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LOCALLY CONVEX SPACES OVER NONSPHERICALLY COMPLETE VALUED FIELDS

by Wim H. SCHIKHOF (*)

Abstract. - A structure theory (part I) and a duality theory (part II) for locally convex spaces over ultrametric nonspherically complete valued fields is developed. As an application nuclearity, reflexivity and the Hahn-Banach property are obtained for certain classes of spaces of C^∞- and analytic functions and their duals.

Introduction.

Consider the following property (*) defined for locally convex spaces E over a non-archimedean, nontrivially valued complete field K.

(*) Let D be a subspace of E, let p be a continuous seminorm on E, let $f \in D'$ with $|f| \leqslant p$ on D. Then f has an extension $\bar{f} \in E'$ such that $|\bar{f}| \leqslant p$ on E.

The non-archimedean Hahn-Banach theorem ([14], Th. 3.5) states that (*) holds if K is spherically (= maximally) complete. From [11] p. 91 & 92 it is not hard to derive the following 'converse'.

PROPOSITION. - Let K be not spherically complete. If E is Hausdorff and (*) is true $\dim E \leqslant 1$.

The purpose of this paper is to show that, for certain classes of spaces over nonspherically complete base fields, a satisfactory duality theory exists. Here the key role is played by the 'polar seminorms' (Definition 3.1). We shall mainly be dealing with the following two classes of spaces.

(i) Strongly polar spaces. - E is strongly polar if each continuous seminorm is a polar seminorm. Strong polarness of E will turn out to be equivalent to the following Hahn-Banach property (**) (Theorem 4.2).

(**) Let D be a subspace of E, let p be a continuous seminorm on E, let $\varepsilon > 0$, let $f \in D'$ with $|f| \leqslant p$ on D. Then f has an extension $\bar{f} \in E'$ such that $|\bar{f}| \leqslant (1+\varepsilon)\, p$ on E.

(ii) Polar spaces. - E is polar if there is a basis P of continuous seminorms consisting of polar seminorms. In general, polar spaces do not have the Hahn-Banach property (**) but the conclusion of (**) does hold if D is onedimensional and if $p \in P$. In particular, the elements of the dual of a Hausdorff polar space separate the points of E.

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In § 2 we shall introduce several examples of spaces of (continuous C^n- , C^∞- , analytic) functions. Throughout, we shall apply the results of our theory to them. (See Example 4.5 (vii), Corollary 4.4 (iii), Proposition 6.9 and the Corollaries 7.9, 8.8, 9.10, 10.11, 11.6.)

Note. - The main theory in this paper holds for spherically complete K as well, although in that case most results are known. Possible exceptions are Proposition 4.11, the Theorems 5.12, 8.3, 8.5, 9.6 and the Corollaries 10.9 (i), 11.4.

1. Preliminaries.

For terms that are unexplained here we refer to [1], [13], [14].

(a) Base field. - Throughout K is a non-archimedean nontrivially valued complete field with valuation $|\ |$. We set $\Gamma_K := \{|\lambda| \ ; \ \lambda \in K\}$, $\overline{\Gamma_K}$:= the closure of $|K|$ in $\mathbb{R}$, $|K^\times| := \{|\lambda| \ ; \ \lambda \in K \ , \ \lambda \neq 0\}$,

$$B(0\ ,\ 1) := \{\lambda \in K \ ; \ |\lambda| \leq 1\} \ , \quad B(0\ ,\ 1^-) := \{\lambda \in K : \ |\lambda| < 1\} \ .$$

The field of the p-adic numbers is denoted $\mathbb{Q}_p$, the completion of its algebraic closure is $\mathbb{C}_p$, with valuation $|\ |_p$.

(b) Convexity. - Let E be a vector space over K . A nonempty subset A of E is absolutely convex if $x\ ,\ y \in A$, $\lambda\ ,\ \mu \in B(0\ ,\ 1)$ implies $\lambda x + \mu y \in A$. An absolutely convex set is an additive subgroup of E . For a nonempty set $X \subset E$ its absolutely convex hull $\operatorname{co} X$ is the smallest absolutely convex set that contains X . We have

$$\operatorname{co} X = \{\lambda_1 x_1 + \dots + \lambda_n x_n : \ n \in \mathbb{N} \ , \ x_1 \ , \dots \ , x_n \in X \ , \ \lambda_1 \ , \dots \ , \lambda_n \in B(0\ ,1)\}.$$

If X is a finite set $\{x_1 \ , \dots \ , x_n\}$ we sometimes write $\operatorname{co}(x_1 \ , \dots \ , x_n)$ instead of $\operatorname{co} X$. A subset A of E is edged if A is absolutely convex and for each $x \in E$ the set $\{|\lambda| : \ \lambda \in K \ , \ \lambda x \in A\}$ is closed in $|K|$. If the valuation on K is discrete every absolutely convex set is edged. If K has a dense valuation an absolutely convex set A is edged if and only if from $x \in E$, $\lambda x \in A$ for all $\lambda \in B(0\ ,\ 1^-)$ it follows that $x \in A$. Intersections of edged sets are edged. For an absolutely convex set A we define A^e to be the smallest edged subset of E that contains A . If the valuation of K is dense we have

$$A^e = \{x \in E \ ; \ B(0\ ,\ 1^-)\ x \subset A\} = \bigcap\{\lambda A : \ \lambda \in K : \ |\lambda| > 1\}$$

(and $A^e = A$ if the valuation is discrete). An absolutely convex set $A \subset E$ is absorbing if $E = \bigcup\{\lambda A : \ \lambda \in K\}$.

(c) Seminorms. - For technical reasons we slightly modify the usual definition as follows. A seminorm on a K-vector space E is a map $p : E \longrightarrow \mathbb{R}$ (we apologize for using the same symbol as in $\mathbb{Q}_p$ and $|\ |_p$) satisfying

(i) $p(x) \in \overline{\Gamma_K}$

(ii) $p(\lambda x) = |\lambda|\ p(x)$

(iii) $p(x + y) \leq \max(p(x) , p(y))$.

for all $x , y \in E$, $\lambda \in K$. Observe that if K carries a dense valuation (i) is equivalent to $p(x) \geq 0$ whereas for a discretely valued field K condition (i) is equivalent to $p(x) \in |K|$. It is easily seen that if $p : E \longrightarrow [0 , \infty)$ satisfies (ii) and (iii) and K has a discrete valuation the formula

$$q(x) = \inf\{s \in |K| : p(x) \leq s\}$$

defines a seminorm q which is equivalent to p . The set of all seminorms on E is closed for suprema (i. e. if P is a collection of seminorms on E and if $q(x) := \sup\{p(x) : p \in P\} < \infty$ for each $x \in E$ then q is a seminorm on E). For each absolutely convex absorbing set $A \subset E$ the formula

$$p_A(x) = \inf\{|\lambda| : \lambda \in K , x \in \lambda A\}$$

defines a seminorm p_A on E , <u>the seminorm associated to</u> A . Then

$$\{x : p_A(x) < 1\} \subset A \subset \{x : p_A(x) \leq 1\} .$$

The proof of the following proposition is elementary.

PROPOSITION 1.1. - <u>The map</u> $A \longmapsto p_A$ <u>is a bijection of the collection of all edged absorbing subsets of</u> E <u>onto the collection of all seminorms on</u> E . <u>Its inverse is given by</u> $p \longmapsto \{x \in E : p(x) \leq 1\}$.

(d) <u>Locally convex topologies</u>. - A topology τ on E (not necessarily Hausdorff) is a <u>locally convex topology</u> (and $E = (E , \tau)$ is a <u>locally convex space</u>) if τ is a vector space topology for which there exists a basis of neighbourhoods of 0 consisting of absolutely convex (or, equivalently, edged) sets. A locally convex topology is induced by a collection of seminorms in the usual way. The closure of a subset A of a locally convex space is denoted $\bar{A}$. The following terms are directly taken over from the 'classical' theory (see for example [16], also [14]) and are given without further explanation. <u>Barrel</u>; <u>barreled space</u>, <u>complete</u>, <u>quasicomplete</u>, <u>bounded set</u>, <u>bounded linear map</u>, <u>bornological space</u>, <u>normed</u> (<u>normable</u>) <u>space</u>, <u>Banach space</u>, <u>Fréchet space</u>, <u>projective limit topology</u> (in particular, <u>product</u> and <u>subspace</u>), <u>inductive limit topology</u> (in particular, <u>direct sum</u>, <u>quotient</u>, <u>strict inductive limit of a sequence of locally convex spaces</u>, <u>LF-space</u>). Let E , F be locally convex spaces over K , let $L(E , F)$ be the K-linear space consisting of all continuous linear maps $E \longrightarrow F$. Let G be a nonempty collection of bounded subsets of E . The <u>topology</u> on $L(E , F)$ <u>of uniform convergence on members of</u> G is the locally convex topology induced by the seminorms

$$T \longmapsto \sup\{q(Tx) : x \in A\} \quad (T \in L(E , F)) ,$$

where q runs through the collection of continuous seminorms of F and where A runs through G . We write $E' := L(E , K)$. (The algebraic dual of E is denoted E^* .) As in the classical theory we have the weak topologies $\sigma(E' , E)$ on E' and

and $\sigma(E, E')$ on E, and the <u>strong topology</u> $b(E', E)$ on E' (they may be not Hausdorff). Sometimes we write E'_σ to indicate $(E', \sigma(E', E))$. Similarly, $E'_b := (E', b(E', E))$.

(c) <u>Compactoids</u> ([6], 1.1). - A subset A of a locally convex space E over K is <u>(a) compactoid</u> if for each neighbourhood U of 0 there exist $x_1, \dots, x_n \in E$ such that $A \subset U + co(x_1, \dots, x_n)$. Subsets, absolutely convex hulls and closures of compactoids are compactoids. If A is an absolutely convex compactoid then so is A^e. If A is a compactoid in E and $T \in L(E, F)$ then TA is a compactoid in F. Each precompact set is a compactoid, each compactoid is bounded. A map $T \in L(E, F)$ is <u>compact</u> if there exists a neighbourhood U of 0 in E such that TU is a compactoid in F.

(f) <u>Nuclear spaces</u> ([6], 3.2).

(i) Let E be a locally convex space over K. For each continuous seminorm p let E_p be the space $E/\mathrm{Ker}\, p$ normed with the quotient norm $\bar{p}$ induced by p, let $E_p^\wedge$ be its completion (whose norm is again denoted $\bar{p}$). The canonical maps $\pi_p : E \mapsto E_p^\wedge$ (where p runs through the collection Γ of all continuous seminorms on E) **induce** a map $E \to \prod_{p\in\Gamma} E_p^\wedge$. In a similar way one can obtain a map $E \to \prod_{p\in\Gamma} E_p$. It is easy to see that, if E is a Hausdorff space, these maps are linear homeomorphisms into.

(ii) Let E, Γ be as in (i). For $p, q \in \Gamma$, $p \leqslant q$ the natural map $E_q \to E_p$ (or sometimes $E_q^\wedge \to E_p^\wedge$) is denoted Φ_{pq}. E is a <u>nuclear space</u> if E is Hausdorff and if to every $p \in \Gamma$ there exists $q \in \Gamma$, $q \geqslant p$ such that the map Φ_{pq} is compact. (By [6], Lemma 2.5 compactness of $\Phi_{pq} : E_q \to E_p$ is equivalent to compactness of $\Phi_{pq} : E_q^\wedge \to E_p^\wedge$.)

The following proposition is easy to prove.

PROPOSITION 1.2. - <u>Subspaces and quotients by closed subspaces of nuclear spaces are nuclear. In a nuclear space each bounded subset is a compactoid.</u>

PROPOSITION 1.3. - <u>For a locally convex space E over K the following are equivalent.</u>

(α) E <u>is nuclear</u>.

(β) E <u>is linearly homeomorphic to a subspace of some power of</u> c_0.

<u>Each element of</u> $L(E, c_0)$ <u>is compact</u>.

<u>Proof</u>. See [4], § 4, Proposition 2. - (The assumptions that K be spherically complete and that the valuation be dense, made throughout [4], are easily seen to be redundant in this case.)

2. Examples.

First we recall the definition of a C^n- (C^∞-) function ([12], § 8 or [13]). Let X be a nonempty subset of K, without isolated points. For $n \in \mathbb{N}$ set

$$\vee^n X = \{(x_1, \dots, x_n) \in X^n : i \neq j \text{ implies } x_i \neq x_j\} .$$

For a function $f : X \to K$ define $\Phi_n f : \vee^{n+1} X \to K \in (n \in \{0, 1, 2, \dots\})$ inductively by $\Phi_0 f := f$ and, for $n \geq 1$,

$$\Phi_n f(x_1, \dots, x_{n+1})$$
$$:= (x_1 - x_2)^{-1}\{\Phi_{n-1} f(x_1, x_3, x_4, \dots, x_{n+1}) - \Phi_{n-1} f(x_2, x_3, x_4, \dots, x_{n+1})\}$$

f is a C^n-function if $\Phi_n f$ can be extended to a continuous function $\bar{\Phi}_n f$ on X^{n+1} (observe that this extension is unique since $\vee^{n+1} X$ is dense in X^{n+1}). We equip the space $C^n(X \to K)$ of all C^n-functions $X \to K$ with the locally convex topology of uniform convergence of $\bar{\Phi}_i f$ on compact subsets of X^{i+1} for $i \in \{0, 1, \dots, n\}$. $C^\infty(X \to K) := \bigcap C^n(X \to K)$, with the topology of uniform convergence of $\bar{\Phi}_i f$ on compact subsets of X^{i+1} for all $i \in \{0,1,2,\dots\}$, The following examples will serve as illustration material throughout.

2.1. The spaces $C^n(X \to K)$ for compact X. — For $n \in \{0, 1, 2, \dots\}$ and compact X the topology on $C^n(X \to K)$ can be described by the single norm $\| \|_n$ given by

$$\|f\|_n = \max_{0 \leq k \leq n} \|\Phi_k f\|_\infty \quad (f \in C^n(X \to K)) .$$

It is shown in [12], 8.5 and 8.22 that $C^n(X \to K)$ is a Banach space of countable type, hence linearly homeomorphic to c_0.

2.2. The space $C^\infty(X \to K)$ for compact X. - It is easily seen that the topology on $C^\infty(X \to K)$ is defined by the (semi-) norms $\| \|_n$ $(n \in \{0, 1, 2, \dots\})$. It is shown in [12], 12.1, that $C^\infty(X \to K)$ is a Fréchet space and in [6], 3.5, that $C^\infty(X \to K)$ is nuclear.

2.3. The space $C^\infty(U \to K)$ where U is an open subset of $\mathbb{Q}_p$. - Let U be a nonempty open subset of $\mathbb{Q}_p$, suppose $K \supset \mathbb{Q}_p$. For $n \in \{0, 1, 2, \dots\}$ set $U_n := \{x \in U : B(x, p^{-n}) \subset U, |x|_p \leq p^n\}$ where

$$B(x, p^{-n}) := \{y \in \mathbb{Q}_p : |y - x|_p \leq p^{-n}\} .$$

Then each U_n is compact and open in $\mathbb{Q}_p$, $U_1 \subset U_2 \subset \dots$, $\bigcup_n U_n = U$. It is not hard to see that the topology on $C^\infty(U \to K)$ is defined by the seminorms p_{jn} $(j, n \in \{0, 1, 2, \dots\})$ where

$$p_{jn}(f) = \|f|U_j\|_n \quad (f \in C^\infty(U \to K)) .$$

From this and [12], 12.1, it follows that $C^\infty(U \to K)$ is a Fréchet space.

Nuclearity of $C^\infty(U \to K)$ is proved in [6], 5.4.

2.4. <u>The space</u> $C_c^\infty(U \to K)$ <u>where</u> U <u>is an open subset of</u> $\mathbb{Q}_p$. - Let K , U , U_n be as in 2.3. As a vector space, let $C_c^\infty(U \to K)$ be the space of all $f \in C^\infty(U \to K)$ with compact support. To be able to put a decent inductive limit topology on $C_c^\infty(U \to K)$ we need the following lemma.

LEMMA 2.4. - <u>Let</u> $V \subsetneqq W$ <u>be nonempty open compact subsets of</u> $\mathbb{Q}_p$. <u>For each</u> $f : V \to K$ <u>let</u> $\tilde{f} : W \to K$ <u>be defined by</u> $\tilde{f}(x) := f(x)$ <u>if</u> $x \in V$, $\tilde{f}(x) := 0$ <u>if</u> $x \in W \setminus V$. <u>Then the map</u> $f \mapsto \tilde{f}$ <u>is a linear homeomorphism of</u> $C^\infty(V \to K)$ <u>into</u> $C^\infty(W \to K)$.

<u>Proof</u>. - The local character of the C^∞-property ([12], 8.12) guarantees that $f \in C^\infty(V \to K)$ implies $\tilde{f} \in C^\infty(W \to K)$. The linearity of $f \mapsto \tilde{f}$ is clear. Let $d' := \inf\{|y - z|_p : y \in W \setminus V , z \in V\}$. Then $d' > 0$. Set $d := \max(1, 1/d')$. Let $f \in C^\infty(V \to K)$. We shall prove by induction on $n \in \{0 , 1 , 2 , \dots\}$ that

$$\|\tilde{f}\|_n \leqslant d^n \|f\|_n .$$

(This, together with the obvious inequality $\|f\|_n \leqslant \|\tilde{f}\|_n$ will finish the proof.) The case $n = 0$ is trivial. For the step from $n - 1$ to n first observe that $\|\tilde{f}\|_n = \|\bar{\Phi}_n \tilde{f}\|_\infty \vee \|\tilde{f}\|_{n-1}$ and, by the induction hypothesis,

$$\|\tilde{f}\|_{n-1} \leqslant d^{n-1} \|f\|_{n-1} \leqslant d^n \|f\|_n .$$

It suffices therefore to prove that for $(x_1 , x_2 , \dots , x_{n+1}) \in \nabla^{n+1} W$ we have

$$(*) \qquad |\Phi_n \tilde{f}(x_1 , x_2 , \dots , x_{n+1})| \leqslant d^n \|f\|_n .$$

Now $(*)$ is true if all $x_1 , \dots , x_{n+1}$ are in V and also if all $x_1 , \dots , x_{n+1}$ are in $W \setminus V$ (in the latter case the left hand side of $(*)$ is 0). For the remaining case we may, by symmetry of $\Phi_n \tilde{f}$, assume that $x_1 \in V$, $x_2 \in W \setminus V$. Then $|x_1 - x_2| \geqslant d'$ so $|x_1 - x_2|^{-1} \leqslant d$ and

$$|\Phi_n \tilde{f}(x_1 , x_2 , \dots , x_{n+1}| =$$
$$|(x_1 - x_2)^{-1} \{\Phi_{n-1} \tilde{f}(x_1 , x_2 , \dots , x_{n+1}) - \Phi_{n-1} \tilde{f}(x_2 , x_3 , \dots , x_{n+1})\}|$$
$$\leqslant d \|\Phi_{n-1} \tilde{f}\|_\infty \leqslant d \|\tilde{f}\|_{n-1} \leqslant d^n \|f\|_n ,$$

which completes the proof of Lemma 2.4.

Now we define the topology on $C_c^\infty(U \to K)$ as follows. For each $n \in \mathbb{N}$ let $E_n := \{f \in C_c^\infty(U \to K) : \operatorname{supp} f \subset U_n\}$. There is an obvious algebraic isomorphism between E_n and $C^\infty(U_n \to K)$ which induces a locally convex topology τ_n on E_n for which it is a nuclear Fréchet space by 2.2. We have $E_1 \subset E_2 \subset \dots$ and $\bigcup E_n = C_c^\infty(U \to K)$. By Lemma 2.4 we have $\tau_{n+1}|E_n = \tau_n$ for each n . Define on

$C_c^\infty(U \to K)$ the inductive limit topology which respect to the inclusion maps $E_n \hookrightarrow C_c^\infty(U \to K)$. We may conclude ([14], Def. 3.3, Th. 3.13) that $C_c^\infty(U \to K)$ is an LF-space (hence complete, barreled, nonmetrizable). We shall see in Corollary 4.14 (iii) that $C_c^\infty(U \to K)$ is nuclear.

2.5. <u>The space</u> $C_0^n(\mathbb{Q}_p \to K)$ <u>and</u> $C_0^\infty(\mathbb{Q}_p \to K)$. - (These are slight generalizations of the spaces in 2.1 and 2.2.) Let $K \supset \mathbb{Q}_p$, $n \in \{0 , 1 , 2 , \ldots\}$. Let

$$C_0^n(\mathbb{Q}_p \to K) := \{f \in C^n(\mathbb{Q}_p \to K) : \lim_{|z|\to\infty} \bar{\Phi}_i f(z) = 0 \quad \text{for} \quad 0 \leq i \leq n\} .$$

(Here $|\ |$ is the usual norm on $\mathbb{Q}_p^{i+1}$). With the norm $\|\ \|_n$ defined by

$$\|f\|_n := \max_{0\leq i\leq n} \|\bar{\Phi}_i f\|_\infty \quad (f \in C_0^n(\mathbb{Q}_p \to K))$$

$C_0^n(\mathbb{Q}_p \to K)$ is easily seen to be a Banach space of countable type. We define $C_0^\infty(\mathbb{Q}_p \to K) := \bigcap C_0^n(\mathbb{Q}_p \to K)$ with the topology induced by the (semi) norms $\|\ \|_n$ $(n \in \{0 , 1 , 2 , \ldots\})$. It takes only obvious modifications of the methods referred to in 2.2 to prove that $C_0^\infty(\mathbb{Q}_p \to K)$ is a nuclear Fréchet space.

2.6. <u>The space</u> $S(\mathbb{Q}_p \to K)$ <u>of</u> C^∞<u>-functions rapidly decreasing at infinity</u>. - Again, let $K \supset \mathbb{Q}_p$. We define $S(\mathbb{Q}_p \to K)$ to be the linear space of all functions $f : \mathbb{Q}_p \to K$ for which $Pf \in C_0^\infty(\mathbb{Q}_p \to K)$ for each polynomial function P . Its topology is defined by the seminorms p_{nk} $(n , k \in \{0 , 1 , 2 , \ldots\})$ where

$$p_{nk}(f) = \|\chi^k f\|_n \quad (f \in S(\mathbb{Q}_p \to K))$$

(here χ denotes the identity polynomial). A standard reasoning shows that $S(\mathbb{Q}_p \to K)$ is a Fréchet space. For a fixed k the map $f \mapsto \chi^k f$ is a linear homeomorphism of the space $S(\mathbb{Q}_p \to K)$, with the topology defined by the seminorms p_{01} , p_{1k} , p_{2k} , ... into $C_0^\infty(\mathbb{Q}_p \to K)$. From this fact, together with the nuclearity of $C_0^\infty(\mathbb{Q}_p \to K)$, one easily derives the nuclearity of $S(\mathbb{Q}_p \to K)$.

Finally we consider two spaces of analytic functions.

2.7. <u>The space</u> $A_\infty(K)$ of <u>entire functions</u>. - Let K have a dense valuation. Let $A_\infty(K)$ be the space of all entire functions $f : K \to K$ with the topology of uniform convergence on bounded subsets of K . By [5], 3.6, $A_\infty(K)$ is a nuclear Fréchet space.

2.8. <u>The space</u> $A_1(K)$ <u>of analytic functions on the open unit disc</u>. - Let K have a dense valuation. Let $A_1(K)$ be the space of all analytic functions $f : B(0 , 1^-) \to K$ with the topology of uniform convergence on proper subdiscs of $B(0 , 1^-)$. By [5], 3.6, $A_1(K)$ is a nuclear Fréchet space.

3. Polarity.

Definition 3.1. - A seminorm p on a K-vector space E is a polar seminorm if $p = \sup\{|f| : f \in E^*, |f| \leq p\}$.

The collection of all polar seminorms on E is closed under suprema and under multiplication by elements of $|K|$. If p is a polar seminorm on a K-vector space F and if $T : E \to F$ is a K-linear map then $p \circ T$ is a polar seminorm on E.

It is an easy consequence of the Hahn-Banach theorem ((*), Introduction) that if K is spherically complete each seminorm on E is polar. For nonspherically complete K we have $(1^\infty/c_0)' = (0)$ ([11], 4.15) so the canonical norm on $1^\infty/c_0$ is not polar.

PROPOSITION 3.2. - Let p be a seminorm on a K-vector space E, let $A := \{x \in E : p(x) \leq 1\}$ be its unit semiball. The following are equivalent.

(α) p is polar.

(β) If $a \in E$, $\lambda \in K$, $|\lambda| < p(a)$ then there exists an $f \in E^*$ with $f(a) = \lambda$ and $|f| \leq p$.

(γ) For each onedimensional subspace D, for each $\varepsilon > 0$, for each $f \in D^*$ with $|f| \leq p$ on D there is an extension $\bar{f} \in E^*$ of f such that $|\bar{f}| \leq (1+\varepsilon)p$ on E.

(δ) For each $a \in E \setminus A$ there is an $f \in E^*$ with $|f(A)| \leq 1$ and $|f(a)| > 1$.

Observe that for spherically complete K the properties (α)-(δ) are true for each seminorm p (and we may even allow $|\lambda| \leq p(a)$ in (β) and $\varepsilon \geq 0$ in (γ)). The proof of Proposition 3.2 (by the above it suffices to consider only dense valuations) is straightforward and is omitted.

Definition 3.3. - Let E be a locally convex space over K, let $A \subset E$. Set $A^0 := \{f \in E' : |f(x)| \leq 1$ for all $x \in A\}$, $A^{00} := \{x \in E : |f(x)| \leq 1$ for all $f \in A^0\}$. A is a polar set if $A = A^{00}$.

The following consequences are immediate.

PROPOSITION 3.4. - Let E be a locally convex space over K.

(i) Each polar set in E is closed and edged.

(ii) A continuous seminorm p on E is polar if and only if $\{x \in E : p(x) \leq 1\}$ is a polar set.

(iii) A subset A of E is a polar set if and only if there is a collection P of polar continuous seminorms such that $A = \bigcap_{p \in P}\{x \in E : p(x) \leq 1\}$.

(iv) A subset A of E is a polar set if and only if there is a collection

$F \subset E'$ <u>such that</u> $A = \bigcap_{f \in F} \{x \in E : |f(x)| \leq 1\}$.

If K is spherically complete each closed edited set is polar ([14], Th. 4.6, Th. 4.8, 'closed & edged' = ' l'-closed'). If K is not spherically complete the 'closed' unit ball of l^∞/c_0 is closed and edged but not polar. See, however, Theorem 4.7. In the following definition we select the two classes of spaces we shall be dealing with throughout.

<u>Definition</u> 3.5. - Let E be a locally convex space over K . E is a <u>strongly polar space</u> if every continuous seminorm on E is polar. E is a <u>polar space</u> if its topology is defined by a family of polar seminorms.

4. Strongly polar spaces.

If K is spherically complete each locally convex space over K is strongly polar. If K is not spherically complete l^∞/c_0 (hence, l^∞) is not strongly polar. It is easy to see that the image of a strongly polar space under a continuous linear map (in particular, a quotient of a strongly polar space) is again strongly polar. We also have:

PROPOSITION 4.1. - <u>Each subspace of a strongly polar space is strongly polar.</u>

<u>Proof</u>. - Let D be a linear subspace of a strongly polar space E , let p be a continuous seminorm on D . Since D carries the relative topology there is a continuous seminorm q on E such that $p \leq q$ on D . The formula

$$r(x) = \inf_{d \in D} \max(p(d),\ q(x - d))$$

defines a seminorm r on E . We have $r \leq q$ (so that r is continuous) and $r = p$ on D . r is polar. Hence, so is its restriction p .

THEOREM 4.2. - <u>For a locally convex space</u> E <u>over</u> K <u>the following are equivalent.</u>

(α) E <u>is strongly polar.</u>

(β) (<u>Hahn-Banach property</u>). <u>For each linear subspace</u> D , <u>for each continuous seminorm</u> p , <u>for each</u> $\epsilon > 0$, <u>for each</u> $f \in D'$ <u>with</u> $|f| \leq p$ <u>on</u> D <u>there is an extension</u> $\bar{f} \in E'$ <u>of</u> f <u>such that</u> $|\bar{f}| \leq (1 + \epsilon)\, p$ <u>on</u> E .

<u>Proof</u>. - The implication (β) $\Longrightarrow$ (α) follows from Proposition 3.2 (γ) $\Longrightarrow$ (α), so we prove (α) $\Longrightarrow$ (β). We may assume that $f \neq 0$. Set $S := \operatorname{Ker} f$, let $\pi : E \to E/S$ be the quotient map, choose $x \in D \setminus S$. Let g be the K-linear map defined on $K\pi(x)$ that sends $\pi(x)$ into $f(x)$. For each $y \in E$ with $\pi(y) = \pi(x)$ we have $y \in D$ and $f(x) = f(y)$ so that, by $|f| \leq p$ on D ,

$$|g(\pi(x))| = |f(x)| \leq \inf_{y \in E} \{p(y) : \pi(y) = \pi(x)\} .$$

It follows that $|g| \leq \tilde{p}$ on $K\pi(x)$, where $\tilde{p}$ is the quotient seminorm of p on

E/S . As E/S is a strongly polar space we can use Proposition 3.2 (α) $\Longrightarrow$ (γ) to obtain an extension $\bar{g} \in (E/S)'$ of g such that $|\bar{g}| \leq (1+\varepsilon)\,\tilde{p}$ on E/S . Set $\bar{f} := \bar{g} \circ \pi$. Then $\bar{f} \in E^{\times}$, $\bar{f}$ extends f and for each $z \in E$ we have

$$|\bar{f}(z)| = |g(\pi(z))| \leq (1+\varepsilon)\,\tilde{p}(\pi(z)) \leq (1+\varepsilon)\,p(z) .$$

Definition 4.3. - A normed space over K is of countable type if there exists a countable subset whose linear span is dense ([11], p. 66). A locally convex space E over K is of countable type if for each continuous seminorm p the normed space E_p (see § 1 (f) (i)) is of countable type.

E is of countable type if and only if for each continuous seminorm p there exists a linear subspace, whose dimension is at most countable, that is p-dense in E (i. e. dense with respect to the topology induced by the single seminorm p). It is not hard to see that in the above we may replace without harm 'continuous seminorm p ' by 'continuous seminorm p belonging to some basis P of continuous seminorms'.

THEOREM 4.4. - A locally convex space of countable type is strongly polar.

Proof. - Let p be a continuous seminorm on the space E . It suffices to check that E_p is strongly polar. E_p is of countable type hence so is the Banach space $\hat{E}_p$. By [11], 3.16 (vi), this space is strongly polar. Then E_p is strongly polar (Proposition 4.1).

Open problem. - Let K be not spherically complete (e. g. $K = \mathbb{C}_p$) . Is every strongly polar space of countable type ?

4.5. Examples. - The following spaces are of countable type and therefore have the Hahn-Banach property 4.2 (β). (See also Corollary 4.14 (i).)

(i) Finite dimensional spaces, spaces with countable dimension.

(ii) Locally convex spaces with a Schauder basis.

(iii) The weak dual E'_σ of any locally convex space E over K .

(iv) Any locally convex space E with the weak topology $\sigma(E , E')$.

(v) For any ultrametric space X , the space $C(X \longrightarrow K)$ of all continuous functions $X \longrightarrow K$ with the topology of uniform convergence on compact subsets.

(vi) For any metrizable locally convex space E , its dual E' with the topology of uniform convergence on compact subsets.

(vii) The spaces $C^n(X \longrightarrow K)$, $C^\infty(X \longrightarrow K)$ ($X \subset K$ compact), $C^\infty(U \longrightarrow K)$, $C^\infty_c(U \longrightarrow K)$ ($U \subset \mathbb{Q}_p$ open) , $C^n_0(\mathbb{Q}_p \longrightarrow K)$, $C^\infty_0(\mathbb{Q}_p \longrightarrow K)$, $S(\mathbb{Q}_p \longrightarrow K)$, $A_\infty(K)$, $A_1(K)$ of § 2.

(For (iii) and (iv), observe that Ker p has finite codimension for each weakly

continuous seminorm p ; for (v), (vi) use the isomorphism $C(Y \to K) \simeq c_0$ ([11], 3.T) for an infinite compact ultrametrizable space Y ; (v) is a stepping stone for (vii).)

In view of the open problem above we are urged to study strongly polar spaces and spaces of countable type separately.

Further properties of strongly polar spaces.

It is quite easy to see that, if D is a dense subspace of a locally convex space E , then D is strongly polar if and only if E is strongly polar. In particular the completion of a Hausdorff strongly polar space is strongly polar. More generally we have the following.

PROPOSITION 4.6. - Let E be a locally convex space with strongly polar subspaces $E_1 \subset E_2 \subset \dots$ such that $\bigcup E_n$ is dense in E . Then E is strongly polar. In particular, the strict inductive limit of a sequence of strongly polar spaces is strongly polar.

Proof. - It suffices to prove the first statement. By the above remark we may assume $E = \bigcup E_n$. Let p be a continuous seminorm on E , let $x \in E$, $\varepsilon > 0$ and let $f \in (Kx)'$ such that $|f| \leq p$ on Kx . We extend f to an $\bar{f} \in E^{\times}$ with $|\bar{f}| \leq (1 + \varepsilon)\, p$ on E as follows. We have $x \in E_n$ for some n . Let $\varepsilon_n , \varepsilon_{n+1}, \dots$ be positive numbers such that $\prod_{j \geq n} (1 + \varepsilon_j) \leq 1 + \varepsilon$. By the strong polarity of E_n there is an extension $f_n \in E_n'$ of f with $|f_n| \leq (1 + \varepsilon_n)\, p$ on E_n . By Theorem 4.2 f_n extends to an $f_{n+1} \in E_{n+1}'$ such that

$$|f_{n+1}| \leq (1 + \varepsilon_{n+1})(1 + \varepsilon_n)\, p \quad \text{on} \quad E_{n+1} .$$

Inductively we arrive at an extension $\bar{f} \in E^{\times}$ of f such that

$$|\bar{f}| < \left(\prod_{n \geq j} (1 + \varepsilon_n)\right) p \leq (1 + \varepsilon)\, p \quad \text{on} \quad E .$$

Open problem. - Is the product of two strongly polar spaces again stronlgy polar ?

THEOREM 4.7 (Compare Proposition 3.4 (i)). - A locally convex space E over K is strongly polar if and only if each closed edged subset is polar.

Proof. - If each closed edged set is polar then, for each continuous seminorm p the set $\{x \in E : p(x) \leq 1\}$ is polar. Hence p is polar by Proposition 3.4 (ii). Conversely, let E be strongly polar, let A be a closed edged subset of E . We shall prove that for each $x \in E \setminus A$ there exists a continuous seminorm p such that $p(A) \leq 1$, $p(x) > 1$. (Then, by Proposition 3.4 (iii), we are done.) We may assume that K is not spherically complete, hence that the valuation is dense. There is a $\mu \in B(0 , 1^-)$ such that $\mu x \notin A$. Set $x_0 := \mu x$. Since A is closed

there is a continuous seminorm q on E such that

$$B := \{y \in E : q(y - x_0) < 1\}$$

does not meet A. Then $V := x_0 - B + A$ is open, absolutely convex. V contains A, $x_0 \notin V$. Let p be the seminorm associated to V. Then p is continuous, $p(A) \leq 1$ and $p(x_0) \geq 1$. It follows that $p(x) = p(\mu^{-1} x_0) > 1$.

COROLLARY 4.8. - <u>An edged subset of a locally convex space E is weakly closed if an only if it is a polar set.</u>

<u>Proof</u>. - From Proposition 3.4 (i) & (iv) it follows that polar sets are weakly closed and edged. If, conversely, A is weakly closed and edged then Theorem 4.7, applied to $\sigma(E, E')$ (which is strongly polar by 4.5 (iv)), implies that A is a polar set in $(E, \sigma(E, E'))$. But the dual of this space equals E' so that, by definition, A is also a polar set with respect to the initial topology of E.

COROLLARY 4.9. - <u>Every closed edged subset of a strongly polar space is weakly closed. In particular, closed linear subspaces of a strongly polar space are weakly closed.</u>

For a subset A of a locally convex space, let $\bar{A}$ be its closure, let $\bar{A}^\sigma$ be its weak closure. We have the following general relation.

PROPOSITION 4.10. - <u>Let A be an absolutely convex subset of a locally convex space. Then</u> $(\bar{A}^\sigma)^e = A^{00}$.

<u>Proof</u>. - $(\bar{A}^\sigma)^e$ is weakly closed, edged, hence polar by Corollary 4.8. As A^{00} is the smallest polar set containing A we have $A^{00} \subset (\bar{A}^\sigma)^e$. On the other hand, A^{00} is weakly closed, edged, $(\bar{A}^\sigma)^e$ is the smallest edged, weakly closed set containing A so that $(\bar{A}^\sigma)^e \subset A^{00}$.

Now let A be an absolutely convex subset of a strongly polar space. Then $(\bar{A})^e$ is closed, edged, hence polar by Corollary 4.9. With Proposition 4.10 we arrive at

$$(\bar{A})^e = (\bar{A}^\sigma)^e = A^{00}.$$

In particular, if A is also closed for the initial topology we obtain

$$A \subset \bar{A}^\sigma \subset A^e$$

so that (for densely valued fields K) we have

$$A \subset \bar{A}^\sigma \subset \lambda A$$

for each $\lambda \in K$, $|\lambda| > 1$. This leads to the following.

<u>Open problem</u>. - Characterize the weakly closed absolutely convex sets of a strongly polar space over a nonspherically complete K. (If K is spherically complete it is shown in [15] that an absolutely convex subset of a locally convex

space is closed if and only if it is weakly closed.) Theorem 5.13 (iv) offers a partial answer.

Related to this problem is the following proposition, extending [1], Proposition 5, that will be used later on (Corollary 10.9).

PROPOSITION 4.11. - Let E be a strongly polar Hausdorff space. Then each weakly convergent sequence in E is convergent.

Proof. - Let us say that a locally convex space has (OP) (Orlicz-Pettis) if each weakly convergent sequence is convergent. A standard reasoning shows that if each member of a family of locally convex spaces has (OP) then so has their product. Further, subspaces of spaces having (OP) have (OP). Now let E be strongly polar, Hausdorff. For each continuous seminorm p the space $E_p^\wedge$, hence $E_p^\wedge$ is strongly polar. By [9], 5.2, $E_p^\wedge$ has (OP). Then also E, being isomorphic to a subspace of $\prod_p E_p^\wedge$, has (OP).

Remark. - The conclusion of Proposition 4.11 holds for every Hausdorff locally convex space over a spherically complete K. If K is not spherically complete then ℓ^∞ is not strongly polar (but ℓ^∞ is polar) ; the sequence $(1,0,0,0,\ldots)$, $(0,1,0,0,\ldots)$, $(0,0,1,0,\ldots)$ converges weakly (to 0) but not strongly.

Further properties of spaces of countable type.

We denote the class of spaces of countable type over K by (S_0). We have the following stability properties.

PROPOSITION 4.12. - Let E be a locally convex space over K.

(i) If $E \in (S_0)$ and D is a linear subspace of E then $D \in (S_0)$ and $E/D \in (S_0)$.

(ii) If D is a dense linear subspace of E then $D \in (S_0)$ if and only if $E \in (S_0)$.

(iii) (S_0) is closed for products and for countable (locally convex) direct sums.

(iv) Let $E_1 \subset E_2 \subset \ldots$ be subspaces in (S_0) of E such that $\bigcup E_n$ is dense in E. Then $E \in (S_0)$.

Proof.

(i) Let p be a continuous seminorm on D. There is a continuous seminorm q on E whose restriction to D is p. In the commutative diagram (see § 1,(f))

$$\begin{array}{ccc} E & \xrightarrow{\pi_q} & E_q^\wedge \\ \uparrow i & & \uparrow j \\ D & \xrightarrow{\pi_p} & D_p^\wedge \end{array}$$

(where i is the inclusion map) the map j is a linear isometry. $E_q^\wedge$ is a Banach space of countable type. Hence ([11], 3.16) so is $D_p^\wedge$. It is an easy exercise to show that the normed space D_p (a dense subspace of $D_p^\wedge$) is also of countable type. It follows that $D \in (S_0)$. Similarly, if p is a continuous seminorm on E/D then $p \circ \pi$ (where $\pi : E \to E/D$ is the quotient map) is a continuous seminorm on E. The map $E_{p\circ\pi} \to (E/D)_p$ is surjective, continuous ; the normed space $E_{p\circ\pi}$ is of countable type. Hence, so is $(E/D)_p$. It follows that $E/D \in (S_0)$.

(ii) If D is dense the map j in the above diagram is a surjective isometry. So if $D \in (S_0)$ then for each continuous seminorm q on E (with restriction p on D) the space $D_p^\wedge$ is of countable type. Hence, so are $E_q^\wedge$ and E_q. Thus, $E \in (S_0)$.

(iii) We first prove that $E_1 \in (S_0)$, $E_2 \in (S_0)$ implies $E_1 \times E_2$ (S_0). It suffices to prove that $(E_1 \times E_2)_{p_1\times p_2} \in (S_0)$ for p_1 (p_2) a continuous seminorm on E_1 (E_2) and where $(p_1 \times p_2)(x, y) := \max(p_1(x), p_2(y))$ $((x,y)$ $E_1\times E_2)$. But $(E_1 \times E_2)_{p_1\times p_2}$ is isometrically isomorphic to $(E_1)_{p_1} \times (E_2)_{p_2}$ which latter space can be embedded into $c_0 \times c_0 \simeq c_0$. It follows that (S_0) is closed for finite products. For the general case let I be an indexing set and, for each $i \in I$, let E_i be a space in (S_0). Let p be a continuous seminorm on $\prod_i E_i$. Since the unit semiball of p is open in the product space it contains a subset of the form $\prod U_i$ where U_i is open E_i for each i and where $U_i \neq E_i$ only for $i \in \{i_1, \dots, i_n\}$ for some $n \in \mathbb{N}$. Thus p factors through $\prod_{j=1}^n E_{i_j}$:

$$\begin{tikzcd} \prod_i E_i \arrow[rr, "g"] \arrow[dr, "p"'] & & \prod_{j=1}^n E_{i_j} \arrow[dl, "\tilde{p}"] \\ & \mathbb{K} & \end{tikzcd}$$

where the map g in the diagram is the canonical projection. The seminorm $\tilde{p}$ is continuous so by what we just have proved there is a countable set A in $\prod_{j=1}^n E_{i_j}$ such that $[\![A]\!]$ (the K-linear span of A) is $\tilde{p}$-dense in $\prod_{j=1}^n E_{i_j}$. Then, if B is a countable set in $\prod E_i$ such that $g(B) = A$, the set $[\![B]\!]$ is p-dense in $\prod_i E_i$. Thus, (S_0) is closed for products. Now let $E := \bigoplus_{i\in\mathbb{N}} E_i$ be the locally convex direct sum of the space $E_1, E_2, \dots \in (S_0)$. For $n \in \mathbb{N}$, set $F_n := \bigoplus_{i=1}^n E_i$, considered as a subspace of E. Then $F_n \simeq \prod_{i=1}^n E_i$ so that, by the preceding proof, $F_n \in (S_0)$. The result $E \in (S_0)$ now follows from (iv).

(iv) By (ii) we may assume $\bigcup E_n = E$. Let p be a continuous seminorm on E. For each $n \in \mathbb{N}$, choose a countable set $A_n \subset E_n$ such that $[\![A_n]\!]$ is p-dense in E_n. Then $A := \bigcup A_n$ is countable and $[\![A]\!]$ is p-dense in E.

As a corollary we obtain the following characterisation.

THEOREM 4.13. - <u>A locally convex Hausdorff space is of countable type if and only if it is linearly homeomorphic to a subspace of c_0^I for some set I.</u>

Proof. - By Proposition 4.12 each subspace of c_0^I is in (S_0). Conversely, if $E \in (S_0)$, E Hausdorff, then each $E_p^\wedge$ (§ 1, (f)) is linearly homeomorphic to a subspace of c_0. One easily constructs an embedding $E \to c_0^\Gamma$ where Γ is the collection of continuous seminorms of E.

It turns out that the class (S_0) coincides with the class (S_0) of [4], § 3, Definition 1. This, combined with Proposition 1.3 yields the following.

COROLLARY 4.14.

(i) Each nuclear space is of countable type.

(ii) The strict inductive limit of a sequence of nuclear spaces is nuclear.

(iii) $C_c^\infty(U \to K)$ (see § 2.4) is nuclear.

Proof. - (i) is immediate, (iii) follows from (ii). To prove (ii), let E be the strict inductive limit of the nuclear spaces $E_1 \subset E_2 \subset \ldots$ By Proposition 4.12 (iv) E is of countable type. To prove that E also satisfies the second condition of Proposition 1.3 (β), let $T \in L(E, c_0)$.

For each n there is an absolutely convex neighbourhood U_n of 0 in E_n such that TU_n is a compactoid subset of the unit ball of c_0. Choose $\lambda_1, \lambda_2, \ldots \in K^\times$ with $\lim_{n\to\infty} \lambda_n = 0$ and set $U := \sum \lambda_n U_n$. Then U is absolutely convex and open in the inductive limit topology. For each $\varepsilon > 0$ we have for almost all n that $\lambda_n TU_n \subset \{x \in c_0 : \|x\| < \varepsilon\}$. It follows readily that $TU = \sum \lambda_n TU_n$ is a compactoid.

In § 8 we shall discuss spaces whose duals are of countable type.

5. Polar spaces.

5.1. Examples. - The following spaces are polar (Definition 3.5).

(i) Strongly polar spaces (see 4.5).

(ii) ℓ^∞. More generally:

(iii) Function spaces (E, τ) of the following type. Let X be a set, let E be a linear space of functions $X \to K$, let B be a collection of subsets of X such that each $f \in E$ is bounded on each element of B, let τ be the topology of uniform convergence on members of B. (For example, the Banach space $BC(X \to K)$ ([11], 3.D) is polar, the strong dual of a locally convex space is polar.)

(iv) For each locally convex space E and each polar space F, the space $L(E,F)$ with the topology of uniform convergence on members of anu class of bounded subsets of E.

(Direct verification.)

The proofs of the next two propositions are also straightforward.

PROPOSITION 5.2. - For a locally convex space E the following are equivalent.

(α) E is a polar space.

(β) For each continuous seminorm q on E there is a polar continuous seminorm p on E such that $p \geq q$.

(γ) The polar neighbourhoods of 0 form a neighbourhood basis of 0 for the topology of E.

PROPOSITION 5.3. - Projective limits (in particular, subspaces and products) of polar spaces are polar.

Quotients of polar spaces need not be polar (ℓ^∞/c_0 for a nonspherically complete base field). But we do have the following.

PROPOSITION 5.4. - The (locally convex) direct sum of any collection of polar spaces is polar.

Proof. - Let E be the direct sum of the polar spaces E_i $(i \in I)$ where I is an indexing set and let $\Phi_i : E_i \to E$ $(i \in I)$ be the canonical injections. Let q be a continuous seminorm on E. For each $i \in I$ the seminorm $q \circ \Phi_i$ is continuous, so there is a continuous polar seminorm p_i on E_i with $p_i \geq q \circ \Phi_i$. Each $x \in E$ has a unique representation $x = \sum_i \Phi_i(x_i)$ where $x_i \in E_i$ for each i and where $\{i : x_i \neq 0\}$ is finite. Set $p(x) := \max_i p_i(x_i)$. Then p is a seminorm on E. It is continuous since $p \circ \Phi_i = p_i$ for each $i \in I$. For each $x \in E$ we have $q(x) = q(\sum \Phi_i(x_i)) \leq \max q \circ \Phi_i(x_i) \leq \max p_i(x) = p(x)$ so that $p \geq q$. We finish the proof by showing that p is polar unsing Proposition 3.2 (β) $\Longrightarrow$ (α). Let $x = \sum \Phi_i(x) \in E$, let $\lambda \in K$, $|\lambda| < p(x)$. We have $p(x)=p_j(x_j)$ for some $j \in I$. There is an $f_j \in E_j'$ such that $f_j(x_j) = \lambda$, $|f_j| \leq p_j$ on E_j. Define $f \in E^*$ by the formula $f(\sum \Phi_i(y_i)) = f_j(y_j)$. Then $|f| \leq p$, $f(x)=\lambda$ and we are done.

PROPOSITION 5.5. - Let E be a dense linear subspace of a locally convex space F. Then E is polar if and only if F is polar. In particular, the completion of a Hausdorff polar space is polar.

Proof. - Left to the reader.

PROPOSITION 5.6. - Let D be a finite dimensional subspace of a Hausdorff polar space E.

(i) Each $f \in D'$ can be extended to an $\bar{f} \in E'$.

(ii) D is a polar set.

Proof. - $\sigma(E, E')$ is Hausdorff, so D is weakly closed, hence polar by Corollary 4.8. To prove (i), let $f \in D'$. By (ii), $\operatorname{Ker} f$ is a polar set. For $a \in D \setminus \operatorname{Ker} f$ there is a $g \in E'$ with $g = 0$ on $\operatorname{Ker} f$ and $g(a) = f(a)$. Then g is an extension of f.

Remark. - It is not possible to extend Proposition 5.6 by admitting D to be of countable type : let K be not spherically complete, set $E := \ell^\infty$, $D := c_0$. Then ([11], 4.15 $(\epsilon) \Longrightarrow (\gamma)$, $(\delta) \Longrightarrow (\gamma)$) $D^{00} = E$, the map $x \longmapsto \sum x_n$ $(x \in D)$ cannot be extended to an element of E' .

It is our purpose to prove the fundamental Theorem 5.12 for polar spaces. First some general observations and lemmas. Let U be a polar zero neighbourhood in a locally convex E , let $a \in E$. It is somewhat doubful whether $U + co(a)$ is again polar, or weakly closed, or edged. (Of course, $U + co(a)$ is absolutely convex and strongly open and closed.) But we can prove the following.

LEMMA 5.7. - Let U be a polar neighbourhood of 0 in a locally convex space E, let $a \in E$. Then $(U + co(a))^{00} = (U + co(a))^e$.

Proof. - If K is spherically complete then E is strongly polar and the equality follows from Theorem 4.7. So assume that the valuation is dense. It suffices to prove $(U + co(a))^{00} \subset (U + co(a))^e$, i. e. that for each $\lambda \in K$, $|\lambda| > 1$ the weak closure of $U + co(a)$ is contained in $\lambda(U + co(a))$ (Proposition 4.10). So let $x \in \overline{U + co(a)}^\sigma$; there is a net (x_i) $(i \in I)$ in $U + co(a)$ converging weakly to x . The seminorm p associated to U is polar so there is an $f \in E'$ with $|f| \leqslant p$ and $|f(a)| \geqslant |\lambda^{-1}|\, p(a)$. For each $i \in I$ we have a decomposition $x_i = u_i + \xi_i\, a$ $(u_i \in U$, $\xi_i \in B(0 , 1))$. So, for each i , $j \in I$

$$f(x_i) - f(x_j) = f(u_i - u_j) + (\xi_i - \xi_j)\, f(a) .$$

There is an i_0 such that $|f(x_i) - f(x_j)| \leqslant 1$ for all i , $j \geqslant i_0$. Since also $|f| \leqslant 1$ on U we find

$$|(\xi_i - \xi_{i_0})\, f(a)| \leqslant 1 \quad (i \geqslant i_0) .$$

We have $p((\xi_i - \xi_{i_0})a) \leqslant |\lambda|\, |\xi_i - \xi_{i_0}|\, |f(a)| \leqslant |\lambda|$. It follows that $(\xi_i - \xi_{i_0})\, a \in \lambda U$ for $i \geqslant i_0$. From

$$x_i - \xi_{i_0}\, a = u_i + (\xi_i - \xi_{i_0})\, a$$

we obtain $x_i - \xi_{i_0}\, a \in U + \lambda U = \lambda U$. Since λU is weakly closed we have

$$x - \xi_{i_0}\, a \in \lambda U, \text{ i. e. } x \in \lambda U + co(a) \subset \lambda(U + co(a)) .$$

COROLLARY 5.8. - Let U be a polar neighbourhood of 0 in a locally convex space E , let $x_1 , \dots , x_n \in E$. Then $(U + co(x_1 , \dots , x_n))^{00} = (U + co(x_1 , \dots , x_n))^e$.

Proof. - By induction. The case $n = 1$ is Lemma 5.7. Suppose the statement is true for $n = m - 1$. Let $x_1 , \dots , x_m \in E$. Set $V := U + co(x_1 , \dots , x_{m-1})$. By the induction hypothesis $V^{00} \subset V^e$. So we get, using Lemma 5.7 again,

$$(U + co(x_1 , \dots , x_m))^{00}$$
$$= (V + co(x_m))^{00} \subset (V^{00} + co(x_m))^{00} \subset (V^{00} + co(x_m))^e \subset (V^e + co(x_m))^e$$
$$= (V + co(x_m))^e = (U + co(x_1 , \dots , x_m))^e .$$

LEMMA 5.9. - Let U be a polar neighbourhood of 0 in a locally convex space E, let $a \in E$. Choose $\lambda \in K$; $|\lambda| > 1$ if the valuation is dense, $\lambda = 1$ if the valuation is discrete. If (x_i) is a net in $U + co(a)$ converging weakly to 0 then $x_i \in \lambda U$ for large i.

Proof. - Let p be the seminorm associated to U. There is an $f \in E'$ with $|f| \leqslant p$, $|f(a)| \geqslant |\lambda^{-1}|\, p(a)$. As in the proof of Lemma 5.7 we have the decompositions $x_i = u_i + \xi_i a$. Since $f(x_i) \longrightarrow 0$ and $|f(u_i)| \leqslant 1$ we have $|f(\xi_i a)| \leqslant 1$ for large i. Thus, $p(\xi_i a) \leqslant |\lambda|$, $|f(\xi_i a)| \leqslant |\lambda|$, i. e. $\xi_i a \in \lambda U$ for large i. It follows that $x_i \in \lambda U$ for large i.

COROLLARY 5.10. - Let U, a, λ be as in the previous lemma. Let (x_i) be a net in $(U + co(a))^e$ converging weakly to 0. Then $x_i \in \lambda U$ for large i.

Proof. - It suffices to consider the case where the valuation is dense. Choose $\mu \in K$, $1 < |\mu| < |\lambda|$. Then $\lambda^{-1} \mu x_i \in U + co(a)$ for all i so that, by Lemma 5.9, $\lambda^{-1} \mu x_i \in \mu U$, i. e. $x_i \in \lambda U$ for large i.

COROLLARY 5.11. - Let U, λ be as in Lemma 5.9, let $a_1 , \dots , a_n \in E$. If (x_i) is a net in $(U + co(a_1 , \dots , a_n))^e$ converging weakly to 0 then $x_i \in \lambda U$ for large i.

Proof. - If the valuation is dense, choose $\lambda_1 , \dots , \lambda_n \in K$ such that $|\lambda_j| > 1$ for all j and $\prod |\lambda_j| \leqslant |\lambda|$. If the valuation is discrete set $\lambda_j := 1$ for all $j \in \{1 , \dots , n\}$. We have

$$(U + co(a_1 , \dots , a_n))^e = [(U + co(a_1 , \dots , a_{n-1}))^e + co(a_n)]^e .$$

By Lemma 5.7 the set $(U + co(a_1 , \dots , a_{n-1}))^e$ is a polar neighbourhood of 0 so by Corollary 5.10 we have $\lambda_n^{-1} x_i \in (U + co(a_1 , \dots , a_{n-1}))^e$ for large i. Inductively we arrive at $\lambda_1^{-1} \lambda_2^{-1} , \dots , \lambda_n^{-1} x_i \in U$ for large i implying that $x_i \in \lambda U$ for large i.

THEOREM 5.12. - Let E be a polar space. Then, on compactoids, the weak topology and the initial topology coincide.

Proof. - Let A be a compactoid. We may assume that A is absolutely convex so that it suffices to prove that, for a net (x_i) in A, $x_i \longrightarrow 0$ weakly implies $x_i \longrightarrow 0$ strongly. Let V be a neighbourhood of 0 in E, let $\lambda \in K$, $|\lambda| > 1$. E is a polar space so there is a polar neighbourhood U of 0 such that $U \subset \lambda^{-1} V$. There exist $a_1 , \dots , a_n \in E$ such that $A \subset U + co(a_1 , \dots , a_n)$. By Corollary 5.11 we have $x_i \in \lambda U \subset V$ for large i. It follows that $x_i \longrightarrow 0$ strongly.

The previous machinery yields the following.

THEOREM 5.13. - Let E be a polar space, let $A \subset E$ be a compactoid.

(i) A^{00} is a compactoid.

(ii) A is closed if and only if A is weakly closed.

(iii) If A is closed and absolutely convex then $A^{00} = A^{e}$.

(iv) If A is closed and edged then $A^{00} = A$.

(v) If A is absolutely convex then A is complete if and only if A is weakly complete.

Proof.

(i) Let U be a polar neighbourhood of 0 , let $\lambda \in K$, $0 < |\lambda| < 1$. There are $x_1 , \dots , x_n \in E$ such that $A \subset \lambda U + co(x_1 , \dots , x_n)$. Using Corollary 5.8 we obtain

$$A^{00} \subset (\lambda U + co(x_1 , \dots , x_n))^{00}$$
$$= (\lambda U + co(x_1 , \dots , x_n))^{e} \subset \lambda^{-1}(\lambda U + co(x_1 , \dots , x_n))$$
$$= U + co(\lambda^{-1} x_1 , \dots , \lambda^{-1} x_n).$$

(ii) From (i) it follows that $\bar{A}^{\sigma}$ is a compactoid. Now apply Theorem 5.12.

(iii) and (iv) follow from (ii) and Proposition 4.10. The proof of (v) is standard.

6. Polarly barreled and bornological spaces.

Definition 6.1. - A locally convex space over K is polarly barreled if every polar barrel is a neighbourhood of 0 .

This notion will suit our purposes in duality theory rather than just 'barreled'. Obviously each barreled space is polarly barreled so that Banach spaces, Fréchet spaces, LF-spaces (see [14]) are polarly barreled. Hence, the spaces of § 2, (2.1)-(2.8) are (polarly) barreled. In general, we have :

PROPOSITION 6.2. - A strongly polar space is barreled if and only if it is polarly barreled.

Proof. - Suppose E is strongly polar, polarly barreled. Let A be a barrel in E . Then A^{e} is edged, closed, so by Corollaries 4.9 and 4.8, A^{e} is a polar set. Then A^{e} is a neighbourhood of 0 . For each $\lambda \in K$, $|\lambda| > 1$ we have $A^{e} \subset \lambda A$. It follows that A is a neighbourhood of 0 so that E is barreled.

Open problem. - Do there exist (polar) spaces that are polarly barreled but not barreled ?

The proofs of the following two propositions are easy and therefore omitted.

PROPOSITION 6.3. - For a locally convex space E over K the following are equivalent.

(α) E is polarly barreled.

(β) If $F \subset E'$ and $p := \sup\{|f| : f \in F\}$ exists then p is continuous.

(γ) The set of all polar continuous seminorms on E is closed for suprema.

PROPOSITION 6.4. - Inductive limits (in particular, direct sums and quotients) of polarly barreled spaces are polarly barreled.

PROPOSITION 6.5. - Let E be a polarly barreled space, let $H \subset E'$. The following are equivalent.

(α) H is bounded for $\sigma(E', E)$.

(β) H is a compactoid for $\sigma(E', E)$.

(γ) H is bounded for $b(E', E)$.

(δ) H is equicontinuous.

(ε) There is a continuous polar seminorm p on E such that $|f| \leq p$ for all $f \in H$.

Proof. - (α) $\Longrightarrow$ (β) is a consequence of the fact that Ker p has finite codimension in E' for each $\sigma(E', E)$-continuous seminorm p on E'. For (α) $\Longrightarrow$ (ε) observe that $p(x) := \sup\{|f(x)| : f \in H\} < \infty$ for each $x \in E$. By Proposition 6.3 p is continuous. From the definition of p it follows that p is polar. The proofs of the remaining implications are either obvious or are similar to the corresponding 'classical' proofs (see [16] Ch. 33).

THEOREM 6.6. - Let E be a polarly barreled space. Then E'_σ and E'_b are (polar) quasicomplete Hausdorff spaces.

Proof. - Similar to the proof of [16], 34.3, Corollary 2.

We briefly consider a polar version of the notion of a bornological space.

Definition 6.7. - A nonempty subset A of a vector space E over K is K-polar if for each $x \in E \setminus A$ there exists an $f \in E^*$ such that $|f(A)| \leq 1$, $|f(x)| > 1$. A locally convex space E over K is polarly bornological if every K-polar set that absorbs every bounded set is a neighbourhood of 0.

Moving along the line of the 'classical' theory the following proposition is not hard to prove.

PROPOSITION 6.8. - Let E be a polarly bornological space. Then E'_b is complete.

PROPOSITION 6.9. - The spaces of § 2, (2.1)-(2.8) are bornological, their strong duals are complete.

Proof. [14] and Proposition 6.8.

7. Topologies compatible with a duality.

Let E be a locally convex space over K. For each $x \in E$ we define the map $j_E(x) : E' \to K$ by the formula $j_E(x)(f) = f(x)$ $(f \in E')$.

LEMMA 7.1. - Let E be a locally convex space over K.

(i) j_E is a linear map of E onto $(E'_\sigma)'$.

(ii) If E is Hausdorff and polar then j_E is a bijection of E onto $(E'_\sigma)'$.

Proof. - For (i) see [14], Th. 4.10. To prove (ii), let $x \in E$, $x \neq 0$. There exists a polar continuous seminorm p with $p(x) \neq 0$. We have

$$p = \sup\{|f| : f \in E', |f| \leq p\}$$

so there exists an $f \in E'$ with $f(x) \neq 0$. It follows that j_E is injective.

From now on in § 7, E is a polar Hausdorff locally convex space over K with topology τ_0, also called the initial topology. $E' := (E, \tau_0)'$.

Definition 7.2. - A polar topology ν on E is τ_0-compatible if $(E, \nu)' = E'$. If there exists a strongest τ_0-compatible topology on E it is the Mackey topology.

It is proved in [14] (Th. 4.18.a) that for a locally convex space over a spherically complete K the Mackey topology exists and equals the topology induced by the seminorms $x \mapsto \sup\{|f(x)| : f \in A\}$ where A runs through the collection of all subsets of E' that are bounded and c-compact for the topology $\sigma(E', E)$.

Open problem. - For a polar Hausdorff locally convex space over a nonspherically complete field, does there exist a Mackey topology ?

(For a partial answer see Corollary 7.8.)

Fortunately it will turn out in the subsequent sections that a full answer is not needed to set up a decent duality theory.

Definition 7.3. - A special covering of E' is a covering G of E' such that

(i) each member of G is edged, $\sigma(E', E)$-bounded, $\sigma(E', E)$-complete,

(ii) for each $A, B \in G$ there is a $C \in G$ such that $A \cup B \subset C$,

(iii) for each $A \in G$ and $\lambda \in K$ there is a $B \in G$ with $\lambda A \subset B$.

For a special covering G of E' the G-topology on E is the topology induced by the seminorms $x \mapsto \sup\{|f(x)| : f \in A\}$ where A runs through G.

By identifying E to the dual of E'_σ (as vector spaces) by means of the map j_E (Lemma 7.1 (ii)) we may view a G-topology as the topology on $(E'_\sigma)'$ of uniform convergence on members of G. Hence, (Example 5.1 (iv)) a G-topology is polar. It is also Hausdorff since G is a covering of E'.

PROPOSITION 7.4. - <u>For a (polar, Hausdorff) locally convex topology ν on E the following are equivalent.</u>

(α) ν <u>is τ_0-compatible.</u>

(β) ν <u>is a G-topology for some special covering G of E'.</u>

<u>Proof.</u> - $(\alpha) \Longrightarrow (\beta)$. For each ν-continuous polar seminorm p set

$$A_p := \{f \in E^* : |f| \leqslant p\} .$$

Then, by (α), $A_p \subset E'$. It takes a standard reasoning to show that (β) is true for $G := \{A_p : p$ is a ν-continuous polar seminorm on $E\}$. $(\beta) \Longrightarrow (\alpha)$. We have seen that ν is polar, Hausdorff. It is easy to see that the weak topology $\sigma = \sigma(E, E')$ is τ_0-compatible and that ν is stronger than σ. Hence we have an inclusion map $E' \hookrightarrow (E, \nu)'$ which is obviously a homeomorphism into with respect to the weak topologies induced by E. Let $g \in (E, \nu)'$; we shall prove that $g \in E'$. There are $A_1, \dots, A_n \in G$ and $\lambda \in K$ such that

$$|g| \leqslant |\lambda| \max(p_1, \dots, p_n)$$

where $p_i(x) = \sup\{|f(x)| : f \in A_i\}$ $(x \in E)$ for each $i \in \{1, \dots, n\}$. There is a $B \in G$ with $B \supset \lambda(A_1 \cup \dots \cup A_n)$. Set $p(x) = \sup\{|f(x)| : f \in B\}$ $(x \in E)$. Then $|g| \leqslant p$. B is complete in E'_σ and therefore closed in $(E, \nu)'_\sigma$. As B is also edged and $(E, \nu)'_\sigma$ is strongly polar (Example 4.5 (iii)) we have by Theorem 4.7 that B is a polar set in $(E, \nu)'_\sigma$. It therefore suffices to prove that $g \in B^{00}$ (where B is considered as a subset of $(E, \nu)'_\sigma$), i. e. that $|\vartheta(g)| \leqslant 1$ for all $\vartheta \in B^0$. By Lemma 7.1 (i) each element of the dual of $(E', \nu)'_\sigma$ has the form δ_x for some $x \in E$, where $\delta_x(h) = h(x)$ $(h \in (E, \nu)'_\sigma)$. Thus, let $\vartheta \in B^0$, $\theta = \delta_x$. Then $|\delta_x(h)| = |h(x)| \leqslant 1$ for all $h \in B$ so that $p(x) \leqslant 1$. Hence, $|\vartheta(g)| = |g(x)| \leqslant p(x) \leqslant 1$ and we are done.

As a corollary we obtain the following non-archimedean version of Mackey's theorem.

THEOREM 7.5. - <u>All τ_0-compatible topologies on E have the same bounded sets.</u>

<u>Proof.</u> - Since the weak topology $\sigma(E, E')$ is τ_0-compatible it suffices to prove that any $\sigma(E, E')$-bounded set $B \subset E$ is ν-bounded for any τ_0-compatible topology ν. By Proposition 7.4 ν is a G-topology for some special covering G of E'. We prove that for each $C \in G$ the seminorm

$$p : x \longmapsto \sup\{|f(x)| : x \in C\}$$

is bounded on B. Now C is absolutely convex, bounded and complete in E'_σ and B^0

is a barrel in E'_σ . By Lemma 7.6 below there is a $\lambda \in K$ such that $C \subset \lambda B^0$ and it follows that $p \leq |\lambda|$ on B .

LEMMA 7.6. - Let B be a barrel in a Hausdorff locally convex space, let C be bounded, absolutely convex, complete. Then there is a $\lambda \in K$ such that $C \subset \lambda B$.

Proof. - Similar to the proof in the complex case (see for example [16], Lemmas 36.2, 36.1, 34.2) and therefore omitted.

COROLLARY 7.7. - A subset of E is bounded if and only if it is weakly bounded.

On the existence of the Mackey topology we have the following result.

COROLLARY 7.8. - Suppose E has the following property. Each polar barrel that absorbs every bounded set is a neighbourhood of 0 . Then the Mackey topology for E exists and is equal to the initial topology τ_0 .

Proof. - Let ν be any τ_0-compatible topology on E ; we prove that $\nu \leq \tau_0$.

Let U be a ν-open neighbourhood of 0 . To prove that U is also τ_0-open we may assume that U is a polar set for ν . Then U is also polar for τ_0 and U is a polar barrel in E . U absorbs every ν-bounded set, hence every τ_0-bounded set by Mackey's Theorem 7.5. By assumption U is a τ_0-neighbourhood of 0 .

COROLLARY 7.9. - The conclusion of Corollary 7.8 holds for polarly barreled spaces and also for polarly bornological spaces. In particular, each one of the spaces of § 2,,(2.1)-(2.8) has a Mackey topology which equals the initial topology.

8. Duals of countable type.

If K is not spherically complete the spaces c_0 and ℓ^∞ are strong duals of one another ([11], 4.17). c_0 is of countable type, ℓ^∞ is not. In this section we derive conditions on E in order that E'_b be of countable type. To this end we establish some properties of compactoids in general locally convex spaces first.

LEMMA 8.1. - Let A be an absolutely convex compactoid in a locally convex space E over K . Let $\lambda \in K$, $|\lambda| > 1$ if the valuation is dense, $\lambda = 1$ if the valuation is discrete. Then for each neighbourhood U of 0 there exist $x_1, \dots, x_n \in \lambda A$ such that $A \subset U + co(x_1 , \dots , x_n)$.

Proof. - (A proof appears in [8] but is also included here for the reader's convenience.) There is a continuous seminorm p such that $\{x : p(x) \leq 1\} \subset U$. We use the notations of § 1,(f). $\pi_p(A)$ is an absolutely convex compactoid in $E_p^\wedge$. By [6], Proposition 1.6, $\pi_p(A)$ is also a compactoid in $F := \overline{[\pi_p(A)]}$. By [11], 4S (viii), the space F is of countable type so that by [11], 4.37 (b), every t-orthogonal sequence in $\pi_p(A)$ tends to 0 . By [11], 4.36 A and C, there exist

$e_1, e_2, \ldots \in \lambda\pi_p(A)$ with $\lim_{n\to\infty} e_n = 0$ in $E_p^\wedge$ and such that $\pi_p(A) \subset \overline{co(e_1, e_2, \ldots)}$. We have $\bar{p}(e_m) \leqslant 1$ for large m so there is an $n \in \mathbb{N}$ such that $\pi_p(A) \subset \{x : \bar{p}(x) \leqslant 1\} + co(e_1, \ldots, e_n)$.

Choose $x_1, \ldots, x_n \in \lambda A$ with $\pi_p(x_i) = e_i$ for $i \in \{1, \ldots, n\}$. Then, using the fact that $\operatorname{Ker} \pi_p = \operatorname{Ker} p$ we arrive easily at

$$A \subset \{x : p(x) \leqslant 1\} + co(x_1, \ldots, x_n) \subset U + co(x_1, \ldots, x_n).$$

PROPOSITION 8.2. - <u>Let A be a metrizable absolutely convex compactoid in a locally convex space E over K. Let λ be as in Lemma 8.1. Then there is a compact set $X \subset \lambda A$ such that $A \subset \overline{co\, X}$. For X we may choose a set of the form $\{0, e_1, e_2, \ldots\}$ where $\lim_{n\to\infty} e_n = 0$.</u>

<u>Proof</u>. - (Our proof is a slight modification of the one of [7], 2.3.) There is a sequence $V_1 \supset V_2 \supset \ldots$ of absolutely convex neighbourhoods of 0 in E such that for each neighbourhood U of 0 there exists an $n \in \mathbb{N}$ such that $U \cap A \supset V_n \cap A$. Choose $\rho_1, \rho_2, \ldots \in K$ such that $|\rho_n| > 1$ for all n, $\prod |\rho_n| \leqslant |\lambda|$ if K has a dense valuation and such that $\rho_n = 1$ for all n if K has a discrete valuation. By the previous lemma there is a finite set $F_1 \subset \rho_1 A$ such that $A \subset V_1 + co\, F_1$. Since $(A + co\, F_1) \cap V_1$ is an absolutely convex compactoid there is a finite set $F_2 \subset \rho_2[(A + co\, F_1) \cap V_1]$ such that

$$(A + co\, F_1) \cap V_1 \subset V_2 + co\, F_2.$$

Inductively we obtain a sequence $F_1, F_2, \ldots$ of finite sets such that for each $n \in \mathbb{N}$

$$(*) \qquad (A + co(F_1 \cup \ldots \cup F_n)) \cap V_n \subset V_{n+1} + co\, F_{n+1}$$

$$(**) \qquad F_{n+1} \subset \rho_{n+1}[(A + co(F_1 \cup \ldots \cup F_n)) \cap V_n].$$

We claim that $X := \{0\} \cup \bigcup_n F_n$ has the required properties :

(i) From $F_1 \subset \rho_1 A$ and $(**)$ we obtain inductively that $F_n \subset \rho_n \rho_{n-1} \ldots \rho_1 A$ $(n \in \mathbb{N})$ so that $X \subset \lambda A$.

(ii) Let U be a neighbourhood of 0 ; we prove that $X \setminus U$ is finite. (Then it follows that $X = \{0, e_1, e_2, \ldots\}$ for some sequence $e_1, e_2, \ldots$ with limit 0 and X is compact.) Choose a neighbourhood U' of 0 such that $\lambda U' \subset U$. There is an $n \in \mathbb{N}$ such that $V_n \cap A \subset U' \cap A$. For $m \geqslant n$ we have, by $(**)$, $F_{m+1} \subset \rho_{m+1} V_m \subset \lambda V_n$. Using (i) we get $\lambda^{-1} F_{m+1} \subset \lambda^{-1} X \subset A$ so that $\lambda^{-1} F_{m+1} \subset V_n \cap A \subset U' \cap A \subset U'$. We find $F_{m+1} \subset \lambda U' \subset U$ for all $m \geqslant n$, i. e. $X \setminus U \subset F_1 \cup \ldots \cup F_n$, a finite set.

(iii) Finally we prove $A \subset \overline{co\, X}$. Let $x \in A$. Since $A \subset V_1 + co\, F_1$ there is an $x_1 \in co\, F_1$ such that $x - x_1 \in V_1 \cap (A + co\, F_1)$. By $(*)$ there is an $x_2 \in co\, F_2$ such that $x - x_1 - x_2 \in V_2 \cap (A + co\, F_1 + co\, F_2)$. Inductively we find $x_1, x_2, \ldots \in X$

such that $x - \sum_{i=1}^{n} x_i \in V_n$ for each n. Hence $\lambda^{-1}(x - \sum_{i=1}^{n} x_i) \in V_n \cap A$ for each n so that $x = \sum_{i=1}^{\infty} x_i$. It follows that $x \in \overline{\text{co}\, X}$.

THEOREM 8.3. - Let E be a Hausdorff polar space. The following conditions are equivalent.

(α) E'_b is of countable type.

(β) Each bounded subset of E is $\sigma(E, E')$-metrizable.

Proof. - (α) ==> (β). Let B be a bounded absolutely convex set in E. The semi-norm p on E' defined by the formula $p(f) = \sup\{|f(x)| : x \in B\}$ is continuous for $b(E', E)$. By (α) there are $f_1, f_2, \ldots$ in E' such that $[\![f_1, f_2, \ldots]\!]$ is p-dense in E'_b. Without loss we may assume that $p(f_i) \leqslant 1$ for each i. It takes a standard reasoning to show that $d : B \times B \to \underset{\sim}{R}$ defined by the formula $d(x, y) = \max_i |f_i(x) - f_i(y)|\, 2^{-i}$ is an ultrametric on B whose induced topology equals the $\sigma(E, E')$-topology on B. To prove (β) ==> (α) consider a bounded subset A of E. The same argument as used in the proof of Proposition (6.5) (α)==>(β) yields that A is a compactoid for $\sigma(E, E')$. By (β) the set A is metrizable for $\sigma(E, E')$. By Proposition 8.2 there is a (bounded) sequence $e_1, e_2, \ldots$ converging weakly to 0 such that $A \subset \overline{\text{co}(e_1, e_2, \ldots)}^{\sigma}$. It follows that the topology $b(E', E)$ equals the topology of uniform convergence on (bounded) weakly compact sets. Also we know that each such weakly compact set is metrizable. A slight and obvious generalization of the proof needed for Example 4.5 (vi) shows that E'_b is of countable type.

To obtain an interesting characterization in the spirit of Theorem 8.3 for a restricted class of spaces we first prove the following variant of Proposition 4.11.

LEMMA 8.4. - Let E be a polar Hausdorff space. Suppose that each bounded subset of E is a compactoid. Then each weakly convergent sequence in E is convergent.

Proof. - Let $x_1, x_2, \ldots \in E$, $\lim_{n\to\infty} x_n = 0$ weakly. Then $\{0, x_1, x_2, \ldots\}$ is weakly bounded hence bounded by Corollary 7.7, hence compactoid by assumption. By Theorem 5.12 the weak and strong topologies coincide on $\{0, x_1, x_2, \ldots\}$. Hence $\lim_{n\to\infty} x_n = 0$ strongly.

THEOREM 8.5. - For a polar Hausdorff space E the following are equivalent.

(α) E'_b is of countable type. Each weakly convergent sequence in E is convergent.

(β) Each bounded subset of E is a metrizable compactoid.

Proof. - (α) ==> (β). Let A be a bounded subset of E. From the second part of the proof of Theorem 8.3 we obtain a sequence $e_1, e_2, \ldots$ such that $e_n \to 0$ weakly and $A \subset \overline{\text{co}(e_1, e_2, \ldots)}^{\sigma}$. Now we have $e_n \to 0$ strongly so that

$co(e_1, e_2, \ldots)$ is a compactoid. By Theorem 5.13 (i) the set $co(e_1, e_2, \ldots)^{00}$ is a compactoid, hence so is $\overline{co(e_1, e_2, \ldots)}^{\sigma}$ by Proposition 4.10. It follows that A is a compactoid. A is metrizable for $\sigma(E, E')$ by Theorem 8.3. By Theorem 5.12, A is also metrizable for the initial topology. Now suppose (β). By Theorem 5.12 each bounded set is metrizable for $\sigma(E, E')$ so, by Theorem 8.3, E'_b is of countable type. The second condition of (α) follows from Lemma 8.4.

Observe that, for a nonspherically complete K, ℓ^{∞} satisfies the conditions of Theorem 8.3 but not the ones of Theorem 8.5.

COROLLARY 8.6. - Let E be a polar Hausdorff space satisfying one of the conditions (α), (β) of Theorem 8.5.

(i) For each bounded set $A \subset E$ there exist $e_1, e_2, \ldots \in E$ with $\lim_{n\to\infty} e_n = 0$ such that $A \subset \overline{co(e_1, e_2, \ldots)}$.

(ii) The $b(E', E)$-topology on E' equals the topology of uniform convergence on compact sets.

(iii) Each weakly compact set in E is compact. Each $\sigma(E, E')$-compactoid is a compactoid.

Proof. - For (i) combine Theorem 8.5 and Proposition 8.2. The assertion (ii) follows from (i). For (iii) observe that weak compactoids are bounded by Corollary 7.7, hence they are compactoids by (β) of Theorem 8.5. For the first statement, apply Theorem 5.12.

COROLLARY 8.7. - The strong dual of a nuclear space in which each bounded set is metrizable is of countable type. In particular the strong duals of nuclear Fréchet spaces or nuclear LF-spaces are of countable type.

Proof. - Proposition 1.2 and Theorem 8.5 $(\beta) \Longrightarrow (\alpha)$ take care of the first statement. For the 'LF-part' of the second statement use [14], Theorem 3.14, 1°.

COROLLARY 8.8. - The duals of the spaces $C^{\infty}(X \to K)$ (X compact), $C^{\infty}(U \to K)$, $C_c^{\infty}(U \to K)$ (U open in $\mathbb{Q}_p$), $C_0^{\infty}(\mathbb{Q}_p \to K)$, $S(\mathbb{Q}_p \to K)$, $A_{\omega}(K)$, $A_1(K)$ of § 2 are of countable type.

9. Reflexivity.

For a locally convex space E over K we denote, as usual, the space $(E'_b)'_b$ by E''. We have the inclusion $(E'_{\sigma})' \subset E''$ as linear spaces so that, by Lemma 7.1, j_E maps E linearly into E''.

Definition 9.1. - A locally convex space E over K is reflexive if

$$j_E : E \to E''$$

is a surjective homeomorphism.

LEMMA 9.2. - Let E be a Hausdorff polar space. Then $j_E : E \to E''$ is injective and its inverse : $j_E(E) \to E$, is continuous.

Proof. - Injectivity follows from Lemma 7.1 (ii). Let (x_i) be a net in E for which $j_E(x_i) \to 0$ in E'' ; we prove that $x_i \to 0$ in E . Let p be a polar continuous seminorm on E . Then $B := \{f \in E' : |f| \leq p\}$ is equicontinuous hence bounded in E'_b . Hence $j_E(x_i) \to 0$ uniformly on B . But then, since $p = \sup\{|f| : f \in B\}$ we have also $p(x_i) \to 0$. By polarness of E , $x_i \to 0$.

LEMMA 9.3. - Let E be a polarly barreled space. Then $j_E : E \to E''$ is continuous.

Proof. - Let (x_i) be a net in E , converging to 0 , let B be a bounded subset of E'_b . By Proposition 6.5 there exists a polar continuous seminorm p on E such that $|f| \leq p$ for all $f \in B$. As $p(x_i) \to 0$ we have $j_E(x_i) \to 0$ uniformly on B . It follows that $j_E(x_i) \to 0$ in E'' .

LEMMA 9.4. - If E is reflexive then so is E'_b .

Proof. - The map $(j_E)' \circ j_{E'} : E'_b \to E''' \to E'_b$ is the identity. Now j_E is an isomorphism of locally convex spaces hence so is its adjoint $(j_E)'$. It follows that $j_{E'}$ is an isomorphism, i. e. that E'_b is reflexive.

LEMMA 9.5. - Let E be a polar space such that $j_E : E \to E''$ is surjective. Then E'_b is polarly barreled.

Proof. - Let B be a polar barrel in E'_b . From surjectivity of j_E it follows that $B = A^0$ for some set $A \subset E$. Since B is absorbing we have that $f(A)$ is bounded in K for each $f \in E'$ i. e. A is bounded in the topology $\sigma(E , E')$. By Corollary 7.7 A is bounded for the initial topology implying that $B = A^0$ is neighbourhood of 0 in E'_b .

THEOREM 9.6. - For a locally convex space E over K the following are equivalent.

(α) E is reflexive.

(β) E is a Hausdorff, polarly barreled, polar space. E is weakly quasicomplete.

Proof. - (α) $\Longrightarrow$ (β) . E is isomorphic to a dual space, hence E is Hausdorff and polar (Example 5.1 (iv)). By Lemma 9.4 the space E'_b is reflexive and E'' is polarly barreled by Lemma 9.5. Hence, so is E . From Theorem 6.6 and the polar barreledness of E'_b we obtain that E'' is $\sigma(E'' , E')$-quasicomplete. j_E is a homeomorphism of $(E , \sigma(E , E'))$ onto $(E'' , \sigma(E'' , E'))$.

(β) $\Longrightarrow$ (α). From the Lemmas 9.2 and 9.3 it follows that j_E is a homeomorphism of E into E'' . To prove that j_E is onto it suffices, by Lemma 7.1 (ii), to prove that $(E'_\sigma)' = (E'_b)'$ as sets. In other words we must prove that $b(E' , E)$ is $\sigma(E' , E)$-compatible. Now, the topology $b(E' , E)$ equals the topology of

uniform convergence on all $\sigma(E, E')$-bounded subsets of E (Corollary 7.7). This is also the topology of uniform convergence on members of $G := \{A \subset E : A$ edged, A closed and bounded for $\sigma(E, E')\}$. By assumption each member of G is complete for $\sigma(E, E')$. Using the map j_E of Lemma 7.1 (ii) (which is obviously a homeomorphism of $(E, \sigma(E, E'))$ onto $(E'_\sigma)'_\sigma$) we find that $G_1 := \{j_E(A) : A \in G\}$ is a special covering of $(E'_\sigma)'$ and that $b(E', E)$ is the G_1-topology. By Proposition 7.4 we then have $(E'_\sigma)' = (E'_b)'$.

As an application we shall prove Theorem 9.8 extending [11], 4.17 (which says that if K is not spherically complete, every K-Banach space of countable type is reflexive).

LEMMA 9.7. - Let E be a strongly polar space with a quasicomplete linear subspace D. If E is $\sigma(E, E')$-quasicomplete then D is $\sigma(D, D')$-quasicomplete.

Proof. - Let (x_i) be a net in D that is $\sigma(D, D')$-bounded and $\sigma(D, D')$-Cauchy ; we prove that it is $\sigma(D, D')$-convergent. Obviously (x_i) is bounded and Cauchy for $\sigma(E, E')$ so there is an $x \in E$ such that $x_i \to x$ for $\sigma(E, E')$. To prove that $x \in D$ consider $X := (\overline{co(x_i)_i})^e$ where the bar indicates the closure in D for the initial (relative) topology. D is strongly polar by Proposition 4.1. X is bounded by Corollary 7.7. Then, since X is also closed in D, X is complete, hence closed in E. X is also edged and we have by Corollary 4.9 that X is $\sigma(E, E')$-closed. It follows that $x = \sigma(E, E') - \lim_i x_i \in X \subset D$. To prove that $x_i \to x$ also for $\sigma(D, D')$, let $f \in D'$. By Theorem 4.2 f has an extension $\bar{f} \in E'$. We have $\bar{f}(x_i) \to \bar{f}(x)$, hence $f(x_i) \to f(x)$ which finishes the proof.

THEOREM 9.8. - For a locally convex space E of countable type over a nonspherically complete K the following are equivalent.

(α) E is reflexive.

(β) E is Hausdorff, quasicomplete, (polarly) barreled.

Proof.- (α) $\Longrightarrow$ (β) follows from Theorem 9.6 and the observation that a reflexive space is quasicomplete since it is the strong dual of a polarly barreled space (Theorem 6.6). We prove (β) $\Longrightarrow$ (α). By Theorem 4.13 there is a set I such that E can be viewed as a subspace D of c_0^I. As c_0 is reflexive it is weakly quasicomplete by Theorem 9.6 ; it is an easy exercice to show that also c_0^I is weakly quasicomplete. From Lemma 9.7 we obtain weak quasicompleteness of D. Now apply Theorem 9.6 to D.

COROLLARY 9.9. - Each Fréchet space of countable type over a nonspherically complete K is reflexive. Countable strict inductive limits of such spaces are reflexive.

Proof. - The conditions of Theorem 9.8 are satisfied. (See [14]. Theorem 3.13, 3°, Theorem 3.16 and Proposition 4.12 (iv) of this paper.)

COROLLARY 9.10. - Let K be not spherically complete. Then all the spaces of § 2, (2.1)-(2.8) are reflexive.

If K is spherically complete no infinite dimensional normable space is reflexive ([11], 4.16). In that case the spaces $C^n(X \to K)$ (§ 2.1), $C_0^n(\mathbb{Q}_p \to K)$ (§ 2.5) are not reflexive. We will see in the next section that the remaining spaces of (2.1)-(2.8) are reflexive.

10. Montel spaces.

Definition 10.1. - A locally convex space over K is a Montel space if it is Hausdorff, polar, polarly barreled and if each closed bounded subset is a complete compactoid.

It follows from the definition that quasicomplete barreled nuclear spaces are Montel spaces so that all the spaces of § 2, (2.1)-(2.8), with the exceptions $C^n(X \to K)$, $C_0^n(\mathbb{Q}_p \to K)$, are Montel spaces.

LEMMA 10.2. - Let E be a polar space for which each bounded subset is a compactoid. Then E is weakly quasicomplete if and only if E is quasicomplete.

Proof. - Suppose E is quasicomplete and let A be a set which is closed and bounded for $\sigma(E, E')$. Let B be the weak closure of the absolutely convex hull of A. Then B is weakly bounded hence bounded by Corollary 7.7, and B is a compactoid. B is closed in E hence complete. By Theorem 5.13 (v) B is weakly complete. Then A, being weakly closed in B, is weakly complete. Conversely, assume that E is weakly quasicomplete, let A be a strongly closed and bounded subset of E. Then A is compactoid. By Theorem 5.13 (ii) A is weakly closed (and weakly bounded) so that A is weakly complete by assumption. According to a standard reasoning A is also strongly complete.

THEOREM 10.3. - A Montel space is reflexive.

Proof. - Lemma 10.2 and Theorem 9.6.

In this context it is interesting to quote the following 'converse' in case K is spherically complete.

THEOREM 10.4. - Each reflexive space over a spherically complete field is a Montel space.

Proof. -([1], Proposition 4 a).

The example c_0 shows that if K is not spherically complete there exist reflexive

spaces which are not Montel spaces.

To prove Theorem 10.7 we need the following two general lemmas.

LEMMA 10.5. - Let $\tau_1 < \tau_2$ be locally convex topologies on a K-vector space E such that $\tau_1 = \tau_2$ on τ_1-compactoids. Then (E, τ_1) and (E, τ_2) have the same compactoid sets.

Proof. - We prove that an absolutely convex τ_1-compactoid A is also τ_2-compactoid. Choose $\lambda \in K$, $|\lambda| > 1$. Then $\tau_1 = \tau_2$ on λA. Let U be a τ_2-neighbourhood of 0 in E. Then there exists a τ_1-neighbourhood V of 0 in E such that $V \cap \lambda A \subset U \cap \lambda A$. Since A is τ_1-compactoid there exist, by Lemma 8.1, $x_1, \dots, x_n \in \lambda A$ such that $A \subset V + co(x_1, \dots, x_n)$. Then, by convexity, $A \subset \lambda A$ and $A \subset (V \cap \lambda A) + co(x_1, \dots, x_n) \subset (U \cap \lambda A) + co(x_1, \dots, x_n) \subset U + co(x_1, \dots, x_n)$.

LEMMA 10.6. - (Compare [2], Proposition 13). Let E be a locally convex space over K. Then on equicontinuous sets of E' the topology $\sigma(E', E)$ coincides with the topology of uniform convergence on compactoids.

Proof. - Let $H \subset E'$ be equicontinuous, let (f_i) be a net in H converging to $f \in H$ for $\sigma(E', E)$. Let A be a compactoid, let $\varepsilon > 0$. We prove that $|f - f_i| \leq \varepsilon$ on A for large i. By equicontinuity there is a neighbourhood U of 0 in E such that $|(f_i - f)(U)| < \varepsilon$ for all i.

There exist $x_1, \dots, x_n \in E$ such that $A \subset U + co(x_1, \dots, x_n)$. There is an i_0 such that $|(f_i - f)(x_j)| \leq \varepsilon$ for $j \in \{1, 2, \dots, n\}$ and $i \geq i_0$. It follows that $|f_i - f| \leq \varepsilon$ on $U + co(x_1, \dots, x_n)$, hence on A, for $i \geq i_0$.

THEOREM 10.7. - The strong dual of a Montel space is a Montel space.

Proof. - Let E be a Montel space. By Theorem 10.3 and Lemma 9.4 its strong dual E'_b is reflexive. It therefore suffices to show that a bounded subset of E'_b is a compactoid. Consider the topologies $\tau_1 = \sigma(E', E)$ and $\tau_2 = b(E', E)$ on E'. On τ_1-compactoids (i. e., on equicontinuous sets of E', see Proposition 6.5) τ_1 coincides with the topology of uniform convergence on compactoids (Lemma 10.6). It follows that $\tau_1 = \tau_2$ on τ_1-compactoids.

By Lemma 10.5 each τ_1-compactoid is also a τ_2-compactoid. Thus, each bounded subset of E'_b is a compactoid.

THEOREM 10.8. - Each quasicomplete polarly barreled subspace of a Montel space is a Montel space.

Proof. - The statement follows directly from the definitions after observing the following fact which is a consequence of Lemma 8.1. If A is a compactoid in E, $A \subset D \subset E$, D is a subspace of E then A is a compactoid in D.

A combinaison of the theory of § 8 and § 10 yields the following corollaries.

COROLLARY 10.9. - Let E be a Hausdorff, polar, polarly barreled, quasicomplete space for which each weakly convergent sequence is convergent. If E'_b is of countable type then E is a Montel space. In particular we have the following.

(i) Let K be spherically complete. If E is a Fréchet space or an LF-space and E'_b is of countable type then E is reflexive.

(ii) Let E be a Fréchet space or an LF-space, of countable type, whose strong dual is also of countable type. Then E is a Montel space.

Proof. - Theorem 8.5 yields the general statement. For (i) and (ii) use Proposition 4.11 and observe that, for spherically complete K, each locally convex space over K is strongly polar.

COROLLARY 10.10. - Let χ be the class of all reflexive locally convex spaces E over K such that both E and E'_b are of countable type. For a locally convex space E over K the following are equivalent.

(α) $E \in \chi$.

(β) E is a Montel space. Each bounded subset of E is metrizable. E is of countable type.

(γ) E and E'_b are of countable type. E is Hausdorff, barreled, quasicomplete.

COROLLARY 10.11. - The spaces $C^\infty(X \dashrightarrow K)$ (X compact), $C^\infty(U \dashrightarrow K)$, $C^\infty_c(U \dashrightarrow K)$ (U open in $\mathbb{Q}_p$), $C^\infty_0(\mathbb{Q}_p \to K), S(\mathbb{Q}_p \to K), A_\infty(K)$, $A_1(K)$ of § 2 are members of the class χ of Corollary 10.10.

Remark. - From the definition of a Montel space it follows, with Theorem 5.13 (iv), that each closed, edged, bounded set in a Montel space is a polar set. With an eye on Proposition 4.7 one may wonder whether every Montel space is strongly polar. The following example shows that this is not so. Let K be not spherically complete, let E be the K-vector space ℓ^∞ on which we put the strongest locally convex topology (i. e. the topology induced by all seminroms on ℓ^∞). The canonical norm on ℓ^∞/c_0 is not polar so that E is not strongly polar. On the other hand it is easily seen that E is a Montel space (bounded sets are finite dimensional, E can be viewed as a locally convex direct sum of one-dimensional spaces, Proposition 5.4, the seminorm associated to some barrel is continuous).

11. Nuclear duals.

In the spirit of § 8 we derive conditions on E in order that E'_b be nuclear, using Proposition 1.3.

LEMMA 11.1. - Let E be a metrizable or an LF-space over K. Let $x_1, x_2, \ldots$ be a sequence in E converging to 0. Then there exist $\lambda_1, \lambda_2, \ldots \in K$ with $\lim_{n\to\infty} |\lambda_n| = \infty$ and $\lim_{n\to\infty} \lambda_n x_n = 0$.

Proof. - By [14], Theorem 3.14, Corollaire, it suffices to consider the case where E is metrizable. There are absolutely convex neighbourhoods $U_1 \supset U_2 \supset \dots$ of 0 forming a neighbourhood basis at 0. Choose $\rho \in K$, $|\rho| > 1$. There are $N_1 < N_2 < \dots$ in $\mathbb{N}$ such that for each m, $n \in \mathbb{N}$

$$m > N_n \Longrightarrow \rho^n x_n \in U_n .$$

Choose $\lambda_1 = \lambda_2 = \dots = \lambda_{N_1} = 1$, $\lambda_{N_1+1} = \dots = \lambda_{N_2} = \rho$, $\lambda_{N_2+1} = \dots = \lambda_{N_3} = \rho^2$, etc. One checks easily that $\lim_{n\to\infty} \lambda_n x_n = 0$.

LEMMA 11.2. - Let E be a metrizable or an LF-space over K. Suppose E is semireflexive (i. e. $j_E : E \to E''$ is surjective) and also that each weakly convergent sequence is strongly convergent. Then every $T \in L(E'_b \to c_0)$ is compact.

Proof. - For every n the map $f \mapsto (Tf)_n$ $(f \in E'_b)$ is in E'', hence by semireflexivity there is an $x_n \in E$ such that $(Tf)_n = f(x_n)$ for all $f \in E'$. Thus, T has the form

$$f \mapsto (f(x_1), f(x_2), \dots) \quad (f \in E'_b) .$$

As T maps into c_0 we have $x_n \to 0$ weakly and, by assumption, $x_n \to 0$ strongly. By the previous lemma there exist $\lambda_1, \lambda_2, \dots \in K^\times$ with $\lim_{n\to\infty} |\lambda_n| = \infty$ and $B := \{\lambda_n x_n : n \in \mathbb{N}\}$ is bounded. Then B^0 is a neighbourhood of 0 in E'_b and $T(B^0) \subset \{(\xi_1, \xi_2, \dots) \in c_0 : |\xi_n| \leq |\lambda_n|^{-1}$ for all $n\}$. The latter set is easily seen to be a compactoid in c_0.

THEOREM 11.3. - Let E be a polar semireflexive locally convex space over K. Suppose that each bounded subset of E is a compactoid and that E is either metrizable or an LF-space. Then E'_b is nuclear.

Proof. - It suffices to combine Lemma 8.4, Lemma 11.2, Theorem 8.5 $(\beta) \Longrightarrow (\alpha)$, and Proposition 1.3 (for E'_b).

COROLLARY 11.4. - (Extension of [5], Proposition 5.7 (ii)). Let K be spherically complete. Then the strong dual of any reflexive metrizable or LF-space is nuclear.

Proof. - Theorem 10.4 and Theorem 11.3.

COROLLARY 11.5. - The strong dual of a Montel space which is either metrizable or LF is nuclear. The strong dual of a nuclear Fréchet space or a nuclear LF-space is nuclear.

Proof. - The first statement follows from Theorem 11.3. For the second statement observe that the space is barreled and complete. Hence, by Proposition 1.2, it is a Montel space.

COROLLARY 11.6. - The duals of the spaces $C^\infty(X \to K)$ (X compact), $C^\infty(U \to K)$

$C_c^\infty(U \to K)$ (U open in $\mathbb{Q}_p$), $C_0^\infty(\mathbb{Q}_p \to K)$, $S(\mathbb{Q}_p \to K)$, $A_\infty(K)$, $A_1(K)$ of § 2 are nuclear.

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