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Exposé n° 11

CONGRUENCE PROPERTIES OF COEFFICIENTS OF SOLUTIONS OF  
PICARD/FUCHS EQUATIONS

by F. Beukers

Let

$$(1) \quad F(t) = \sum_{n=1}^{\infty} f(n)t^{n-1}, \quad f(n) \in \mathbb{Z}_p$$

be a power series which is a solution of a  $p$ -adic differential equation coming from algebraic geometry. It is a well-known fact that  $F(t)$  can be approximated  $p$ -adically by algebraic functions. A theorem of J. Denef and L. Lipschitz [DL] states that the coefficients of an algebraic function mod  $p^S$  can be generated by a  $p^S$ -automaton, in other words:

$$(2) \quad \begin{aligned} &\forall s \in \mathbb{N}, \exists r \in \mathbb{N}, \forall i \in \mathbb{Z} \text{ with } 0 \leq i < p^r \text{ we find } r' \in \mathbb{N} \text{ with } r' < r \\ &\text{and } i' \in \mathbb{Z} \text{ with } 0 \leq i' < p^{r'} \text{ such that } \forall m \in \mathbb{N} \text{ we have} \end{aligned}$$

$$f(mp^r + i) \equiv f(mp^{r'} + i') \pmod{p^S}$$

It turns out that in special examples of (1) the statement (2) can be remarkably refined in certain cases. The purpose of this paper is to study a number of such examples. A number of the results mentioned here are due M. Coster, and they will appear in his thesis.

The examples we have in mind fall roughly under two headings (Case I and II).

Case I. Suppose  $F(t)dt$  is the differential of a formal group law of height one over  $\mathbb{Z}_p$ .

Let  $A$  be the Hasse-Witt coefficient of this formal group. An alternative way of describing Case I is,

$$\exists A \in \mathbb{Z}_p, \exists \vartheta(t) \in \mathbb{Z}_p[[t]] \text{ such that } F(t)dt = AF(t^p)t^{p-1}dt + d\vartheta(t).$$

(see [H]). Comparison of coefficients yields the following equivalent statement,

$$\exists A \in \mathbb{Z}_p \text{ such that } f(mp^r) \equiv Af(mp^{r-1}) \pmod{p^r} \quad \forall m, r \in \mathbb{N}.$$

Examples of this case are  $f(n) = a(n), b(n), c(n)$ , where

$$a(n) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k}, \quad b(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}, \quad c(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \dots$$

These numbers arise as denominators for approximations in the irrationality proofs of  $\log 2$ ,  $\zeta(2)$  and  $\zeta(3)$  respectively (see [P], [B1]). Since

$$\frac{dt}{\sqrt{1-6t+t^2}} = \sum_{n=1}^{\infty} a(n-1)t^{n-1}dt$$

and  $dt/\sqrt{1-6t+t^2}$  is a rational differential, the corresponding formal group is just the multiplicative one for every odd prime  $p$ . Hence

$$a(mp^r-1) \equiv a(mp^{r-1}-1) \pmod{p^r} \quad \forall m, r \in \mathbb{N}.$$

In [B2] it is shown that the same congruence also holds for  $b(n)$ ,  $c(n)$ . However, it turns out that for  $p \geq 5$  we have

$$b(mp^r-1) \equiv b(mp^{r-1}-1) \pmod{p^{3r}} \quad \forall m, r \in \mathbb{N}$$

and the same congruence holds for  $c(n)$  as well. As far as we know this stronger congruence cannot be explained yet on the basis of a general theory. Subsequent generalisations are given by M.Coster [C1].

Let  $p$  be a prime  $\equiv 1 \pmod{4}$ . Then the canonical differential form  $dt/\sqrt{1-6t^2+t^4}$  on the elliptic curve  $E: y^2 = t^4 - 6t^2 + 1 \simeq \mathbb{C}/\mathbb{Z}[i]$  corresponds to the formal addition law on  $E$  with Hasse invariant  $\not\equiv 0 \pmod{p}$  since  $p \equiv 1 \pmod{4}$ . As a consequence we find

$$(3) \quad a\left(\frac{mp^r-1}{2}\right) \equiv (\alpha + \beta i) a\left(\frac{mp^{r-1}-1}{2}\right) \pmod{p^r} \quad \forall m, r \in \mathbb{N}, m \text{ odd.}$$

where  $p = \alpha^2 + \beta^2$ ,  $\alpha \equiv 1 \pmod{4}$ . Numerical experience suggests however the following conjecture,

CONJECTURE: Congruence (3) is true mod  $p^{2r}$ .

It was proved by Van Hamme for  $m=r=1$ . A similar, non-conjectural, statement is

THEOREM. Let  $p, \alpha, \beta$  be as in (3). Then

$$\left(\frac{-1/2}{\frac{mp^r-1}{4}}\right) \equiv (\alpha + \beta i) \left(\frac{-1/2}{\frac{mp^{r-1}-1}{4}}\right) \pmod{p^{2r}} \quad \forall m, r \in \mathbb{N} \\ m \equiv 1 \pmod{4}.$$

The proof, given by Chowla, Dwork, Evans for  $m=r=1$  [CDE] and by Van Hamme and M.Coster in general, consists of showing that

$$\left(\frac{-1/2}{\frac{mp^r-1}{4}}\right) \Big/ \left(\frac{-1/2}{\frac{mp^{r-1}-1}{4}}\right) \equiv \Gamma_p^2(1/4) / \Gamma_p(1/2) \pmod{p^{2r}}$$

and then noticing that the latter number is a Jacobi sum equal to  $\alpha + \beta i$ , [GK]. Using the same line of argument it is possible to prove similar congruences for other binomial coefficients (M.Coster [C3]).

We now turn our attention to  $b(n)$  and  $c(n)$ .

THEOREM (Stienstra-Beukers [58]) Let notations be as in (3). Then

$$(4) \quad b\left(\frac{mp^r-1}{2}\right) \equiv (\alpha + \beta i)^2 b\left(\frac{mp^{r-1}-1}{2}\right) \pmod{p^r} \quad \forall m, r \in \mathbb{N} \\ m \text{ odd}$$

CONJECTURE. Congruence (4) is true mod  $p^{2r}$ .

This conjecture was proved by Van Hamme for the case  $m=r=1$ .

Now take  $p$  an odd prime and let

$$\sum_{n=1}^{\infty} \gamma_n q^n = q \prod_{n=1}^{\infty} (1-q^{2n})^4 (1-q^{4n})^4$$

Let  $\pi \in \mathbb{Z}_p$  be the zero of  $X^2 - \gamma_p X + p^3$  such that  $|\pi|_p = 1$  (if it exists). Then

THEOREM (Beukers [63])

$$(5) \quad c\left(\frac{mp^r-1}{2}\right) \equiv \pi c\left(\frac{mp^{r-1}-1}{2}\right) \pmod{p^r} \quad \forall m, r \in \mathbb{N}, m \text{ odd.}$$

CONJECTURE. Congruence (5) is true mod  $p^{2r}$ .

In all conjectures there seems to be no known explanation of why these stronger congruences should occur.

We now describe the second class of examples.

Case II. Suppose  $p$  prime and  $F(t) \equiv Q(t)Q(t^p)Q(t^{p^2}) \dots \pmod{p}$

where  $Q(t) \in \mathbb{Z}_p[t]$ ,  $\deg Q \leq p-1$ .

Put  $Q(t) = \sum_{n=0}^{p-1} q(n)t^n$ , then, as a consequence of (6), we find that if  $n = n_0 + n_1 p + n_2 p^2 + \dots + n_t p^t$  is the expansion of  $n$  in base  $p$ , then  $f(n) \equiv q(n_0)q(n_1) \dots q(n_t) \pmod{p}$ . In particular, if  $p \mid q(n_i)$  for some  $i$ , then  $f(n) \equiv 0 \pmod{p}$ . In many examples the factorisation (6) exists.

Let  $\mu_p(d, n)$  be the number of digits  $d$  occurring in  $n$  in base  $p$ . It turns out that often the occurrence of  $p \mid q(n_i)$  for some  $i$  implies a much stronger

divisibility property when  $\mu_p(n_1, n)$  increases. For example, let  $f(n) = c(n)$ , where  $c(n)$  is as above. Then we have a factorisation of type mentioned above. If  $p=5$  then  $Q(t) = 1 - 2t^2 + t^4$ , i.e.  $5 \mid q(1)$ ,  $5 \mid q(3)$ .

CONJECTURE: let  $q = \mu_5(1, n) + \mu_5(3, n)$ , then  $5^q \mid c(n)$ .

For  $p=11$  we have similarly,

CONJECTURE: Let  $q = \mu_{11}(5, n)$ , then  $11^q \mid c(n)$ .

Both conjectures were positively verified for all  $n \leq 1000$ .

Now let  $b(n)$  be as above. Then, for any  $p$  we have a factorisation of the form above [SB]. Moreover, if  $p \equiv 3 \pmod{4}$  then  $b(\frac{p-1}{2}) \equiv 0 \pmod{p}$  [SB], i.e. the  $(p-1)/2$ -th coefficient of  $Q(t)$  is zero mod  $p$ . This suggests the following conjecture, which was verified for a large number of cases,

CONJECTURE: Let  $p$  be a prime and  $p \equiv 3 \pmod{4}$ . Let  $q = \mu_p(\frac{p-1}{2}, n)$ . Then  $p^q \mid b(n)$ .

Now consider the  $a(n)$  which can also be defined by

$$\sum_{n=0}^{\infty} a(n)t^n = \frac{1}{\sqrt{1-6t+t^2}}$$

We deal with the more general

$$\sum_{n=0}^{\infty} u(n)t^n = (1 + \alpha_1 t + \dots + \alpha_{p-1} t^{p-1})^{\frac{1}{1-p}}$$

where  $\alpha_i \in \mathbb{Z}_p$  and  $p$  is an odd prime.

THEOREM (M.Coster) Let  $J = \{1 \leq j \leq p-1 \mid p \text{ divides } \alpha_j\}$  (set of relevant digits). Let

$$q = \sum_{j \in J} \mu_p(j, n).$$

Then

$$\left[ \frac{q+1}{2} \right]_p \text{ divides } u(n).$$

COROLLARY. Let  $p \equiv 3 \pmod{4}$ , and  $q = \mu_p(\frac{p-1}{2}, n)$ . Then  $p^{\left[ \frac{q+1}{2} \right]}$  divides  $a(n)$ .

Proof of the Corollary. According to Coster's theorem we have to show that the coefficient of  $x^{\frac{1}{2}(p-1)}$  in  $(x^2-6x+1)^{\frac{1}{2}(p-1)}$  is divisible by  $p$ . Since the elliptic curve  $y^2 = x^4-6x^2+1$  is isomorphic to  $\mathbb{C}/\mathbb{Z}[i]$ , its Hasse-invariant is zero if  $p \equiv 3 \pmod{4}$ . This is equivalent to saying that the coefficient of  $x^{p-1}$  in  $(x^4-6x^2+1)^{\frac{1}{2}(p-1)}$  is divisible by  $p$ , as asserted.

In special cases Coster's theorem can be strenghtened.

THEOREM (M.Coster). Suppose  $p \mid \alpha_j$  for  $j=s, s+1, \dots, p-1$ . Let

$$q = \sum_{j=s+1}^{p-1} \mu_p(j, n).$$

Then  $p^q$  divides  $u(n)$ .

EXAMPLE. Notice that  $(1-t)^{-\frac{1}{2}} = ((1-t)^{\frac{1}{2}(p-1)})^{1/1-p}$ . So  $p \mid \alpha_j$  for  $j=\frac{1}{2}(p+1), \dots, p-1$ .

On the other hand

$$(1-t)^{-\frac{1}{2}} = \sum_{n \geq 0} \binom{-\frac{1}{2}}{n} (-t)^n.$$

Let  $\alpha \in \mathbb{Z}_p$ ,  $n \in \mathbb{N}$  have  $p$ -adic expansions  $a_0 + a_1 p + a_2 p^2 + \dots$  and  $n_0 + n_1 p + n_2 p^2 + \dots + a_t p^t$  respectively. Then it is an elementary exercise to show that the number of factors  $p$  in  $\binom{\alpha}{n}$  equals the number of occurrences of  $n_i > \alpha_i$ . Note that this implies the above theorem for this particular case.

The proof of the last two theorems is elementary but very tedious.

One expands  $(1 + \alpha_1 t + \dots + \alpha_{p-1} t^{p-1})^{1/1-p}$  as a powerseries in  $t$  to obtain

$$u(n) = \sum_{\sum b_i = n} \binom{1/(1-p)}{b_1 \dots b_{p-1}} \prod \alpha_i^{b_i}$$

and then study the divisibility properties of the multinomial coefficients

$$\binom{1/(1-p)}{b_1 \dots b_{p-1}} = \frac{1}{1-p} \left( \frac{1}{1-p} - 1 \right) \dots \left( \frac{1}{1-p} - b + 1 \right)}{b_1! b_2! \dots b_{p-1}!}, \quad b = \sum_i b_i.$$

We hope that by these examples we have motivated the following question.

QUESTION. Are the above divisibility properties and congruences special cases of an unrecognised overall  $p$ -adic structure of coefficients of solutions of Picard-Fuchs equations ?

-REFERENCES-

- [B1] F.Beukers, A note on the irrationality of  $\zeta(3)$ ,  
Bull.London Math.Soc.11(1979),268-272.
- [B2] F.Beukers, Some congruences for the Apéry numbers,  
J.Number Theory 21(1985),141-155
- [B3] F.Beukers, Another congruence for the Apéry numbers,  
J.Number Theory 25(1987),201-210.
- [C1] M.Coster, Congruence properties of coefficients of  
certain power series, submitted to Compositio Math.
- [C2] M.Coster, Some congruences for generalised Apéry num-  
bers, preprint.
- [C3] M.Coster, Generalisation of a congruence of Gauss,  
submitted to J.Number Theory.
- [CDE] S.Chowla, B.Dwork, R.Evans, On the mod  $p^2$  determination  
of  $\binom{(p-1)/2}{(p-1)/4}$ , J.Number Theory 24(1986), 188-196.
- [DL] J.Denef, L.Lipschitz, Algebraic power series and dia-  
gonals, preprint.
- [H] M.Hazewinkel, Formal Groups and Applications, Academic  
Press, New York 1978.
- [GK] B.Gross, N.Koblitz, Gauss sums and the  $p$ -adic  $\Gamma$ -function,  
Ann.Math.109(1979), 569-581.
- [P] A.J.van der Poorten, A proof that Euler missed..., Apéry's  
proof of the irrationality of  $\zeta(3)$ , Math.Intelligencer 1  
(1979), 195-203.
- [SB] J.Stienstra, F.Beukers, On the Picard-Fuchs equation and  
the formal Brauer group of certain elliptic K3-surfaces,  
Math.Annalen 271 (1985), 269-304.

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