mu$ whenever $\mu\in I$.

Proposition III.D. *Let* D *be a clopen bounded infraconnected set with a* T-*filter* $\mathcal{F}$ *and* $f\in H(D)$. *We assume the equation* $y' = fy$ *admits a solution* g *strictly annuled by* $\mathcal{F}$. *Then* f *is not annuled by* $\mathcal{F}$.

Proof. We will first assume $\mathcal{F}$ is increasing, of center a, of diameter R. We can obviously assume a = O. Since g is strictly annulled by $\mathcal{F}$, there exists $\lambda > -\text{Log } R$ such that $\lim_{\mu \to -\text{Log}R} v(g,\mu) = +\infty$ with $v(g,\mu) < +\infty$ for $\mu \in]-\text{Log } R,\lambda]$, and then there exists a sequence of couples (λ'_n,λ''_n) with $-\text{Log } R > \lambda'_n > \lambda''_n$, $\lim_{n\to+\infty} \lambda''_n = \lim_{n\to+\infty} \lambda'_n = -\text{Log } R$ and such that $\frac{d}{d\mu}(f,\mu)$ exists and is strictly negative whenever $\mu \in [\lambda'_n,\lambda''_n]$. By Proposition C we know that $v(g',\mu) = v(g,\mu) - \mu$ whenever $\mu \in [\lambda',\lambda'']$ therefore $v(f,\mu) = \mu$ whenever $\mu \in [\lambda'_n,\lambda''_n]$. Thus $v(f,\mu)$ does not go to $+\infty$ when μ approache $-$ log R, which proves f is not annulle by $\mathcal{F}$.

In the case that $\mathcal{F}$ is decreasing we can do the same demonstration in choosing a center of $\mathcal{F}$ (we can take it in a spherically complete extension of K, if required).

Proposition III.E. *Let* D *be a clopen bounded infraconnected set containing* O, *let* f *belong to* H(D) *and let* ϕ *belong to* H(D) *such that* $\phi' = f\phi$. *We assume the function* $\mu \to v(f,\mu)$ *to be linear in an interval* $I = [\lambda',\lambda'']$ *and* $v(\phi,\mu) < +\infty$ *whenever* $\mu \in I$. *The function* $\mu \to v(\phi,\mu)$ *is also linear in* I.

Proof. We assume $v(\phi,\mu)$ to be non linear in I. Then there exists a point $\sigma \in]\lambda',\lambda''[$ such that $v'_d(\phi,\sigma) \neq v'_g(\phi,\sigma)$. With no loss of generality we can suppose $\sigma = O$ in performing a suitable change of variable.

We will first construct an interval $\mathcal{J} = [\mu',\mu'']$ with $\mu' < O < \mu''$ such that the function $\mu \to v(\phi,\mu)$ is linear in both $[\mu',O]$ and $[O,\mu'']$ and $v(\phi',\mu)$ is bounded in $\mathcal{J}$. Since $v(\phi,\mu) < +\infty$ whenever $x \in I$, there exist $\mu_1,\mu_2 \in I$ with $\mu_1 < O < \mu_2$ such that $v(\phi,\mu)$ is bounded by a number L in the interval $\mathcal{J} = [\mu_1,\mu_2]$.

We can obviously choose μ_1,μ_2 close to O enough to have the function $\mu \to v(\phi,\mu)$ linear in each one of the intervals $[\mu_1,O]$ and $[O,\mu_2]$ because it is bounded in $\mathcal{J}$, hence piecewise linear in $\mathcal{J}$. Since $v'_g(\phi,O) \neq v'_d(\phi,O)$, the function $\mu \to v(\phi,\mu)$ is not constant in at least one of the two intervals $[\mu_1,O]$ and $[O,\mu_2]$.

For example suppose first it is not constant in $[\mu_1,O]$. Since it is linear, it is not constant in any one of the intervals included in $[\mu_1,O]$ and then we can apply Proposition III.C which proves $v(\phi',\mu) = v(\phi,\mu) - \mu$ whenever $\mu \in [\mu_1,O]$ and therefore $v(\phi',\mu)$ is bounded in $[\mu_1,O]$ by a number L'_1. In addition, since $v(\phi',O) = v(\phi,O)$ it does exist $\mu'' \in [O,\mu_2]$ such that $v(\phi',\mu)$ is bounded in $[O,\mu'']$ by a number L'_2. Let us put $\mu' = \mu_1$ and

$L' = \max(L_1,L_2)$. The function $\mu \to v(\phi',\mu)$ is then upper bounded by L' in $[\mu',\mu'']$ while $v(\phi,\mu)$ is upper bounded by L.

In the same way, if we suppose $v(\phi,\mu)$ to be non constant in $[0,\mu_2]$ we have a symetric construction and therefore we finally have an upper bound L' for $v(\phi',\mu)$ in the interval $[\mu',\mu'']$ in all cases.

We set $M = \mathrm{Max}(L,L')$. By definition ϕ satisfies

(1) $$v(\phi,\mu) \le M \quad \text{whenever } \mu\in\mathcal{F}$$

(2) $$v(\phi',\mu) \le M \quad \text{whenever } \mu\in\mathcal{F}.$$

Now by Theorem I.6 there exists $\psi \in R(D)$ satisfying $\|\phi-\psi\|_D < \omega^{-M}$ and $\|\phi'-\psi'\|_D < \omega^{-M}$ and therefore we have

(3) $$v(\phi-\psi,\mu) > M \quad \text{whenever } \mu\in\mathcal{F},$$

together with

(4) $$v(\phi'-\psi',\mu) > M \quad \text{whenever } \mu\in\mathcal{F}.$$

Then (1) and (3) imply

(5) $$v(\phi,\mu) = v(\psi,\mu) \text{ whenever } \mu\in\mathcal{F}$$

while (2) and (4) imply $v(\phi',\mu) = v(\psi',\mu)$ whenever $\mu\in\mathcal{F}$, hence

$$v(\frac{\psi'}{\psi},\mu) = v(\frac{\varphi'}{\varphi},\mu) = v(f,\mu) \text{ whenever } \mu\in\mathcal{F},$$

which proves that $v(\frac{\psi'}{\psi},\mu)$ *is linear in* $\mathcal{F}$,
in the form $q\mu+B$ *with* $q\in\mathbb{Z}$.

Now ψ factorizes in the form $\frac{P}{Q}\,\theta$ where P and Q are monic polynomials that have all of their zeros in $C(0,1) = \{x \mid |x| = 1\}$ and θ belongs to $R(D)$ and has no zero in $C(0,1)$.

Since $v'_d(\varphi,0), v'_d(\varphi,0) \ne v'_g(\varphi,o)$, by (5) we have $v'_d(\psi,0)$ so that P and Q don't have the same number of zeros in $C(0,1)$. Let $P(x) = \sum_{j=o}^{m} \alpha_j x^j$ and let $Q(x) = \sum_{j=o}^{m} \beta_j x^j$. Then $m \ne n$ and

(6) $$\alpha_m = \beta_n = 1.$$

On the other hand, since P and Q have all of their zeros in $C(0,1)$ we see

$$|\alpha_j| \le 1 \text{ whenever } j = 0,\dots,m,$$
$$|\beta_j| \le 1 \text{ whenever } j = 0,\dots,n$$

(7) $$v(P,0) = v(Q,0) = 0.$$

$P'Q - PQ'$ is then a polynomial $\sum_{j=o}^{m+n-1} \lambda_j x^j$ with $|\lambda_j| \le 1$ whenever $j = 0,\dots,m+n-1$ and by (6) we have $\lambda_{m+n} = m-n$, hence $|\lambda_{m+n-1}| = 1$.

Then $v(P'Q-PQ',0) = 0$, hence by (7) we see

$$v\left(\frac{P'Q-PQ'}{PQ},0\right) = 0. \tag{8}$$

Now we will show

$$v\left(\frac{\theta'}{\theta},0\right) > 0 . \tag{9}$$

Since θ has neither any zero nor any pole in $C(0,1)$, there exist r', r'' such that $r'<1<r''$ and such that θ has neither any zero nor any pole in $\Gamma(0,r',r'')$ and therefore θ is equal to a Laurent series $\sum_{-\infty}^{+\infty} a_n x^n$ convergent in $\Gamma(0,r',r'')$. Moreover, there exists $t\in\mathbb{Z}$ such that $|a_t||x|^t > |a_n||x|^n$ whenever $x\in\Gamma(0,r',r'')$. Let us factorize θ in the form $x^t\gamma$. Then in $\Gamma(0,r',r'')$, γ is equal to a Laurent series $\sum_{-\infty}^{+\infty} b_n x^n$ with $b_o = a_t$ and we see that

$$v(\gamma',0) = \inf_{n\in\mathbb{Z}} v(nb_n) = \inf_{n\neq o} v(b_n) > v(b_o) = v(\gamma,0)$$

from which $v(\frac{\gamma'}{\gamma},0)>0$. As $\frac{\theta'}{\theta} = \frac{\gamma'}{\gamma} + \frac{t}{x}$ we see

$$v\left(\frac{\theta'}{\theta},0\right) = v\left(\frac{\gamma'}{\gamma},0\right) + v\left(\frac{t}{x},0\right) = v\left(\frac{\gamma'}{\gamma},0\right) > 0$$

which finishes showing (9).

Now let us consider $\frac{\psi'}{\psi} = \frac{\theta'}{\theta} + \frac{P'Q-PQ'}{PQ}$. By (8) and (9) we have $v(\frac{\theta'}{\theta},0) > v(\frac{P'Q-PQ'}{PQ},0)$ and therefore there exists an interval $U = [-\rho,\rho]$ such that $v(\frac{\theta'}{\theta},\mu) > v(\frac{P'Q-PQ'}{PQ},\mu)$ whenever $\mu\in U$. Then we have

$$v\left(\frac{\psi'}{\psi},\mu\right) = v\left(\frac{P'Q-PQ'}{PQ},\mu\right) \text{ whenever } \mu. \tag{10}$$

Let us put $h(x) = \frac{P'Q-PQ'}{PQ}$. Since $P(x)Q(x)$ has exactly $m+n$ zeros in $C(0,1)$, and $P'Q-PQ'$ has at most $m+n-1$ zeros in $C(0,1)$ we see that

$$v'_d(h,0) > v'_g(h,0). \tag{11}$$

Now by (10) we have $v'_d(\frac{\psi'}{\psi},0) = v'_d(h,0)$ and $v'_g(\frac{\psi'}{\psi},0) = v'_g(h,0)$ hence by (11) we obtain $v'_d(\frac{\psi'}{\psi},0) > v'_g(\frac{\psi'}{\psi},0)$ which contradicts the fact $v(\frac{\psi'}{\psi},\mu)$ is a linear function in $\mathcal{I}$, and that finishes proving Proposition III.E.

<u>Proof of the Theorem III</u> .

Let us assume that $(\mathcal{E})$ admits a non identically null solution g and assume g is not invertible in $H(D)$. By Theorem II.2., g is strictly annulled by a T-filter $\mathcal{F}$ on D and by Proposition III.D., f is not strictly

annulled by $\mathcal{F}$.

For example let us assume $\mathcal{F}$ is increasing of center O, of diameter R and assume first f is annulled by $\mathcal{F}$. Since f is not strictly annuled by $\mathcal{F}$, there exists $r \in]0,R[$ such that $v(f,\mu) = +\infty$ whenever $\mu \in [-\text{Log } R, -\text{Log } r]$ hence (1) $v(g',\mu) = +\infty$ whenever $\mu \in [\text{Log } R, -\text{Log } r]$. But since g is strictly annuled by $\mathcal{F}$, there exists an interval in $\mathcal{J} \subset [\text{Log } R, -\text{Log } r]$ such that $v(g,\mu)$ is a linear non constant function in $\mathcal{J}$, and by Proposition III.C we know that $v(g',\mu) = v(g,\mu) - \mu$, which contradicts (1). Thus, f is not annulled by $\mathcal{F}$.

Now we know the function $\mu \to v(f,\mu)$ is linear in an interval $[-\text{Log } R,\lambda]$ $[E_3,E_4,G]$. By Proposition III.E the function $\mu \to v(g,\mu)$ is also linear in $[-\text{Log } R,\lambda]$ and that contradicts the hypothesis "g strictly annuled by $\mathcal{F}$".

In case $\mathcal{F}$ is decreasing we can perform a similar demonstration, (in taking a center in a spherically complete extension of K if required).

Thus g is invertible in H(D) and then by Theorem II.1. we know that the space of solutions of ($\mathcal{E}$) has dimension 1.

IV. $p \neq 0$ AND f NON QUASI-INVERTIBLE

Here we assume $p \neq 0$ and we put $\omega = p$. We will denote by $\mathcal{E}(f)$ the differential equation $y' = fy$ ($f \in H(D)$) and by $\mathcal{S}(f)$ the space of the solutions of $\mathcal{E}(f)$ in H(D). We will prove there does exist infraconnected clopen bounded sets with elements $f \in H(D)$ annulled by a T-filter $\mathcal{F}$ with non identifically null solutions of $\mathcal{E}$ (f) annulled by $\mathcal{F}$ and such a situation may provide any finite dimension for $\mathcal{S}(f)$ and even an infinite dimension.

<u>Theorem IV.1.</u> *There exists an infraconnected clopen bounded set* Δ *with a* T-*filter* $\mathcal{F}$ *and elements* $f \in H(\Delta)$ *strictly annulled by* $\mathcal{F}$ *such that* $\mathcal{E}(f)$ *has solutions strictly annulled by* $\mathcal{F}$ *together with dim* $\mathcal{S}(f) = 1$.

Theorem IV.1. will be proven in constructing concretely the set Δ and the $f \in H(\Delta)$ in Proposition IV.D.

<u>Theorem IV.2.</u> *Let* $n \in \mathbb{N}$; *there exists an infraconnected clopen bounded set* D *and elements* $f \in H(D)$ *strictly annulled by* n *increasing* T-*filters two by two complementary such that* $\dim(\mathcal{S}(f)) = n$.

Theorem IV.3. *There exists an infraconnected clopen bounded set* D *with a sequence of increasing* T-*filters* $(\mathcal{F})_{n\in\mathbb{N}}$ *two by two complementary, and elements* $f\in H(D)$ *strictly annulled by each one of the* $\mathcal{F}_n$, *such that* $\mathcal{S}(f)$ *is isomorphic to the space of the sequences* $(\lambda_n)_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} \lambda_n = 0$.

The proofs of Theorem 2 and 3 will require the following Lemma IV.A, and Proposition IV.B.

Lemma IV.A. *Assume* D *has a sequence of increasing* T-*filters* $(\mathcal{F}_n)_{n\in\mathbb{N}}$ *such that* $\mathcal{C}(\mathcal{F}_n) \cap \mathcal{C}(\mathcal{F}_m) = 0$ *whenever* $n \neq m$. *Let* $f\in H(D)$ *be such that* $f(x) = 0$ *whenever* $x\in \bigcap_{n\in\mathbb{N}} \mathcal{P}(\mathcal{F}_n)$ *and assume there exists a sequence*$(f_n)_{n\in\mathbb{N}}$ *in* H(D) *such that* $f_n \in \mathcal{J}_o(\mathcal{F}_n)$ *and* $f(x) = f_n(x)$ *whenever* $x\in\mathcal{C}(\mathcal{F}_n)$, *for every* $n\in\mathbb{N}$. *Then the series* $\sum_{n\in\mathbb{N}} f_n$ *converges to* f *in* H(D).

Proof. Suppose first the series Σf_n converges to a limit h in H(D). Then h is clearly equal to f because by definition when $x \in \bigcap_{\cap\in\mathbb{N}} \mathcal{P}(\mathcal{F}_n)$, $f_n(x)=0$ for all n, hence $h(x) = f(x) = 0$, and when $x \in \mathcal{C}(\mathcal{F}_n)$ then $x\in\mathcal{P}(\mathcal{F}_n)$ for all $n \neq q$ hence $h(x) = f_q(x) = f(x)$.

Now we only have to prove the series $\sum_{n\in\mathbb{N}} f_n$ converges and that means to prove the sequence $\|f_n\|_D$ goes to zero. Suppose the sequence $\|f_n\|_D$ does have a subsequence $q\to\|f_{n_q}\| \geq \delta$ whenever $q\in\mathbb{N}$. Consider now the sequence of T-filter $(\mathcal{F}_{n_q})_{q\in\mathbb{N}}$ and for each $q\in\mathbb{N}$ let $a_q\in\mathcal{C}(\mathcal{F}_{n_q})$ be such that $|f(a_q)| \geq \delta$.

Suppose first we can extract a subsequence $m\to a_{q_m}$ such that (1) $|a_{q_m} - a_{q_j}| = r$ whenever $j\neq m$. Let us set $b_m = a_{q_m}$, and $\mathcal{G}_m = \mathcal{F}_{n_q}$;. with no loss of genrality we can obviously assume $b_o = 0$.

By classical results [E_3] we know that $v(f,-\text{Log } C) = v(f(b_m))$ is true for all m except maybe a finite number. Hence we see that $v(f,-\text{Log } C) \leq -\text{Log } \delta$. Let $A = v(f,-\text{Log } r)$. We know that $v(f(x)) = A$ is true in $C(0,r) \cap D$ except (maybe) in a finite number of disks $d^-(b_m,r)$.

Consider an $m\in\mathbb{N}$ such that $v(f(x)) = A$ for $x\in d^-(b_m,r) \cap D$. Then trivially $\lim_{\mathcal{G}_m} v(f(x)) = A < +\infty$ hence that contradicts the hypothesis

$f \in \mathcal{S}(\mathcal{F}_n)$ for all n. Hence the sequence $m \to a_{q_m}$ satisfying (1) does not exist. Then we know we can extract a subsequence $m \to a_{q_m}$ such that the sequence $|a_{q_{m+1}} - a_{q_m}|$ is strictly monotonous, of limit $r[E_3]$.

Let us put again $\mathcal{G}_m = \mathcal{F}_{n_{q_m}}$ $(m \in \mathbb{N})$ and $b_m = a_{q_m}$.

Suppose first $r = 0$. The sequence $m \to b_m$ converges to a limit α in D and we can assume $\alpha = 0$. Since f is continuous and $|f(\alpha_{q_m})| \geq \delta$ there exists $\rho > 0$ such that $|f(x) \geq \frac{\delta}{2}$ for $|x| \leq \rho$ hence $|f(x)| \geq \frac{\delta}{2}$ for $x \in \mathcal{C}(\mathcal{G}_m)$ and that contradicts $f \in \mathcal{S}(\mathcal{F}_n)$ for all $n \in \mathbb{N}$ for all $n \in \mathbb{N}$. Hence $r \neq 0$.

Suppose now the sequence $|b_{m+1} - b_m|$ is strictly increasing. We can obviously assume $b_o = 0$ hence the sequence $|b_m|$ is strictly increasing of limit r.

Consider $v(f,\mu)$ when $\mu > -\log r$. Since $v(f(b_m)) \leq -\log \delta$ we know that there exists N such that $v(f,v(b_m)) = v(f(b_m)) \geq N$ and then $v(f,\mu)$ is upperbounded by $-\operatorname{Log} \delta$ in an interval $[\lambda, -\operatorname{Log} r]$, and there exists $\gamma \in]\lambda, -\operatorname{Log} r[$ such that $v(f(x)) = v(f,v(x)) \leq -\operatorname{Log} \delta$ for all $x \in \Gamma(o,p^{-\gamma},r)$. Then that contradicts the hypothesis $f \in \mathcal{J}(\mathcal{G}_m)$ when $b_m \in \Gamma(o,p^{-\gamma},r)$.

Thus we have contradiction when the sequence $|b_{m+1} - b_m|$ is strictly increasing. We obtain a similar contradiction when this sequence is strictly decreasing of limit $r \neq 0$. Finally that finishes proving Lemma IV.A.

Proposition IV.B.

Assume D *has a sequence of increasing* T-*filters* $(\mathcal{F}_n)_{n \in \mathbb{N}}$ *such that* $\mathcal{C}(\mathcal{F}_n) \cap \mathcal{C}(\mathcal{F}_m) = \emptyset$ *whenever* $n \neq m$. *Assume for each* $i \in \mathbb{N}$ *there exists* $f_n \in \mathcal{J}_o(\mathcal{F}_n)$ *such that* $\mathcal{E}(f_n)$ *has a solution* $g_{n \neq o} \in \mathcal{J}_o(\mathcal{F}_n)$ *and such that the series* $\sum_{n \in \mathbb{N}} f_n$ *converges in* H(D). *Let* $f = \sum_{n \in \mathbb{N}} f_n$. *The family* $\{(g_n) n \in \mathbb{N}\}$ *-is a linearly free family of solutions of* $\mathcal{E}(f)$.

Moreover, if the $(\mathcal{F}_n)_{n \in \mathbb{N}}$ *are the only* T-*filters on* D, *then for every solution* g *of* $\mathcal{E}(f)$ *there exists a unique sequence* $(\lambda_n)_{n \in \mathbb{N}}$ *in* K *such that the series* $\sum_{n \in \mathbb{N}} \lambda_n g_n$ *converges to* g *in* H(D).

Proof. The family $(g_n)_{n \in \mathbb{N}}$ is clearly linearly free because the supports of the g_n are two by two disjointed. Let us show if D has no T-filter but

the $(\mathcal{F}_n)_{n\in\mathbb{N}}$, every solution g of $\mathcal{E}(f)$ is the sum of a series $\sum_{n\in\mathbb{N}} \lambda_n g_n$.

Indeed, let $g\in\mathcal{J}(f)$ and first consider g(x) when $x\in\mathcal{C}(\mathcal{F}_q)$ for certain $q\in\mathbb{N}$

We know $f(x) = f_q(x)$ whenever $x\in\mathcal{C}(\mathcal{F}_q)$ and then every $h\in\mathcal{S}(f)$ has a restriction to $\mathcal{C}(\mathcal{F}_q)$ that is a solution of $(\mathcal{E}_q)$. Since D has no T-filter except the $(\mathcal{F}_n)$, and since the $\mathcal{C}(\mathcal{F}_n)$ are two by two disjointed, one sees $\mathcal{C}(\mathcal{F}_q)$ has no T-filter except $\mathcal{F}_q$. Hence $H(\mathcal{C}(\mathcal{F}_q))$ has no divisor of zero hence by Theorem II.3, the differentiel equation $\mathcal{E}(\bar{f}_q)$ defined in $\mathcal{C}(\mathcal{F}_q)$ by the restriction $\bar{f}_q$ of f_q to $\mathcal{C}(\mathcal{F}_q)$ has a space of solution of dimension 1, hence there exists $\lambda_q \in K$ such that $f(x) = \lambda_q g_q(x)$ whenever $x\in\mathcal{C}(\mathcal{F}_q)$.

Now by hypothesis $g_q \in \mathcal{J}_o(\mathcal{F}_q)$ hence $g\in\mathcal{J}(\mathcal{F}_q)$. We will shaw $g(x) = 0$ whenever $x \in \bigcap_{\cap\in\mathbb{N}} \mathcal{P}(\mathcal{F}_q)$. Indeed let $\Delta = \mathcal{C}(\mathcal{F}_q) \cup (\bigcap_{n\in\mathbb{N}} \mathcal{P}(\mathcal{F}_n))$.
Clearly $\mathcal{F}_q$ is secant to Δ and then Δ has no T-filter complementary to $\mathcal{F}_q$. Hence by Lemma II.B we know that $g\in\mathcal{J}_o(\mathcal{F}_q)$ hence $g(x) = 0$ whenever $x \in \mathcal{P}(\mathcal{F}_q) \cap \Delta$ hence $g(x) = 0$ whenever $x \in \bigcap_{n\in\mathbb{N}} \mathcal{P}(\mathcal{F}_n)$ and then $g(x) = 0$.
By Lemma IV.A. we know that g is the sum of the convergent series $\sum_{n\in\mathbb{N}} \lambda_n g_n$

The sequence (λ_n) is clearly unique. Indeed suppose $g = \sum_{n\in\mathbb{N}} \mu_n g_n$. Then when $x\in\mathcal{C}(\mathcal{F}_n)$ we have $g(x) = \mu_n g_n(x)$ because $g_m(x) = 0$ for all $m \neq n$ hence $\mu_n = \lambda_n$.

Proposition IV.C. *Let* h *be a series convergent in* $d^-(0,1)$ *and let* D *be a clopen bounded infraconnected set containing* 0, *of diameter* $R \geq 1$, *such that* h *and* D *satisfy*

(1) $$h(x) \neq 0 \text{ for all } x\in D \cap d^-(0,1)$$

(2) $$\lim_{\substack{|x|\to 1^- \\ x\in D}} |h(x)| = +\infty$$

(3) $$\lim_{\substack{|x|\to 1^- \\ x\in D}} \frac{h'(x)}{h(x)} = 0$$

Let φ *and* f *be the functions defined in* D *by* $\varphi(x) = \frac{1}{h(x)}$, $f(x) = -\frac{h'(x)}{h(x)}$ *whenever* $x\in D \cap d^-(0,1)$. $\varphi(x) = f(x) = 0$ *whenever* $x\in D \setminus d^-(0,1)$. *Then* φ *and* f *belong to* H(D) *and* φ *satisfies* $\mathcal{E}(f)$.

Proof. Since (2) we know that φ belongs to H(D) by results of $[S_2]$. In the same way, since $\lim_{|x|\to 1^-} (\frac{h'(x)}{h(x)}) = 0$ we know that f belongs to H(D) $[S_4]$.

Then φ obviously satisfies ($\mathcal{E}(f)$).

The following Proposition IV.D clearly proves Theorem IV.1 in showing how constructing D, f and the solutions of $\mathcal{E}(f)$.

Proposition IV.D.

Let (a_n) *be the sequence defined as follows :*

$a_o \neq 0$,

$a_n = 1$ *when* n *is neither* 0 *nor in the form* p^m *for any* $m \in \mathbb{N}$

$a_n = \frac{1}{n}$ *when* n *is in the form* p^m *for some* $m \in \mathbb{N}$.

Let $h(x) = \sum_{n=o}^{\infty} a_n x^n$; *then* h *is convergent in* $d^-(0,1)$.

Let Δ *be the set of the* $x \in d^-(0,1)$ *such that* $v(h(x)) = v(h,v(x))$ *and assume* $D \cap d^-(0,1) = \Delta$. *Then*

(1) $h(x) \neq 0$ for all $x \in D$

(2) $$\lim_{\substack{|x| \to 1^- \\ x \in D}} |h(x)| = +\infty$$

(3) $$\lim_{\substack{|x| \to 1^- \\ x \in D}} \frac{h'(x)}{h(x)} = 0$$

Moreover the increasing filter $\mathcal{F}$ *of center* 0, *of diameter 1 is a* T-*filter and it is the only one* T-*filter on* Δ.

In addition, the function f *and* φ *defined in* Δ *by* $f(x) = -\frac{h'(x)}{h(x)}$ and $\varphi = \frac{1}{h}$ *belong to* $H(\Delta)$ *and they both are strictly annulled by* $\mathcal{F}$; φ *is a solution of* $\mathcal{E}(f)$ *and it generates* $\mathcal{S}(f)$.

Proof. First, it is easily seen that h converges for $|x|<1$ because we verify that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

Next, it is well known [A,S_2] that the relation $v(h(x)) = v(h,v(x))$ is true in all $d^-(0,1)$ except in a set included in a union of circles in the form $\bigcup_{n=1}^{\infty} C(0,r_n)$ with $\lim_{n\to\infty} r_n = 1$. This set Δ is obviously infraconnected and clopen.

As $h(0) = a_o \neq 0$, (1) is satisfied by definition of Δ because $v(h(x)) = v(h,v(x)) < +\infty$, for all $x \in \Delta$.

As the sequence $|a_n|$ is not bounded, we know that $\lim_{\mu\to 0^+} v(h,\mu) = +\infty$ [A,S_3], hence Relation (2) is clearly satisfied.

Now h' is bounded in $d^-(0,1)$. Indeed $h'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$ with $|na_n| = |n| \leq 1$ when n is not a p^m and $|p^m a_{p^m}| = p^{-m}p^m = 1$ hence $|na_n| \leq 1$ whenever $n \in \mathbb{N}$. Finally $|h'(x)| \leq 1$ in $d^-(0,1)$ hence (3) is clearly satisfied.

On the other hand, by definition f, φ, φ' are strictly annulled by the increasing filter $\mathcal{F}$ of center 0 of diameter 1. Hence $\mathcal{F}$ is a T-filter. By definition of Δ it is easily seen that Δ has no T-filter different from $\mathcal{F}$.

By Proposition IV.C, φ and f belong to $H(\Delta)$ and by construction they are strictly annulled by $\mathcal{F}$. Hence φ is obviously a solution of $\mathcal{E}(f)$.

On the other hand, since $\mathcal{F}$ is the only one T-filter on Δ , $H(\Delta)$ has no divisor of zero [E_5] hence $\mathcal{S}(f)$ has dimension ≤ 1. As $\varphi \neq 0$, we see that φ generates $\mathcal{S}(f)$, and that finishes proving Proposition IV.D.

Proof of Theorem IV.2. Consider again the set Δ obtained in Proposition IV.D. Let R be>1, and let $a_1,\ldots,a_n$ be points in $d(0,R)$ such that $|a_i - a_j| \geq 1$ whenever $i \neq j$. For all $i = 1,\ldots,n$ let $\Delta_i = a_i + \Delta = \{a_i + x \mid x \in \Delta\}$, let $D_i = (d(0,R) \setminus d^-(a_i,1)) \cup \Delta_i$ and let $D = \bigcap_{i=1}^{n} D_i$.

By Proposition IV.D, the set D has the increasing filters $\mathcal{F}_i$ of center a_i, of diameter 1 (obtained from $\mathcal{F}$ by translation of a_i). Hence for every $i = 1,\ldots,n$ there exists $f_i \underset{\neq 0}{\in} \mathcal{J}_o(\mathcal{F}_i)$ such that $\mathcal{E}(f_i)$ admits solutions $g_i \underset{\neq 0}{\in} \mathcal{J}_o(\mathcal{F}_i)$ strictly annulled by $\mathcal{F}_i$. Since $\mathcal{F}_i$ has diameter 1 and $|a_i - a_j| \geq 1$ for $i \neq j$ the $\mathcal{C}(\mathcal{F}_i)$ satisfy $\mathcal{C}(\mathcal{F}_i)(1 \leq i \leq n)$.

Now let $f = \sum_{i=1}^{n} f_i$. Then it is easily seen that

$$f(x) = f_i(x) \quad \text{whenever} \quad x \in \mathcal{C}(\mathcal{F}_i)$$

$$f(x) = 0 \quad \text{whenever} \quad x \in \bigcap_{i=1}^{n} \mathcal{P}(\mathcal{F}_i).$$

For each $i = 1,\ldots,n$, g_i is then solution of $\mathcal{E}(f_i)$ and therefore it is solution of $\mathcal{E}(f)$.

At last, by Proposition IV.B. the $(g_i)_{1 \leq i \leq n}$ are linearly independant. As $\mathcal{F}_1,\ldots,\mathcal{F}_n$ are the only T-filters on D, $g_1,\ldots,g_n$ is then a base of the space of solutions by the last assertion of Proposition IV.B.

Proof of Theorem IV.3. The proof will roughly follow the same process as in Theorem IV.2, with a sequence of filters $\mathcal{F}_n$ instead of a finite number

of $(\mathcal{F}_1)$. Let R be >1 and consider a sequence $(a_n)_{n\in\mathbb{N}}$ in d(O,R) such that $|a_n-a_n| = 1$ for $n \neq m$. For every $n\in\mathbb{N}$ let h_n be a Taylor series defined like h in Proposition IV.D. in specifying here $h_n(O) = p^{-n}$, and let Δ_n be the set of the $x\in d^-(O,1)$ such that $v(h_n,v(x))$. Then $v(h_n,\mu)\leq -n$ for all $\mu>O$ hence $v(h_n(x)) \leq -n$ and therefore $\left|\frac{h_n'(x)}{h_n(x)}\right| \leq p^{-n}$ whenever $x\in\Delta_n$.

Now we set $\Delta_n = a_n + \Delta_n$ and $B = d(O,R)\setminus(\bigcup_{n=1}^{\infty} d^-(a_n,n))$ and $D = B\cup(\bigcup_{n=1}^{\infty} \Delta_n)$ Obviously B has no T-filter, each Δ_n has an increasing T-filter $\mathcal{F}_n$ of center a_n of diameter 1. Each $\mathcal{F}_n$ induces a T-filter on D and we will still denote it by $\mathcal{F}_n$ and D has no T-filter other than the $\mathcal{F}_n$.

Now we put $f_n(x) = -\frac{h_n'(x-a_n)}{h_n(x-a_n)}$ for $x\in\Delta_n$ and $f_n(x) = O$ for $x\in D\setminus\Delta_n$. By (1) we have $\|f_n\|_D = p^{-n}$ and then the series $\sum_{n=1}^{\infty} f_n$ converges to a limit $f\in H(D)$ such that $f(x) = f_n(x)$ whenever $x\in\Delta_n$ and $f_n(x) = O$ whenever $x\in D\setminus(\bigcup_{n=1}^{\infty}\Delta_n)$.

By Proposition IV.C. for each $n\in\mathbb{N}^*$, there exists a solution $g_n\in(\mathcal{S}(f_n) \cap \mathcal{J}_o(\mathcal{F}_n))\setminus\{O\}$ and we can obviously choose the g_n to satisfy (2) $\|g_n\|_D = 1$. By Proposition IV.B the set $\{g_n|n\in\mathbb{N}\}$ is linearly free and every solution g of $\mathcal{E}(f)$ may be written of a unique matter in the form $\sum_{n=1}^{\infty}\lambda_n g_n$ (with $\lambda_n\in K$).

$\mathcal{S}(f)$ is then isomorphic to a subspace of $K^{\mathbb{N}}$. On the other hand by (2) a séries $\sum_{n=1}^{\infty}\lambda_n g_n$ converges in H(D) if and only if $\lim_{n\to\infty}\lambda_n = O$. If (λ_n) is such a sequence, the series $\sum_{n=1}^{\infty}\lambda_n g_n$ converges to a limit g. Thus $\mathcal{E}(f)$ is isomorphic to the space of the sequences $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\lambda_n = O$ and that ends the proof of Theorem IV.3.

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