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# $p$-adic analysis and operations in $K$-theory

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I want to explain how certain ideas from $p$-adic analysis can be useful to topologists. The results which I will talk about could be formulated (and some have been) without using the language of $p$-adic analysis, but I claim that they are clearer when viewed this way and one is led directly to explicit formulas.

The seminar will fall into three parts. In the first I will describe some algebraic results and problems. In the second I will say briefly why this algebra is of interest to topologists, and in the third I will show that some elementary $p$-adic analysis can help us to understand what is going on here.

Few of the results presented here are new; I have merely tried to thread together a number of known results. The identification of the ring of $p$-adic $K$-theory operations which appears at the end of this paper has not, to my knowledge, been stated in quite this form before, but Ravenel [**11**], especially theorem 7.6, and Bousfield [**3**] section 5 have very closely related statements. I would like to thank Andy Baker for a number of useful conversations around these ideas over several years.

## NUMERICAL AND STABLY NUMERICAL POLYNOMIALS

Let $A$ denote $\{ h(w) \in \mathbf{Q}[w] : h(\mathbf{Z}) \subset \mathbf{Z} \}$, which we refer to as the ring of *numerical* polynomials.

There are some very celebrated elements in $A$, in particular:

$$\binom{w}{i} = \frac{w(w-1)\dots(w-i+1)}{i!} \qquad \text{(Pascal)},$$

and, if $p$ is prime,

$$\frac{w^p - w}{p}, \qquad \text{(Fermat)}.$$

I would like to thank Lionel Schwartz for arranging my very pleasant visit to Paris, and the British Council and the C. N. R. S. for their financial support.

In fact $1$, $w$, $\binom{w}{2}$, $\binom{w}{3}$, ... form a basis for $A$ (as abelian group). This goes back to Newton. To express, for example, $\frac{w^5-w^3}{24} \in A$ in terms of the basis we form the *difference table*:

| $w$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $\frac{w^5-w^3}{24}$ | 0 | 0 | 1 | 9 | 40 | 125 |
| | 0 | 1 | 8 | 31 | 85 | |
| | 1 | 7 | 23 | 54 | | |
| | 6 | 16 | 31 | | | |
| | 10 | 15 | | | | |
| | | 5 | | | | |

Therefore $\frac{w^5-w^3}{24} = \binom{w}{2} + 6\binom{w}{3} + 10\binom{w}{4} + 5\binom{w}{5}$.

In general, if $\Delta h(w) = h(w+1) - h(w)$, the coefficient of $\binom{w}{i}$ in $h(w)$ is $\Delta^i h(w)\big|_{w=0} = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} h(j)$.

Let us form

$$A[w^{-1}] = \{ h(w) \in \mathbf{Q}[w, w^{-1}] : w^m h(w) \in A, \text{ for some } m \geq 0 \},$$
$$= \{ h(w) \in \mathbf{Q}[w, w^{-1}] : h(\tfrac{1}{n}) \in \mathbf{Z}[\tfrac{1}{n}], \text{ for all } n \in \mathbf{Z} \setminus \{0\} \},$$

which we call the ring of *stably numerical* polynomials. Notice that $\{w^{-m}\binom{w}{i} : m, i \geq 0\}$ spans $A[w^{-1}]$ but does not form a basis.

Continuing our list of elements, we have in $A[w^{-1}]$

$$\frac{w^{\phi(k)} - 1}{k}, \qquad \text{(Euler)}$$

and hence

$$\frac{w^n - 1}{m(n)},$$

where $m(n)$ is twice the least common multiple of those $k$ such that $\phi(k)$ divides $n$. There is an extra 2 because of the structure of the 2-adic units.

In fact

$$\nu_p(m(n)) = \begin{cases} 1 + \nu_p(n), & \text{if } p \text{ is odd and } p-1 \text{ divides } n, \\ 0, & \text{if } p \text{ is odd and } p-1 \text{ does not divide } n, \end{cases}$$

$$\nu_2(m(n)) = \begin{cases} 2 + \nu_2(n), & \text{if } n \text{ is even}, \\ 1, & \text{if } n \text{ is odd}. \end{cases}$$

Since $m(n)$ is the denominator of $\frac{B_n}{2n}$, where $B_n$ is the Bernoulli number, (Von Staudt) equivalently we have

$$\frac{B_n}{2n}(w^n - 1) \in A[w^{-1}].$$

Another way to generate these elements is via the following result.

THEOREM 1. *There is a linear homomorphism*

$$\kappa : A \to A[w^{-1}],$$

*such that* $\kappa(w^{n-1}) = (-1)^n \frac{B_n}{n}(w^n - 1)$, $n \geq 1$.

The classical Kummer congruences all derive from this map.

One may prove this theorem as follows. We show that $\kappa = \varphi \circ \beta$, and this is just a matter of calculation, where $\beta : A \to A$ sends $\binom{w}{k}$ to $(-1)^k \left( \binom{w}{k+1} - \binom{w}{k} \right)$, and $\varphi : A \to A[w^{-1}]$ sends $\binom{w}{k}$ to the expression $N_k(w^{-1}\binom{w}{2}, \ldots, w^{-1}\binom{w}{k+1})$. Here $N_k$ is the Newton polynomial which expresses the power sums in terms of the elementary symmetric functions. The roots of this proof lie in topology [4].

There is also a $p$-local version:

THEOREM 2. *There is a linear homomorphism*

$$\kappa_p : A_{(p)}[w^{-1}] \cap \mathbf{Q}[w] \to A_{(p)}[w^{-1}] \cap \mathbf{Q}[w],$$

*such that* $\kappa_p(w^{n-1}) = (-1)^n(1 - p^{n-1})\frac{B_n}{n}(w^n - 1)$, $n \geq 1$.

This follows from the existence of Mazur's Bernoulli measures, see page 200 of [8] where the result is stated almost in this form. However one may prove it along the lines of the previous theorem and then Mazur's measures are an immediate consequence.

THEOREM 3, [2]. $A[w^{-1}]$ *is a free abelian group.*

PROOF: Let $F(n, m)$ denote the subgroup of $A[w^{-1}]$ rationally generated by $w^n$, $w^{n+1}$, ..., $w^{m+1}$. Since $F(0, 0)$ is a copy of $\mathbf{Z}$, generated by 1, it is sufficient to show that each extension $F(n, m) \subset F(n, m + 1)$ and $F(n, m) \subset F(n - 1, m)$ is split. For then we can go up the chain

$$F(0,0) \subset F(0,1) \subset F(-1,1) \subset F(-1,2) \subset F(-2,2) \subset \ldots$$

building a basis.

There are automorphisms of $A[w^{-1}]$ given by $h(w) \mapsto w^n h(w)$ and $h(w) \mapsto h(w^{-1})$ which show that we need only consider the extension $F(1,m) \subset F(0,m)$.

Now $F(0,m) = \{ h(w) \in \mathbf{Q}[w] : h(w) \in A[w^{-1}], \quad \deg h(w) \leq m \}$. Define $\pi : F(0,m) \to \mathbf{Q}$, by $\pi : h(w) \mapsto h(0)$, then $F(1,m) = \ker \pi$. We need to show that $\operatorname{Im} \pi \subset \frac{1}{N}\mathbf{Z}$ for some natural number $N$.

For each prime $p$ choose distinct integers $k_0, k_1, \ldots, k_m$, all of which are non-zero modulo $p$, so that if $h(w) \in A[w^{-1}]$ we have $h(k_j) \in \mathbf{Z}_{(p)}$ for each $j = 0, 1, \ldots, m$. Thus, writing $h(w) = \sum_{i=0}^{m} a_i w^i \in F(0,m)$,

$$ ({k_j}^i) \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{pmatrix} \in {\mathbf{Z}_{(p)}}^{m+1}. $$

Therefore $\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{pmatrix} \in ({k_j}^i)^{-1} {\mathbf{Z}_{(p)}}^{m+1}$. In particular $\pi h = a_0$ belongs to $\dfrac{1}{\det({k_j}^i)} \mathbf{Z}_{(p)}$ for all $h(w) \in F(0,m)$.

If $p \geq m+2$, we can set $k_j = j+1$ then $\det\left( (j+1)^i \right)$ is not divisible by $p$ (Vandermonde) so that $\pi h \in \mathbf{Z}_{(p)}$. This completes the proof.

COROLLARY 4. $\operatorname{Hom}(A[w^{-1}], \mathbf{Z})$ *is uncountable.*

It is this group in which I am interested, for reasons which I will now explain.

## WHY IS THIS ALGEBRA OF INTEREST TO TOPOLOGISTS?

Consider the contravariant functor $K$ : **Top** $\to$ **Ab** defined as the Grothendieck group of the semi-group of complex vector bundles on a space. The natural transformations $\varphi_X : K(X) \to K(X)$ of this functor (necessarily additive since we are thinking of the functor as taking values in the category of abelian groups) are determined by a *universal example* $X = \mathbf{CP}^\infty$, infinite dimensional complex projective space: $K(\mathbf{CP}^\infty) = \mathbf{Z}[[t]]$, where $1 + t =$ [Hopf bundle] and $\varphi$ is uniquely determined by $\varphi_{\mathbf{CP}^\infty}(1+t) \in \mathbf{Z}[[t]]$. This is essentially Atiyah's splitting principle.

For example the Adams operation $\psi^k$ correspond to the series $(1+t)^k$, $k \in \mathbf{Z}$.

But notice that composition of operations does *not* correspond to multiplication in the ring $\mathbf{Z}[[t]]$, for example $\psi^k \psi^l = \psi^{kl}$ not $\psi^{k+l}$.

However from the point of view of *stable* homotopy theory we need more.

The theorem of Bott periodicity says that there is a split short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(X) & \longrightarrow & K(X\times S^2) & \underset{p^*}{\overset{i^*}{\rightleftarrows}} & K(X) & \longrightarrow & 0 \\ & & [E] & \longmapsto & [E\otimes \mathrm{Hopf}] & & & & \\ & & & & X\times S^2 & \xleftarrow{\ i\ } & X & & \\ & & & & x\times * & \longmapsto & x & & \end{array}$$

This enables us to construct the functors $K^i(\ )$ for all $i \in \mathbf{Z}$, with $K^i \cong K^{i+2}$, which fit together to form a generalised cohomology theory. We need to know about the natural transformations of this theory. This means considering those natural transformations $\varphi$ of $K$ such that there exists $\psi$ with the diagram

$$\begin{array}{ccccc} K(X) & \longrightarrow & K(X\times S^2) & \longrightarrow & K(X) \\ \downarrow{\scriptstyle \varphi_X} & & \downarrow{\scriptstyle \psi_{X\times S^2}} & & \downarrow{\scriptstyle \psi_X} \\ K(X) & \longrightarrow & K(X\times S^2) & \longrightarrow & K(X) \end{array}$$

commuting.

Suppose that $\varphi_{\mathbf{CP}^\infty}(1+t) = f(t) \in \mathbf{Z}[[t]]$, $\psi_{\mathbf{CP}^\infty}(1+t) = g(t)$, then, as we shall see in a moment, $f(t) = (1+t)g'(t)$. This extends $\varphi$ to $K^{-1}$ and $K^{-2}$.

For example if $\varphi$ is the Adams operation $\psi^k$, we need $(1+t)^k = (1+t)g'(t)$ so $g(t) = \frac{1}{k}(1+t)^k + \text{constant}$, but $\frac{1}{k}(1+t)^k \notin \mathbf{Z}[[t]]$ unless $k = \pm 1$.

More generally, writing $Tg(t) = (1+t)g'(t)$ we need $g_n(t)$ for all $n \geq 0$ such that $f(t) = T^n g_n(t)$.

Now there is a dual homology theory $K_i(\ )$, and it turns out, see [5] or [12], that $K_0(\mathbf{CP}^\infty) = A$, the ring of numerical polynomials, with the duality $K^0(\mathbf{CP}^\infty) = \mathrm{Hom}(K_0(\mathbf{CP}^\infty), \mathbf{Z})$ working as follows: the series $\sum a_i t^i \in \mathbf{Z}[[t]]$ corresponds to the homomorphism $\binom{w}{i} \mapsto a_i$.

If we want to know how an operation corresponding to $f(t) \in \mathbf{Z}[[t]]$ acts on $K^0(S^{2n})$ the answer is that it is multiplication by the result of evaluating the corresponding homomorphism on $w^n \in A$. In terms of the coefficients of the series $f(t)$ this gives formulas involving Stirling numbers of the second kind, see [5].

Moreover if $\varphi : A \to \mathbf{Z}$, $\binom{w}{i} \mapsto a_i$ is the restriction of $\psi : w^{-1}A \to \mathbf{Z}$, $w^{-1}\binom{w}{i} \mapsto b_i$, the formula

$$\binom{w}{i} = iw^{-1}\binom{w}{i} + (i+1)w^{-1}\binom{w}{i+1} \tag{5}$$

shows that $f(t) = (1+t)g'(t) = Tg(t)$, where $f(t) = \sum a_i t^i$, and $g(t) = \sum b_i t^i$.

This shows that if $B = A[w^{-1}] \cap \mathbf{Q}[w] \supset A$ the group $\operatorname{Im} T^\infty \subset \mathbf{Z}[[t]]$ corresponds to $\operatorname{Hom}(B, \mathbf{Z}) \subset \operatorname{Hom}(A, \mathbf{Z})$.

Note that $B/A$ is a torsion group (In fact for all $h(w) \in \mathbf{Q}[w] \supset B$ there exists $d \in \mathbf{Z}$ such that $dh(w) \in \mathbf{Z}[w] \subset A$.) so that we have a monomorphism $\operatorname{Hom}(B, M) \to \operatorname{Hom}(A, M)$ for any torsion-free $M$.

To be more precise we need to introduce another theory $k^*(\ )$, known as *connective* $K$-theory, and, in summary, we can identify:

$$\begin{aligned} \operatorname{Hom}(A, \mathbf{Z}) &\text{ with the natural transformations of } K(\ ), \\ \operatorname{Hom}(B, \mathbf{Z}) &\text{ with the natural transformations of } k^*(\ ), \\ \operatorname{Hom}(A[w^{-1}], \mathbf{Z}) &\text{ with the natural transformations of } K^*(\ ), \end{aligned}$$

where in the last two cases we are considering only natural transformations of degree zero.

At the moment we understand only the first of these three groups.

The point about introducing the new theory $k^*(\ )$ is that in the above sketch we have considered only extending a natural transformation of $K^0(\ )$ to $K^{-i}(\ )$ for $i > 0$. For $k^*(\ )$ this is sufficient, but for $K^*(\ )$ we need to go the other way too. It will turn out that over the $p$-adic integers the distinction disappears.

The natural map $\operatorname{Hom}(A[w^{-1}], \mathbf{Z}) \to \operatorname{Hom}(B, \mathbf{Z})$ is a monomorphism. We can see this as follows. Suppose that $\varphi : A[w^{-1}] \to \mathbf{Z}$ restricts to zero on $B$, and thus on all non-negative powers of $w$, then since

$$w^{-1}\left(\frac{1 - w^{p-1}}{p}\right)^n \in A[w^{-1}]$$

for all $n \geq 0$, we see that $\varphi(w^{-1})$ is infinitely divisible by $p$, and hence must be zero. This argument may now be repeated to show that $\varphi(w^{-2}) = 0$, and inductively we see that $\varphi = 0$.

We remarked earlier that the group $\operatorname{Hom}(A[w^{-1}], \mathbf{Z})$ is uncountable, since $A[w^{-1}]$ is free on countably many generators. However the only

explicit examples we know are those of the form $a\psi^1 + b\psi^{-1} : h(w) \mapsto ah(1) + bh(-1)$, where $a, b \in \mathbf{Z}$.

As always in topology we are prepared to introduce coefficients. Preferably our coefficients should lie in a ring $R$ so that the multiplicative structure is preserved. If we study $K$-theory with coefficients in $R$ the abelian group $A$ is replaced by the $R$-module $R \otimes A$, and $\mathrm{Hom}(A, \mathbf{Z})$ by $\mathrm{Hom}_R(R \otimes A, R)$ which is isomorphic to $\mathrm{Hom}(A, R)$. Similarly we need to consider $\mathrm{Hom}(A[w^{-1}], R)$ and $\mathrm{Hom}(B, R)$.

In $\mathrm{Hom}(A[w^{-1}], \mathbf{Z}_{(p)})$ we have $h(w) \mapsto h(k)$ for any $k \in \mathbf{Z}_{(p)}^{\times}$. This is the Adams operation $\psi^k$; it acts on $K^0(S^{2n}; \mathbf{Z}_{(p)})$ by multiplication by $k^n$, $n \in \mathbf{Z}$.

There is a theorem of Madsen, Snaith and Tornehave [10] which may help us identify $\mathrm{Hom}(B, \mathbf{Z}_{(p)}) \cong \mathrm{Im}\, T^{\infty}$, where now $T : \mathbf{Z}_{(p)}[[t]] \to \mathbf{Z}_{(p)}[[t]]$. Their theorem tells us that $\varphi : K(\ ; \mathbf{Z}_{(p)}) \to K(\ ; \mathbf{Z}_{(p)})$ extends to a natural transformation of $k^*(\ ; \mathbf{Z}_{(p)})$ if and only if $\varphi$ commutes with transfer.

Let me explain what this means. There are spaces $X_r$, constructed in the following way. The space $\mathbf{CP}^{\infty}$ is defined as $S^{\infty}/U(1)$, where $S^{\infty}$ is the unit sphere in $\mathbf{C}^{\infty}$. Let $C_{p^r}$ denote the cyclic subgroup $\{ z \in U(1) : z^{p^r} = 1 \}$, then $X_r = S^{\infty}/C_{p^r}$, so that there is a fibering $X_r \to \mathbf{CP}^{\infty}$ and a $p$-fold covering $X_r \to X_{r+1}$. This covering defines a map

$$\mathrm{tr} : K(X_r) \to K(X_{r+1})$$

which may be defined by sending the class $[E]$, where $E$ is a vector bundle over $X_r$, to the class represented by the bundle over $X_{r+1}$ which has as its fibre over $x$ the direct sum of the fibres of $E$ over the $p$ points of $X_r$ which cover $x$.

Madsen, Snaith and Tornehave's theorem says that $\varphi$ extends to a stable operation if and only if $\varphi_{X_r}$ and $\varphi_{X_{r+1}}$ commute with tr for all $r \geq 0$.

It is standard topology that

$$K(X_r) \cong \frac{\mathbf{Z}[[t]]}{((1+t)^{p^r} - 1)}$$

and an easy calculation that $\psi^k$ commutes with transfer if and only if $p$ does not divide $k$, see [1], page 180. The various proofs of the theorem of Madsen, Snaith and Tornehave then depend on trying to consider the closure of these Adams operations in some sense.

Now a simple calculation shows that

$$\frac{\mathbf{Z}[[t]]}{((1+t)^{p^r} - 1)} \cong \mathbf{Z} \oplus s\mathbf{Z}_p \oplus s^2\mathbf{Z}_p \oplus \cdots \oplus s^{p^r-1}\mathbf{Z}_p,$$

where $s = 1+t$, which suggests strongly that we should introduce $p$-adic coefficients.

## $p$-ADIC MEASURE THEORY

Consider the space $C(\mathbf{Z}_p, \mathbf{Z}_p)$ of continuous functions from the ring of $p$-adic integers to itself.

THEOREM 6 (MAHLER).

$$C(\mathbf{Z}_p, \mathbf{Z}_p) = \left\{ w \mapsto \sum_{j \geq 0} c_j \binom{w}{j} : c_j \to 0 \right\} \cong c_0(\mathbf{Z}_p).$$

Note that the coefficients $c_j$ may be obtained by Newton interpolation in the same way as we expand elements of $A$ in terms of the binomial coefficient basis.

Let

$$\begin{aligned} M(\mathbf{Z}_p, \mathbf{Z}_p) &= \mathrm{Hom}^c(C(\mathbf{Z}_p, \mathbf{Z}_p), \mathbf{Z}_p), \\ &= l^\infty(\mathbf{Z}_p) \cong \mathbf{Z}_p[[t]], \end{aligned}$$

denote the space of $\mathbf{Z}_p$-valued measures on $\mathbf{Z}_p$ and if $\mu \in M(\mathbf{Z}_p, \mathbf{Z}_p)$ with $h \in C(\mathbf{Z}_p, \mathbf{Z}_p)$ write

$$\mu(h) = \int_{\mathbf{Z}_p} h(w)\, d\mu(w).$$

If $f(t) = \sum_{i \geq 0} a_i t^i \in \mathbf{Z}_p[[t]]$, write $\mu_f \in M(\mathbf{Z}_p, \mathbf{Z}_p)$ for the corresponding measure, which is determined by

$$\int_{\mathbf{Z}_p} \binom{w}{i} d\mu_f(w) = a_i.$$

LEMMA 7.

$$\int_{\mathbf{Z}_p} (1+x)^w\, d\mu_f(w) = f(x) \text{ if } p \text{ divides } x.$$

The proof is nearly formal, see [9] page 98.

We can see now that $\mathrm{Hom}(A, \mathbf{Z}_p) = M(\mathbf{Z}_p, \mathbf{Z}_p)$, with the same identification with $\mathbf{Z}_p[[t]]$.

What does $\psi^k$ correspond to under this identification? The series is $(1+t)^k$ so that $\int_{\mathbf{Z}_p} \binom{w}{i} d\psi^k(w) = \binom{k}{i}$, or $\int_{\mathbf{Z}_p} h(w)\, d\psi^k(w) = h(k)$, that is point measure at $k \in \mathbf{Z}_p$.

It is clear now that the closure we want, the closure of $\{ \psi^k : k \in \mathbf{Z}_p^\times \}$, meaning the smallest closed $\mathbf{Z}_p$-submodule containing this set, is the submodule $M(\mathbf{Z}_p^\times, \mathbf{Z}_p) \subset M(\mathbf{Z}_p, \mathbf{Z}_p)$ of measures whose support lies in $\mathbf{Z}_p^\times$.

In fact, of course, $M(\mathbf{Z}_p, \mathbf{Z}_p) = M(\mathbf{Z}_p^\times, \mathbf{Z}_p) \oplus M(p\mathbf{Z}_p, \mathbf{Z}_p)$, since $\mathbf{Z}_p$ is the disjoint union of $\mathbf{Z}_p^\times$ and $p\mathbf{Z}_p$; and we can be precise about this splitting. A measure is restricted to a subset $X$ by multiplying by $\chi_X$, the characteristic function of $X$.

So if $f(t) = \sum_{i \geq 0} a_i t^i \in \mathbf{Z}_p[[t]]$ and $\mu_f$ is the corresponding measure let us compute $\int_{p\mathbf{Z}_p} h(w)\, d\mu_f(w)$. (I will assume from now on that $p$ is an odd prime. The formulas are almost the same if $p = 2$.)

By Newton interpolation the coefficient of $\binom{w}{j}$ in $h(w)\chi_{p\mathbf{Z}_p}(w)$ is

$$\sum_{s=0}^{j} (-1)^{j-s} \binom{j}{s} h(s) \chi_{p\mathbf{Z}_p}(s) = \sum_{r=0}^{\lfloor j/p \rfloor} (-1)^{j+r} \binom{j}{rp} h(rp).$$

Therefore

$$\int_{p\mathbf{Z}_p} h(w)\, d\mu_f(w) = \sum_{j \geq 0} \sum_{r=0}^{\lfloor j/p \rfloor} (-1)^{j+r} \binom{j}{rp} h(rp) a_j,$$

and $\mu_f|_{p\mathbf{Z}_p}$ will be zero (so that $f(t)$ corresponds to a measure whose support is in $\mathbf{Z}_p^\times$) if and only if $\int_{p\mathbf{Z}_p} \binom{w}{i}\, d\mu_f(w) = 0$ for all $i \geq 0$, that is

$$\sum_{j \geq i} \sum_{i \leq rp \leq j} (-1)^{j+r} \binom{j}{rp} \binom{rp}{i} a_j = 0,$$

for all $i \geq 0$.

Now if $f(t) = Tg(t) = (1+t)g'(t)$ it is easy to verify that

$$\int_{\mathbf{Z}_p} h(w)\, d\mu_f(w) = \int_{\mathbf{Z}_p} wh(w)\, d\mu_g(w).$$

In fact we have already made the relevant calculation in formula (5). So, in general,

$$\int_{\mathbf{Z}_p} h(w)\, d\mu_{T^n g}(w) = \int_{\mathbf{Z}_p} w^n h(w)\, d\mu_g(w).$$

Thus if $f \in \operatorname{Im} T^n$ it follows that $\int_{p\mathbf{Z}_p} h(w)\, d\mu_f(w)$ is divisible by $p^n$ for all $h(w)$. This shows that if $f \in \operatorname{Im} T^\infty$ we have $\int_{p\mathbf{Z}_p} h(w)\, d\mu_f(w) = 0$ for all $h(w)$ and $\mu_f$ is supported in $\mathbf{Z}_p^\times$.

To prove the converse, and illuminate the situation, note that

$$\int_{\mathbf{Z}_p} (1+t)^w\, d\mu_f(w) = f(t) \in \mathbf{Z}_p[[t]].$$

This is almost formal and is an extension of lemma 7, using the $(p,t)$-adic topology on $\mathbf{Z}_p[[t]]$.

Now $T(1+t)^w = w(1+t)^w$, so that if $\mu_f$ has its support in $\mathbf{Z}_p^\times$ we may define

$$g_n(t) = \int_{\mathbf{Z}_p} \frac{(1+t)^w}{w^n}\, d\mu_f(w),$$

and $f(t) = T^n g_n(t)$.

Notice that having identified $M(\mathbf{Z}_p^\times, \mathbf{Z}_p)$ with $\operatorname{Hom}(B, \mathbf{Z}_p)$ there is no difficulty in extending any homomorphism $B \to \mathbf{Z}_p$ to $A[w^{-1}] \supset B$. We simply evaluate $\int_{\mathbf{Z}_p^\times} h(w)\, d\mu(w)$.

However this does not work with $\mathbf{Z}_{(p)}$-coefficients. In fact we have the following commutative diagram, in which the arrows are all proper inclusions,

$$\begin{array}{ccc} \operatorname{Hom}(B, \mathbf{Z}_p) & = & \operatorname{Hom}(A[w^{-1}], \mathbf{Z}_p) \\ \uparrow & & \uparrow \\ \operatorname{Hom}(B, \mathbf{Z}_{(p)}) & \longrightarrow & \operatorname{Hom}(A[w^{-1}], \mathbf{Z}_{(p)}) \end{array}$$

This phenomenon is also discussed in [6].

Since we are assuming $p$ is an odd prime, $\mathbf{Z}_p^\times \cong C_{p-1} \times (1 + p\mathbf{Z}_p)$. The first, cyclic factor gives us idempotent operations which split $p$-adic $K$-theory into $(p-1)$ summands, one of which gives a new pair of multiplicative theories, $G$-theory and $g$-theory. In fact the splitting is rationally defined so that $p$-local $K$-theory is also split.

The ring of operations of $p$-adic $G$-theory, or $g$-theory, is then isomorphic to $M(1+p\mathbf{Z}_p, \mathbf{Z}_p)$, the space of measures on $1+p\mathbf{Z}_p$, which is itself isomorphic to $\operatorname{Hom}(B_0, \mathbf{Z}_p)$, where $B_0 = B \cap \mathbf{Q}[w^{p-1}]$.

I emphasise again that the operations form a ring, under composition, because we are now in a position to understand the multiplicative structure.

The Adams operations, which correspond to point measures, compose as $\psi^k\psi^l = \psi^{kl}$. (This is because $\psi^k$ acts on line bundles as $L \mapsto L^{\otimes k}$.) It is clear then that the product we want is the convolution product of measures with respect to *multiplication* on $\mathbf{Z}_p$. In terms of power series this becomes very complicated. Essentially we need formulas for expanding $\binom{wz}{k}$ as a linear combination of terms of the form $\binom{w}{i}\binom{z}{j}$.

If, however, we restrict to $1 + p\mathbf{Z}_p$ we may use the isomorphism (of topological groups)

$$\begin{aligned} \mathbf{Z}_p &\longrightarrow 1 + p\mathbf{Z}_p \\ w &\longmapsto (1+p)^w \end{aligned}$$

to see that

$$M(1 + p\mathbf{Z}_p, \mathbf{Z}_p) \cong M(\mathbf{Z}_p, \mathbf{Z}_p) \cong \mathbf{Z}_p[[s]].$$

The change of variable here is an attempt to avoid confusion.

Now convolution in $M(\mathbf{Z}_p, \mathbf{Z}_p)$ with respect to *addition* is easy. The appropriate formula is

$$\binom{w+z}{k} = \sum_{i+j=k} \binom{w}{i}\binom{z}{j},$$

and we see that $M(1 + p\mathbf{Z}_p, \mathbf{Z}_p) \cong \mathbf{Z}_p[[s]]$ as *rings*.

It was shown in [7] that this ring of operations is a local ring.

How does the operation corresponding to $f(s) \in \mathbf{Z}_p[[s]]$ act on the group $G^0(S^{2n})$? We simply need to evaluate

$$\int_{\mathbf{Z}_p} (1+p)^{nw}\, d\mu_f(w).$$

Now $(1+p)^{nw} = \big(1+((1+p)^n-1)\big)^w$, with $((1+p)^n-1)$ divisible by $p$. Thus lemma 7 shows that $\mu_f$ acts by multiplication by $f((1+p)^n-1)$, which will always be a convergent series.

Now $f((1+p)^n-1) \equiv f(0) \mod p$, so we see that an operation is invertible if and only if it is invertible on any even-dimensional sphere. This is a strengthened form of a theorem of Keith Johnson [7]. By remarking that the ring of operations in $p$-adic $K$-theory is isomorphic to $\mathbf{Z}_p[C_{p-1}] \otimes \mathbf{Z}_p[[s]]$, one may easily obtain the corresponding result for $p$-adic $K$-theory in its non-split form. Since a $p$-local operation which is invertible as a $p$-adic operation must actually have a $p$-local inverse the statements hold also for the $p$-local theories.

Let me finish by referring briefly to the $p$-local case. It is *not* true that $\mathrm{Hom}(B_0, \mathbf{Z}_{(p)}) \subset \mathrm{Hom}(B_0, \mathbf{Z}_p) = \mathbf{Z}_p[[s]]$ is the subring $\mathbf{Z}_{(p)}[[s]]$, though it does look like this additively because there is another homeomorphism $\mathbf{Z}_p \to 1 + p\mathbf{Z}_p$, $w \mapsto 1 + pw$ which sends $\mathbf{Z}_{(p)}$ to $1 + p\mathbf{Z}_{(p)}$.

It would be nice to have a description of $\mathrm{Hom}(B_0, \mathbf{Z}_{(p)})$ as a subring of $\mathbf{Z}_p[[s]]$, other than the rather abstract one that $f((1+p)^n - 1)$ is rational for all $n \geq 0$. Ultimately it would be nice to know something about $\mathrm{Hom}(B, \mathbf{Z})$.

There are certainly some interesting elements in $\mathrm{Hom}(B, \mathbf{Z}_{(p)})$. Consider the measure constructed by following the map $\kappa_p : B \to B$ of theorem 1 with $\psi^k$. We get a measure whose moments are

$$\int_{\mathbf{Z}_p^\times} w^{n-1}\, d\mu(w) = (-1)^n(1 - p^{n-1})\frac{B_n}{n}(k^n - 1),$$

for $n > 0$. This is Mazur's Bernoulli measure. I don't know whether the moments are rational for $n \leq 0$, which would show that we had an element of $\mathrm{Hom}(A[w^{-1}], \mathbf{Z}_{(p)})$.

## References


1. J. F. Adams, "Infinite Loop Spaces", Princeton University Press, 1978.
2. J. F. Adams and F. W. Clarke, *Stable operations on complex K-theory*, Illinois J. Math. **21** (1977), 826–829.
3. A. K. Bousfield, *On the homotopy theory of K-local spectra at an odd prime*, Amer. J. Math. **107** (1985), 895–932.
4. A. Baker, F. Clarke, N. Ray and L. Schwartz, *On the Kummer congruences and the stable homotopy of BU*, Preprint, Swansea 1985.
5. F. Clarke, *Self maps of BU*, Math. Proc. Cambridge Philos. Soc. **89** (1981), 491–500.
6. K. Johnson, *The relation between stable operations for connective and non-connective K-theory*, Canad. Math. Bull. **29** (1986), 246–255.
7. K. Johnson, *The algebra of stable operations for p-local complex K-theory*, Preprint, Dalhousie 1986.
8. N. M. Katz, *p-adic L-functions for CM fields*, Invent. Math. **49** (1978), 199–297.
9. S. Lang, "Cyclotomic fields", Graduate Texts in Mathematics **59**, Springer-Verlag, New York Heidelberg Berlin, 1978.
10. I. Madsen, V. Snaith and J. Tornehave, *Infinite loop maps in geometric topology*, Math. Proc. Cambridge Philos. Soc. **81** (1977), 399–430.
11. D. C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), 351–414.
12. L. Schwartz, *Opérations d'Adams en K-homologie et applications*, Bull. Soc. Math. France **109** (1981), 237–257.



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