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CAPACITY THEORY ON ALGEBRAIC CURVES AND CANONICAL HEIGHTS

by Robert RUMELY (*)

This note outlines a theory of capacity for adelic sets on algebraic curves. It was motivated by a paper of D. CANTOR [C], where the theory was developed for $\underset{\sim}{P}^1$. Complete proofs of all assertions are given in a manuscript [R], which I hope to publish in the Springer-Verlag Lecture Notes in Mathematics series.

The capacity is a measure of the size of a set which is defined geometrically but has arithmetic consequences. (It goes under several names in the literature, including "Transfinite diameter", "Tchebychev constant", and "Robbins constant", depending on the context.) The introduction to Cantor's paper contains several nice applications, which I encourage the reader to see. I have mainly been concerned with generalizations of the following theorem of Fekete and Szegö [F-S].

THEOREM. - *Let* E *be a compact set in* $\underset{\sim}{C}$, *stable under complex conjugation. Then,*

(A) *If the logarithmic capacity* $\gamma(E)$ *is* < 1 , *there is a neighborhood* U *of* E *which contains only a finite number of complete Galois orbits of algebraic integers.*

(B) *If* $\gamma(E) \geqslant 1$, *then every neighborhood of* E *contains infinitely many complete Galois orbits of algebraic integers.*

Some examples of capacities are : for a circle or disc, its radius R ; for a line segment, $\frac{1}{4}$ of its length ; for two segments $[-b, -a] \cup [a, b]$, $\frac{1}{2}(b^2 - a^2)^{\frac{1}{2}}$; for a regular n-gon inscribed in a circle of radius R ,

$$R \cdot \frac{\Gamma(1 + 1/n)}{\Gamma(1 - 1/n)\,\Gamma(1 + 2/n)} \cdot$$

The capacity $\gamma(E)$ in the theorem should more properly be called the "logarithmic capacity of E with respect to the point ∞ ". The general definition of capacity will be given below. Of equal significance with $\gamma(E)$ is the Green's function $G(z, \infty; E)$. Recall that this is a nonnegative function, harmonic in $\underset{\sim}{C} \setminus E$, with value 0 on E and a logarithmic pole at ∞ , such that $G(z, \infty; E) - \log|z|$ is bounded in a neighborhood of ∞ . The Fekete-Szegö theorem is proved by constructing monic polynomials in $\underset{\sim}{Z}[z]$ whose normalized logarithm $(1/\deg P) \log|P(z)|$ closely approximates $G(z, \infty; E)$. The algebraic integers in the theorem are the roots of the polynomials.

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The condition in the Fekete-Szegö theorems that the numbers be algebraic integers is a restriction on their conjugates at finite primes, just as lying in a neighborhood of E is a restriction at the archimedean prime. CANTOR generalized the classical theory in several directions. First, he gave an adelic formulation, placing all the primes on an equal footing. Second, he defined the capacity of an adelic set with respect to several points, not just one, which gave the theory smooth behavior under pullbacks by rational functions. (In the classical theory over $\mathbb{C}$, if $F(z)$ is a monic polynomial of degree n, then $\gamma(F^{-1}(E)) = \gamma(E)^{1/n}$.) Thirdly, he formulated versions of the theory with rationality conditions : for example, in the Fekete-Szegö theorem, if $E \subset \mathbb{R}$, then the numbers produced in part (B) could be taken to be totally real. (This special case was originally proved by R. ROBINSON.) There were some errors in the proofs of the rationality, but no doubt the results are true. Cantor's definition of the capacity of a set with respect to several points was quite novel, involving the value as a matrix game of a certain symmetric matrix constructed from Green's functions.

The functoriality properties of Cantor's capacity suggested that it should be possible to extend the theory to all curves. In doing so, I have given a different approach to the original results, and found some interesting connections with Néron's canonical heights.

Notation. - Let $\mathcal{C}$ be a smooth, geometrically connected projective curve defined over a number field K. If v is a place of K, we write K_v for the completion of K at v. $\tilde{K}$ will be the algebraic closure of K, $\tilde{K}_v$ the algebraic closure of K_v, and $\hat{K}_v$ the completion of the algebraic closure of K_v. $\mathrm{Gal}(\tilde{K}/K)$ will be the usual Galois group ; $\mathrm{Gal}(\hat{K}_v/K_v)$ the group of continuous automorphisms of $\hat{K}_v/K_v$. If v is nonarchimedean, and lies over a rational prime p, the absolute value on K_v associated to v will be normalized so that $|p|_v = 1/p$; if v is archimedean, then $|x + yi|_v = (x^2 + y^2)^{\frac{1}{2}}$. Thus, we are using the absolute normalization for our absolute values. These absolute values extend in a unique way to absolute values on the $\hat{K}_v$, which we continue to denote by $|x|_v$. For any field F, $\mathcal{C}(F)$ will mean the set of points of $\mathcal{C}$ rational over F, and $F(\mathcal{C})$ the field of algebraic functions on $\mathcal{C}$ rational over F.

Classical theory. - In the classical theory, if $E \subset \mathbb{C}$ is a compact set, then its capacity (with respect to the point ∞) is given by the equivalent definitions

$$\gamma(E) = \lim_{n\to\infty} \max_{\{z_1,\dots,z_n\}\subset E} \prod_{i\neq j} |z_i - z_j|^{1/n(n-1)} \qquad \text{(transfinite diameter)}$$

$$= \lim_{n\to\infty} \min_{P(z)\in\mathbb{C}[z]} \max_{z\in E} |P(z)| \qquad \text{(Tchebychev constant)}$$

$$= e^{-V(E)}, \text{ where} \qquad \text{(logarithmic capacity)}$$

$$V(E) = \inf_{\text{prob. meas. } \nu \text{ on } E} \int_E \int_E - \log|z_1 - z_2| \, d\nu(z_1)\, d\nu(z_2) \qquad \text{(equilibrium potential)}$$

$$= \lim_{z\to\infty} (G(z, \infty ; E) - \log|z|) \qquad \text{(Robbins constant)} .$$

Here a probability measure is a positive measure of total mass 1, and $G(z,\infty;E)$ is the Green's function of the unbounded component of the complement of E. If $\gamma(E) \neq 0$, there is a unique probability measure μ minimizing the integral defining $V(E)$, and Green's function is given by

$$G(z\,,\,\infty\,;\,E) = V(E) + \int_E \log|z - w|\, d\mu(w)\ .$$

Throughout the following, we will implicitly assume that $\gamma(E) \neq 0$. (This is the case, for example, if E contains a one-dimensional continuum.) For non-compact sets F, the capacity and Green's function are defined by limits :

$$\gamma(F) = \sup_{\text{compact } E\subset F} \gamma(E)$$

$$G(z\,,\,\infty\,;\,F) = \inf_{\text{compact } E\subset F} G(z\,,\,\infty\,;\,E)\ .$$

An important class of sets whose capacities are known are PL-domains (Polynomial Lemniscate domains). If $P(z) \in \underline{C}[z]$ is a monic polynomial of degree n, then the set $E = \{z \in \underline{C}\ ;\ |P(z)| \leq R\}$ has capacity $\gamma(E) = R^{1/n}$. This is because the Green's function is $1/n.\log|P(z)|$ for $z \notin E$, and $V(E)$ can be read off as the residue of the Green's function at ∞. In particular, the capacity of a circle is its radius.

The equality of the various definitions of $\gamma(E)$ was proved by FEKETE and SZEGÖ. Each definition of $\gamma(E)$ is useful in a different context. Its role as the Tchebychev constant gives functoriality under pullbacks. Its definition in terms of the measure μ allows the construction of polynomials whose logarithm approximates the Green's function. Its expression in terms of $V(E)$ allows it to be computed for many sets, and was the definition which CANTOR generalized in the adelic theory.

<u>The canonical distance</u>. - In all definitions of capacity in the classical case, the crucial ingredient is the presence of the distance function $|x - y|$ which has a pole at ∞. The connection between the geometric and arithmetic sides of the theory comes from the fact that the distance function can be used to decompose the absolute value of a polynomial in terms of its roots.

In constructing a theory of capacity on curves, the starting point is to find similar functions which can be used to decompose the v-absolute value of algebraic functions on $\mathcal{C}(\hat{K}_v)$ for every place v and every curve $\mathcal{C}$. I call such functions "canonical distance functions", although the term should be understood guardedly since they do not in general satisfy the triangle inequality, but only a weak version of it. For any place v, and any point $\zeta \in \mathcal{C}(\hat{K}_v)$, there is a canonical distance $[z_1\,,\,z_2]_\zeta$ which is unique up to scaling by a constant, and satisfies the following properties :

1° (Positivity) For all $z_1\,,\,z_2 \in \mathcal{C}(\hat{K}_v) \setminus \{\zeta\}$, we have $0 \leq [z_1\,,\,z_2]_\zeta < \infty$, with $[z_1\,,\,z_2]_\zeta = 0$ if, and only if, $z_1 = z_2$.

2° (Normalization at ζ) Let $g(z) \in \hat{K}_v(\mathcal{C})$ have a simple zero at ζ. Then there is a constant $c_g > 0$ such that for any $z_2 \in \mathcal{C}(\hat{K}_v) \setminus \{\zeta\}$,

$$\lim_{z_1 \to \zeta} [z_1 , z_2]_\zeta \, |g(z_1)|_v = c_g .$$

3° (Symmetry) $[z_1 , z_2]_\zeta = [z_2 , z_1]_\zeta$.

4° (Continuity) $[z_1 , z_2]_\zeta$ is continuous as a function of two variables. If v is archimedean, $\log[z_1 , z_2]_\zeta$ is harmonic in each variable separately ; if v is nonarchimedean, $\log[z_1 , z_2]_\zeta$ is locally constant in each variable, provided $z_1 \neq z_2$.

5° (Decomposition of functions) If $f(z) \in \hat{K}_v(\mathcal{C})$ has zeros and poles (with multiplicity) at $a_1 , \dots , a_n$ and $\zeta_1 , \dots , \zeta_n$ respectively, then there is a constant c_f so that for all $z \in \mathcal{C}(\hat{K}_v)$ where $f(z)$ is defined,

$$|f(z)|_v = c_f \prod_{i=1}^n [z , a_i]_{\zeta_i} .$$

6° (Galois invariance) If $\mathcal{C}$ is defined over K_v, and $\zeta \in \mathcal{C}(K_v)$, then for all $\sigma \in \mathrm{Gal}(\hat{K}_v/K_v)$,

$$[\sigma z_1 , \sigma z_2]_\zeta = [z_1 , z_2]_\zeta .$$

7° (Weak triangle inequality) There is a constant M depending only on $\mathcal{C}$ and v such that $[z_1 , z_3]_\zeta \leq M.\max([z_1 , z_2]_\zeta , [z_2 , z_3]_\zeta)$. If v is nonarchimedean and $\mathcal{C}$ has nondegenerate reduction at v, then $M = 1$.

Properties 2°, 4° and a weak version of 5° characterize $[z_1 , z_2]_\zeta$. One can show that for any $z_2 \in \mathcal{C}(\hat{K}_v)$ there is a function $f(z) \in \hat{K}_v(\mathcal{C})$ whose only poles are at ζ and whose zeros all lie in a prespecified ball about z_2. Furthermore, after fixing a uniformizing parameter $g(z)$ at ζ as in property 2° call such an $f(z)$ normalized if $\lim_{z\to\zeta} |f(z)\, g(z)^n|_v = 1$, where $n = \deg(f)$. Then

$$[z_1 , z_2]_\zeta = \lim_{\substack{\text{normalized } f \\ \text{zeros of } f \to z_2}} |f(z_1)|_v^{1/n} .$$

The existence of the limit is a consequence of a maximum modulus principle for algebraic functions on curves. The symmetry comes from Weil reciprocity.

Property 5° suggests a connection with Néron's canonical local height pairing ; and in fact Néron's pairing and the canonical distance can be defined in terms of each other. Recall that Néron's pairing is a real-valued, bilinear function $\langle D,D'\rangle_v$ on divisors of degree 0 in $\mathcal{C}(\tilde{K}_v)$ having coprime support. It has the property that if $D' = \mathrm{div}(f)$ for some function $f(z)$, then, writing $\log_v x$ for the logarithm to the base $p(v)$, where $p(v)$ is the rational prime lying below v (taking $p(v) = e$ if $v = \infty$), we have

$$\langle D , \mathrm{div}(f)\rangle_v = - \log_v |f(D)|_v .$$

It is continuous in both variables, and thus can be regarded as extending functional evaluation to nonprincipal divisors. (It should be noted that there is some variation in the literature concerning the normalization of Néron's pairing. Here we are requiring it to approach $+\infty$ on the diagonal, and to take rational values if v is nonarchimedean). The following clean formulation for the relation between the canonical distance and Néron's pairing was shown to me by B. GROSS : There is a constant C , depending on the choice of uniformizing parameter $g(z)$ at ζ , such that

$$\log[z_1 , z_2]_\zeta + C = \lim_{w\to\zeta}(\langle(z_1) - (w) , (z_2) - (\zeta)\rangle_v + \log_v|g(z)|_v) .$$

This expression allows the finer properties of Néron's pairing, given by intersection theory, to be transferred to the canonical distance.

The facts and formulas above arose from a study of the classical theory whose goal was first to put the point ∞ on an equal footing with the points of $\underset{\sim}{P}^1(\underset{\sim}{C})$, and then to find analogues for all v and all $\mathcal{C}$. In a number of cases special formulas turned up which suggested considering $[z_1 , z_2]_\zeta$ as a distance. Since these formulas also give more insight into the nature of the canonical distance, it seems worth presenting them. In all cases, we obtain an expression of the form

$$[z_1 , z_2]_\zeta = \frac{((z_1 , z_2))_v}{((z_1 , \zeta))_v ((z_2 , \zeta))_v} ,$$

where $((z_1 , z_2))_v$ is continuous, nonnegative, and bounded, with a simple zero along the diagonal.

Special formulas for the canonical distance. - The formulas are more or less explicit, depending on the genus of $\mathcal{C}$.

Genus 0 : The projective line.

- Archimedean case. On $\underset{\sim}{P}^1(\underset{\sim}{C})$ one has the spherical chordal metric, given for $z_1 , z_2 \in \underset{\sim}{C}$ by

$$\|z_1 , z_2\|_v = \frac{|z_1 - z_2|}{(1 + |z_1|^2)^{\frac{1}{2}} (1 + |z_2|^2)^{\frac{1}{2}}}$$

$$\|z_1 , \infty\|_v = \frac{1}{(1 + |z_1|^2)^{\frac{1}{2}}} .$$

Note that $\|z_1 , z_2\|_v$ is invariant when both z_1 and z_2 are inverted, and is uniformly bounded above by 1 . When the plane is identified with a sphere of diameter 1 by stereographic projection, it is the length of the chord from z_1 to z_2 . For notational compatibility with curves of higher genus, write $((z_1 , z_2))_v = \|z_1 , z_2\|_v$. Then

$$[z_1 , z_2]_\zeta = \frac{((z_1 , z_2))_v}{((z_1 , \zeta))_v \, ((z_2 , \zeta))_v} .$$

Observe that $[z_1 , z_2]_\infty = |z_1 - z_2|$, while $[z_1 , z_2]_\zeta = |f(z_1) - f(z_2)|$ for $\zeta \neq \infty$, where $f(z) = (1 + |\zeta|^2)^{\frac{1}{2}}/(z - \zeta)$. We emphasize that $[z_1 , z_2]_\zeta$ is only determined up to scaling by a constant $c(\zeta)$ for each ζ .

- Nonarchimedean case. For finite primes the appropriate analogue of the chordal metric is the p-adic spherical distance, given for $z_1 , z_2 \in \hat{K}_v$ by

$$\|z_1 , z_2\|_v = \frac{|z_1 - z_2|_v}{\max(1 , |z_1|_v) \max(1 , |z_2|_v)}$$

$$\|z_1 , \infty\|_v = \frac{1}{\max(1 , |z_1|_v)}$$

and again putting $((z_1 , z_2))_v = \|z_1 , z_2\|_v$, we have

$$[z_1 , z_2]_\zeta = \frac{((z_1 , z_2))_v}{((z_1 , \zeta))_v \, ((z_2 , \zeta))_v} .$$

From these expressions properties 1°-7° follow easily.

All curves, good reduction. - If $\mathcal{C}/K$ is any curve, then for v of K where $\mathcal{C}$ has nondegenerate reduction with respect to the given embedding in $\underset{\sim}{P}^n$, there is an analogue of the formula above, with $((z_1 , z_2))_v$ given by the v-adic spherical metric on $\underset{\sim}{P}^n$. This is defined as follows : fix a system of affine coordinates on $\underset{\sim}{P}^n(\hat{K}_v)$. Then, for $z_1 , z_2 \in \underset{\sim}{P}^n(\hat{K}_v)$, if there is some affine patch in which both z_1 and z_2 have integral coordinates,

$$\|z_1 , z_2\|_v = \max_i |z_{1i} - z_{2i}|_v$$

(using the coordinates in that patch). Otherwise $\|z_1 , z_2\|_v = 1$. It is easy to check that $\|z_1 , z_2\|_v$ is invariant under a change of coordinates in $PGL(n+1,\hat{\mathcal{O}}_v)$, where $\hat{\mathcal{O}}_v$ is the ring of integers in $\hat{K}_v$.

Genus 1 : Elliptic curves.

- Archimedean case. We use the fact that an elliptic curve over $\underset{\sim}{C}$ is isomorphic to a complex torus $\underset{\sim}{C}/[\omega_1 , \omega_2]$. NÉRON has given an explicit formula for the local height pairing in terms of the Weierstrass σ-function, and by modifying things slightly we get the canonical distance. Let

$$\sigma(u) = u \prod_{0 \neq \omega \in L} (1 - u/\omega)^{u/\omega + \frac{1}{2}(u/\omega)^2}$$

be the σ-function for the lattice $L = [\omega_1 , \omega_2]$. Write η_1 for the period of $\zeta(u) = d/du(\log \sigma(u))$ under ω_1 , and η_2 for the period of $\zeta(u)$ under ω_2 , so that by the Legendre relation, $\eta_2 \omega_1 - \eta_1 \omega_2 = 2\pi i$. Given $u \in \mathbb{C}$, we can uniquely decompose $u = \omega_1 u_1 + \omega_2 u_2$ with $u_1 , u_2 \in \mathbb{R}$. Define $\eta(u) = \eta_1 u_1 + \eta_2 u_2$, and let

$$\underline{k}(u) = e^{-\frac{1}{2}u\eta(u)} \sigma(u) .$$

Then $\underline{k}(u + \omega_1) = - e^{-\pi i u_2} \underline{k}(u)$, and $\underline{k}(u + \omega_2) = - e^{-\pi i u_1} \underline{k}(u)$, so that $|\underline{k}(u)|$ is **periodic** (cf. LANG [L]). Let u_1 and u_2 correspond to z_1 and z_2 under the isomorphism $\mathbb{C}/[\omega_1 , \omega_2] \approx \mathcal{C}(\mathbb{C})$. Defining $((z_1 , z_2))_v = |\underline{k}(u_1 - u_2)|$, we have

$$[z_1 , z_2]_\zeta = \frac{((z_1 , z_2))_v}{((z_1 , \zeta))_v ((z_2 , \zeta))_v} .$$

- Nonarchimedean case, bad reduction. The canonical distance is invariant under base extension, so we can assume in this case that $\mathcal{C}$ is a Tate curve. Then there is some $q \in \hat{K}_v^*$ with $|q|_v < 1$ such that $\mathcal{C}(\hat{K}_v)$ is isomorphic to $\hat{K}_v^*/(q)$. As MANIN [M] has pointed out, one can express Néron's local height pairing in terms of p-adic theta-functions ; similarly, we get the canonical distance. The basic theta-function is

$$\theta(u) = \prod_{n\geq 0} (1 - q^n/u) \prod_{n<0} (1 - q^{-n} u) \quad \text{for} \quad u \in \hat{K}_v^* .$$

Put $vq(u) = \mathrm{ord}_v(u)/\mathrm{ord}_v(q)$, and define the "mollifier"

$$\delta(u) = |u|_v^{\frac{1}{2}(vq(u)^2+vq(u))} ,$$

Then $\underline{k}(u) = \delta(u).|\theta(u)|_v$ is a real-valued function such that for all u , $\underline{k}(qu) = \underline{k}(u^{-1}) = \underline{k}(u)$, as follows from the functional equations of the theta-function. It is well known that algebraic functions on a Tate curve can be expressed in terms of $\theta(u)$, and a short calculation shows that if we put $((z_1 , z_2))_v = \underline{k}(u_1 u_2^{-1})$, then we have the familiar formula

$$[z_1 , z_2]_\zeta = \frac{((z_1 , z_2))_v}{((z_1 , \zeta))_v ((z_2 , \zeta))_v} .$$

Genus $g \geq 2$.

- Archimedean case. Although the formulas are not as explicit as in the previous cases, their theoretical significance is clearer. It turns out that $((z_1 , z_2))_v$ is a multiple of the Arakelov-Green's function $G(z_1 , z_2)$. ARAKELOV introduced his functions in order to extend Néron's pairing from divisors of degree 0 to divisors of arbitrary degree in the archimedean case, in a way modeled on intersection theory (see ARAKELOV [Ar]). Such an extension is not unique ; an Arakelov-Green's function is determined by giving a volume form, or more generally a measure du , normalized so that $\mathcal{C}(\mathbb{C})$ has total mass 1 . Then, there is a unique nonnegative real-valued function $G(z, w)$ on $\mathcal{C}(\mathbb{C})$ such that

(a) $G(z, w)$ is smooth and positive off the diagonal, with a simple zero along the diagonal ;

(b) $(1/4\pi i)(\partial/\partial z)(\partial/\partial\bar{z}) \log G(z, w)\, dz \wedge d\bar{z} = du - \delta_w$ for every w ;

(c) $\int_{\mathcal{C}} \int_{\mathcal{C}(\mathbb{C})} \log G(z, w)\, du(z) = 0$ for each w.

Condition (c) is simply a convenient normalization ; the crucial properties are (a) and (b), which ensure that $\log(G(z, w_1)/G(z, w_2))$ is harmonic for $z \neq w_1, w_2$, with logarithmic singularities of opposite signs at those points. Green's identities show that $G(z, w)$ is symmetric. For any choice of an Arakelov-Green's function, we can put $((z_1, z_2))_v = G(z_1, z_2)$ and get a family of distance functions $[z_1, z_2]_\zeta$ by the usual formula. For curves of genus 0 and 1 above, we have chosen $((z_1, z_2))_v$ to correspond to the constant positive curvature and flat metrics, respectively.

It should be noted that GROSS [Gr-Z] has given a formula for the Arakelov-Green's function of a curve of genus ≥ 2 with the constant negative curvature metric. His formula uses the uniformization of the curve by the upper half-plane, and expresses $G(z, w)$ as the residue at $s = 1$ of a Poincaré series formed from Legendre functions of the second kind.

- Nonarchimedean case. Here the construction of functions $((z_1, z_2))_v$ which decompose the canonical distance depends on intersection theory. If L_w/K_v is a finite extension, put $\mathcal{C}_w = \mathcal{C} \times_{K_v} \operatorname{spec}(L_w)$. Let C_w be the minimal regular model of $\mathcal{C}_w$, and $R(C_w)$ the dual graph to the special fibre of C_w. After a finite extension of the base, $\mathcal{C}$ has semi-stable reduction, and hence the graphs $R(C_w)$ all have the same topology for large L_w. One can form a "reduction graph" $R(\mathcal{C})$, which is essentially the direct limit of the $R(C_w)$ together with a metric on the edges, in such a way that the components of the special fibres of the C_w correspond to a dense set of points on $R(\mathcal{C})$.

The final result is as follows. There is a decomposition

$$-\log_v[x, y]_\zeta = i_v(x, y)_\zeta + j_v(x, y)_\zeta ,$$

in which $i_v(x, y)$ and $j_v(x, y)_\zeta$ take rational values. Furthermore,

$$i_v(x, y)_\zeta = i_v(x, y) - i_v(x, \zeta) - i_v(y, \zeta)$$

and

$$j_v(x, y)_\zeta = j_v(x, y) - j_v(x, \zeta) - j_v(y, \zeta) .$$

Here $i_v(x, y)$ is a purely "local" term : it is 0 unless both x and y reduce to the same nonsingular point on the special fibre of some C_w, and then it is $i_v(x, y) = -\log_v |g_y(x)|_v$ for an appropriate local uniformizer $g_y(x)$ at y. On the other hand, $j_v(x, y)$ is not unique, but is specified by giving a measure of total mass 1 on $R(\mathbb{C})$. As a function of x and y, it depends only on the special fibre to which x and y reduce. Thus, it and $j_v(x, y)_\zeta$ can be regarded as functions on $R(\mathbb{C})$. For any two points $\bar{y}$, $\bar{\zeta}$ of $R(\mathbb{C})$, $j_v(\bar{x}, \bar{y})_{\bar{\zeta}}$ is a piecewise linear function of $\bar{x} \in R(\mathbb{C})$ taking its maximum at $\bar{y}$ and its minimum at $\bar{\zeta}$. It obeys a mean-value property like that of harmonic functions.

A remark on the triangle inequality. - The constant M in the weak triangle inequality (property 7° of the canonical distance) is definitely sometimes greater than 1. For a Tate curve isomorphic to $\hat{K}_v^*/(q)$, $M = |q|_v^{-1/16}$. Thus, M can be arbitrarily large. In the Archimedean case, numerical computations for elliptic curves $\mathbb{C}/[\tau , 1]$ yield lower bounds for an $\bar{M}$ such that

$$[z_1 , z_3]_\zeta \leq \bar{M}([z_1 , z_2]_\zeta + [z_2 , z_3]_\zeta) .$$

Some values are in the following table.

τ	$\bar{M} \geq$
.5 + .866i	1.000
0 + 1.0i	1.007
- .2 + 2.1i	1.167
.3 + 3.0i	1.486
.5 + 3.5i	1.725

Construction of local Green's functions. - Given the canonical distance, for each place v and set $F_v \subset \mathbb{C}(\hat{K}_v)$, one can define the capacity and construct Green's functions following the classical pattern.

For a compact set E_v, and a point $\zeta \in \mathbb{C}(\hat{K}_v)$ not in E_v, let

$$V_\zeta(E_v) = \inf_{\text{prob. meas. } \nu \text{ on } E_v} \int_{E_v} \int_{E_v} - \log_v[z_1 , z_2]_\zeta \, d\nu(z_1) \, d\nu(z_2)$$

$$\gamma_\zeta(E_v) = e^{-V_\zeta(E_v)} .$$

If $V_\zeta(E_v) \neq \infty$, there is a unique measure μ_ζ, the "equilibrium distribution", for which the inf is achieved. Define the Green's function by

$$G(z , \zeta ; E_v) = V_\zeta(E_v) + \int \log_v[z , w]_\zeta \, d\mu_\zeta(w) .$$

We understand this to mean ∞ if $V_\zeta(E_v) = \infty$, and z, ζ are not in E_v. Define $G(z , \zeta ; E_v) = 0$ if either z or ζ belongs to E_v. (Note that $V_\zeta(E_v)$ and $\gamma_\zeta(E_v)$ depend on the normalization chosen for the canonical distance $[z , w]_\zeta$, but $G(z , \zeta ; E_v)$ and μ_ζ do not.) It can be shown that if $E_{v1} \subset E_{v2}$, then for all z, ζ the inequality $G(z , \zeta ; E_{v1}) \geq G(z , \zeta ; E_{v2})$ holds. For an arbitrary set F_v, put

$$G(z , \zeta ; F_v) = \inf_{\substack{E_v \subset F_v \\ E_v = \text{compact}}} G(z , \zeta ; E_v) .$$

At this point there is a complication, similar to the one in defining a measurable set in Lebesgue's theory. Above we have defined the capacity $\gamma_\zeta(E_v)$ for a compact set. Another class of sets for which the capacity can be defined naturally are PL_ζ-domains : sets $U_v = \{z \in \mathcal{C}(\hat{K}_v) \; ; \; |f(z)|_v \leq R_v\}$ where $f(z)$ is an algebraic function on $\mathcal{C}$ whose only poles are at ζ, and R_v belongs to the value group of $\hat{K}_v^*$. If $f(z)$ has degree n, and is normalized so $\lim_{z\to\zeta}|f(z)|_v/[z, w]_\zeta^n = 1$, then it is natural to require that $\gamma_\zeta(U_v) = R_v^{1/n}$. Furthermore, for $z \notin U_v$, one wants $G(z, \zeta ; U_v) = (1/n)\log_v(|f(z)|_v/R_v)$. The sets for which the formulas for compact sets and PL_ζ-domains are compatible will be called <u>capacitable</u>. For an arbitrary set F_v, let the "inner" and "outer" capacities of F_v be

$$\underline{\gamma}_\zeta(F_v) = \sup_{\substack{E_v \subset F_v \\ E_v = \text{compact}}} \gamma_\zeta(E_v)$$

$$\bar{\gamma}_\zeta(F_v) = \inf_{\substack{U_v \supset F_v \\ U_v = PL_\zeta\text{-domain}}} \gamma_\zeta(U_v) \; .$$

F_v is capacitable if for all ζ in the complement of F_v, $\underline{\gamma}_\zeta(F_v) = \bar{\gamma}_\zeta(F_v)$. (In a manuscript of this paper, I called such sets <u>admissible</u>.) An RL-domain (Rational Lemniscate domain) is a set of the form $\{z \in \mathcal{C}(\hat{K}_v) \; ; \; |f(z)|_v \leq R_v\}$ for some $f(z) \in \hat{K}_v(\mathcal{C})$. Finite unions of compact sets and RL-domains are capacitable. An example of a non-capacitable set is a set containing one point in infinitely many residue classes (mod v) of $\mathcal{C}(\hat{K}_v)$. If F_v is capacitable,

$$G(z, \zeta ; F_v) = \sup_{\substack{U_v \supset F_v \\ U_v = PL_\zeta\text{-domain}}} G(z, \zeta ; U_v) \; .$$

However, the collection of capacitable sets is not closed under intersection.

The Green's function of F_v has the following properties :

1° (Positivity) $G(z_1, z_2 ; F_v) \geq 0$; and if z_1 or $z_2 \in F_v$, then $G(z_1, z_2 ; F_v) = 0$.

2° (Symmetry) $G(z_1, z_2 ; F_v) = G(z_2, z_1 ; F_v)$.

3° (Transitivity) If F_v is capacitable, and if $G(z_1, z_2 ; F_v)$ and $G(z_2, z_3 ; F_v)$ are both > 0, then $G(z_1, z_3 ; F_v) > 0$.

4° (Galois statility) For all $\sigma \in \mathrm{Gal}(\hat{K}_v/K_v)$,

$$G(\sigma z_1, \sigma z_2 ; \sigma F_v) = G(z_1, z_2 ; F_v) \; .$$

5° (Approximability) For any RL-domain U_v contained in the <u>complement</u> of F_v, any $\epsilon > 0$, and any ζ, there is an algebraic function $f(z) \in \hat{K}_v(\mathcal{C})$ whose only poles are at ζ, such that for all $z \in U_v$, $z \neq \zeta$,

$$|G(z, \zeta ; F_v) - (1/n)\log_v|f(z)|_v| < \epsilon \quad (n = \deg f) \; .$$

6° (Continuity) $G(z_1 , z_2 ; F_v)$ is continuous as a function of two variables for $z_1 \neq z_2$ in the complement of F_v. If v is archimedean, it is harmonic in each variable separately.

The ideas behind the proofs. - In the archimedean case, the development of capacity theory on Riemann surfaces is done by following the proofs in TSUJI's book [Ts] which are all coordinate-free. Once one has the canonical distance, there is nothing new. The main fact used is the maximum modulus principle for harmonic functions.

In the nonarchimedean case, the goals of the theory are the same, but the methods are somewhat different. The primary difficulty is relating local behavior of functions to global behavior, and this is done by using the rigidity of algebraic functions. There are four main tools, two local and two global.

1° Local parametrizability of $\mathcal{C}(\hat{K}_v)$ by power series. - Suppose $\mathcal{C}$ is embedded in $\underset{\sim}{P}^n$, and let $\|z_1 , z_2\|_v$ be the v-adic distance on $\mathcal{C}(\hat{K}_v)$ induced from $\underset{\sim}{P}^n(\hat{K}_v)$. There is a $\delta > 0$ such that for any point $z_0 \in \mathcal{C}(\hat{K}_v)$, the ball

$$B(z_0 , \delta) = \{z \in \mathcal{C}(\hat{K}_v) ; \|z , z_0\|_v \leq \delta\}$$

can be parametrized by convergent power series. This is well known, and is proved by using Hensel's lemma to refine approximate parametrizations. The uniformity comes by using a lemma of Weil from the theory of heights to show that the v-adic singularness of $\mathcal{C}(\hat{K}_v)$ is bounded.

2° The "Jacobian construction principle" for functions. - Let $a \neq \zeta$ be two arbitrary points of $\mathcal{C}(\hat{K}_v)$, and let U_v be a neighborhood of a. Then, there is a function $f(z) \in \hat{K}_v(\mathcal{C})$, all of whose zeros lie in U_v and whose only poles are at ζ. This circumvents the difficulty that for genus $g \geq 1$, not every divisor is principal. It is proved by using the fact that the Abel map

$$\mathcal{C}(\hat{K}_v)^g \longrightarrow \mathcal{J}(\hat{K}_v)$$

is open for the v-adic topology except on a set of codimension 1. (This is well known for the Zariski topology, and it follows for the v-topology by the implicit function theorem for power series.) One takes the image of the divisor $(a) - (\zeta)$ in the Jacobian. By choosing n appropriately, one can arrange that $n[(a) - (\zeta)]$ be arbitrarily near the origin of $\mathcal{J}(\hat{K}_v)$. Then wiggling a few of the copies of (a) within U_v gives a principal divisor.

3° The maximum modulus principle. - Actually two maximum modulus principles are used : a local one for power series, which is well known ; and a global one for algebraic functions over RL-domains. It is as follows. If $g(z) \in \hat{K}_v(\mathcal{C})$ is non-constant, put

$$D = \{z \in \mathcal{C}(\hat{K}_v) ; |g(z)|_v \leq 1\}$$
$$\partial D = \{z \in \mathcal{C}(\hat{K}_v) ; |g(z)|_v = 1\} .$$

Then, if $f(z) \in \hat{K}_v(\mathcal{C})$ has no poles in D, $|f(z)|_v$ achieves its maximum value for $z \in D$ at a point of ∂D.

This is proved by considering the equation relating $f(z)$ and $g(z)$ (which exists because $\mathcal{C}$ has dimension 1), examining its Newton polygon for a fixed z, and reducing to the case of $\underset{\sim}{P}^1$. On $\underset{\sim}{P}^1$ it is due to CANTOR, who used the factorization of $|f(z)|_v$ in terms of its zeros and poles, and a decomposition theorem showing that an RL-domain is a finite union of "punctured discs".

4° The intersection theory formula for Néron's pairing (see GROSS [Gr]). - This gives an expression for the canonical distance which lets one generalize Cantor's decomposition theorem for RL-domains to arbitrary curves. One of its consequences is that finite intersections and unions of RL-domains are again RL-domains. The intersection formula is mainly used in studying the canonical distance on curves with bad reduction. A weak version of the theory can be established without it, restricting to compact sets at places where $\mathcal{C}$ has bad reduction. We will not discuss it further here.

In developing the theory, the general technique is to reduce questions about functions on $\mathcal{C}$ to questions on balls around their zeros, by the maximum modulus principle. On the balls, algebraic functions can be expanded in power series, and so controlled. Compact sets play a key role, because they can be covered with a finite union of parametrizable balls, and such sets are RL-domains.

In his original theory for $\underset{\sim}{P}^1$, CANTOR used a different approach. He took the capacities and Green's functions of RL-domains as basic, rather than those of compact sets. He could do so because his decomposition theorem for RL-domains (proved using the global coordinate system on $\underset{\sim}{P}^1$) allowed him to construct RL-domains in profusion. On arbitrary curves, at least at the start, one does not know enough about RL-domains to get anywhere. The key idea is to replace the global existence problem with a local one, which can be solved using the Jacobian construction principle. Using capacity theory for compact sets, one gradually gains more and more global information, until finally it can be seen that Cantor's approach would have succeeded after all. However, the capacities of both compact sets and RL-domains are important, for it turns out that inner capacity is the correct notion for proving one half of the Fekete-Szegö theorem, and outer capacity for the other.

Definition of the global adelic capacity. - Once one has the local Green's functions, Cantor's formalism for the extended global capacity goes through unchanged.

In the previous sections, we have stated local capacity theory for sets in $\mathcal{C}(\hat{K}_v)$, but we could equally well have done so for sets in $\mathcal{C}(\tilde{K}_v)$. There are no differences. For the global theory it is convenient to restrict to that case.

It is useful to introduce a crude adelization of $\tilde{K}$ relative to K. For each place v of K, fix an embedding of $\tilde{K}$ into $\tilde{K}_v$, and let $\tilde{\mathcal{O}}_v$ be the ring of integers of $\tilde{K}_v$. Define $\tilde{K}_A$ to be the restricted direct product of the $\tilde{K}_v$ relative to the $\tilde{\mathcal{C}}_v$, where the product is taken over the places of K . $\tilde{K}$ embeds naturally in $\tilde{K}_A$ "on the diagonal". We will be dealing with sets of the form $\prod_v E_v \subset \mathcal{C}(\tilde{K}_A)$ in which each E_v is stable under $\mathrm{Gal}(\tilde{K}_v/K_v)$, so the choices of the embeddings will not matter.

Suppose $\mathfrak{X} = \{x_1 , \dots , x_r\} \subset \mathcal{C}(\tilde{K})$ is a finite set of global algebraic points on $\mathcal{C}$, stable under $\mathrm{Gal}(\tilde{K}/K)$. For each place v of K , let a set $E_v \subset \mathcal{C}(\tilde{K}_v)$ be given. We assume :

(a) Each E_v is capacitable, disjoint from $\mathfrak{X}$, and stable under $\mathrm{Gal}(\tilde{K}_v/K_v)$;

(b) All but finitely many E_v are "trivial with respect to $\mathfrak{X}$" in the following sense : v is a place such that $\mathcal{C} \subset \underset{\sim}{P}^n$ has nondegenerate reduction, $x_1,\dots,x_r$ reduce to distinct points (mod v) , and E_v is the set of points on $\mathcal{C}(\tilde{K}_v)$ which do not reduce to the same point as one of the x_i . Equivalently, if $\|z , w\|_v$ is the v-adic spherical metric on $\underset{\sim}{P}^n(\tilde{K}_v)$, then

$$E_v = \mathcal{C}(\tilde{K}_v) \setminus \textstyle\bigcup_{i=1}^r \{z \in \mathcal{C}(\tilde{K}_v) \ ; \ \|z , x_i\|_v < 1\} \ .$$

Given such a collection, write $\underset{\sim}{E} = \underset{\sim}{E}_K = \prod_v E_v$. We are going to define the capacity $\gamma(\underset{\sim}{E} , \mathfrak{X})$ of $\underset{\sim}{E}$ with respect to $\mathfrak{X}$.

If L/K is a finite extension, there is a natural way to associate an adelic set $\underset{\sim}{E}_L \subset \mathcal{C}(\tilde{L}_A)$ to $\underset{\sim}{E}_K$. Namely, for each place w of K lying over v of K , fix an isomorphism of $\tilde{L}_w$ with $\tilde{K}_v$, and put $E_w = E_v$. The capacity will be defined so as to be invariant under base extension. Hence, without loss, we can suppose that each of the $x_i \in \mathfrak{X}$ is rational over K .

For each point x_i , choose a function $g_i(z)$ on $\mathcal{C}$, rational over K and having a simple zero at x_i . (It is really only the choice of a global tangent vector that matters, not the uniformizing parameter.)

Now, for each v , define a "local Green's matrix" Γ_v , which will be an r by r symmetric matrix with nonnegative entries off the diagonal, given by

$$\Gamma_{v,ij} = \begin{cases} G(x_i , x_j ; E_v) & \text{if } i \neq j \\ V_{x_i}(E_v) = \lim_{z\to x_i} G(z , x_i ; E_v) + \log_v|g_i(z)|_v & \text{if } i = j \ . \end{cases}$$

All but finitely many of the Γ_v are the zero matrix. Let $N_v = [K_v : \mathbb{Q}_{p(v)}]$ and $N = [K : \mathbb{Q}]$ be the local and global degrees, respectively, where $p(v)$ is the rational prime below v .

The "global Green's matrix" $\Gamma = \Gamma(\underset{\sim}{E} , \mathfrak{X})$ will be

$$\Gamma = \sum_v (N_v/N)\, \Gamma_v \log(p(v))$$

where if $v = \infty$ we understand $p(v) = e$. By the product formula, Γ is independent of the choice of $g_i(z)$'s . It is again a symmetric $r \times r$ real matrix with nonnegative off-diagonal entries. We let $V(\underset{\sim}{E} , \mathfrak{X})$ be the value of Γ as a matrix game, defined by

$$V(\underset{\sim}{E}\,,\,\mathfrak{X}) = \max_{x\in\mathcal{P}} \min_{y\in\mathcal{P}}\ {}^t x\,\Gamma\,y\ .$$

Here $\mathcal{P}$ is the set of r-element real probability vectors : vectors with nonnegative entries adding up to 1 . Finally, the global capacity is

$$\gamma(\underset{\sim}{E}\,,\,\mathfrak{X}) = e^{-V(\underset{\sim}{E},\mathfrak{X})}\ .$$

The most unusual thing in the definition is the value of Γ as a matrix game. First, it should be noted that the formula is forced if pullbacks are to work properly. However, it would be hard to anticipate the generalization from the capacity with respect to one point, and doing so is one of Cantor's main achievements. Second, most definitions in the subject depend upon the equivalence of two extremal properties, for example the principle "minimizing is the same as equalizing" for the magnitude of oscillations of Tchebychev polynomials. $V(\underset{\sim}{E}\,,\mathfrak{X})$ is a quantity of that type. If $V(\underset{\sim}{E}\,,\,\mathfrak{X}) < 0$, there is a unique probability vector w such that all the entries of Γw are equal ; and their value in that case is exactly $V(\underset{\sim}{E}\,,\,\mathfrak{X})$. Nonetheless, the true meaning of the global capacity remains mysterious.

<u>Functoriality properties of the global capacity</u>. - $\gamma(\underset{\sim}{E}\,,\,\mathfrak{X})$ has nice functoriality properties. From the weights which go into the definition of the Green's matrix, it is invariant under base change. That is, if L/K is a finite extension, then $\gamma(\underset{\sim}{E}_L\,,\,\mathfrak{X}) = \gamma(\underset{\sim}{E}_K\,,\,\mathfrak{X})$. Furthermore, it behaves smoothly under pullbacks. Suppose $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is a nonconstant rational map between two curves defined over K . Let $\underset{\sim}{E}$ and $\mathfrak{X}$ be given on $\mathcal{C}_2$, and let F have degree n . Then

$$\gamma(F^{-1}(\underset{\sim}{E})\,,\,F^{-1}(\mathfrak{X})) = \gamma(\underset{\sim}{E}\,,\,\mathfrak{X})^{1/n}\ .$$

Lastly, if $\underset{\sim}{E} = \prod_v E_v$ and $\underset{\sim}{F} = \prod_v F_v$ are two adelic sets in $(\tilde{K}_A)$ whose capacities with respect to $\mathfrak{X}$ are defined, and if for each v , $E_v \subset F_v$, then $\gamma(\underset{\sim}{E}\,,\,\mathfrak{X}) \leqslant \gamma(\underset{\sim}{F}\,,\,\mathfrak{X})$. Monotonicity in the variable $\mathfrak{X}$ is an open question ; some care is needed even to formulate what it should mean, since $\gamma(\underset{\sim}{E}\,,\mathfrak{X})$ has only been defined if almost all of the E_v are "trivial with respect to $\mathfrak{X}$ " .

In the case of $\underset{\sim}{P}^1$, CANTOR proved a "separation inequality". Suppose $\underset{\sim}{E}$ and $\mathfrak{X}$ are such that $\mathfrak{X}$ can be partitioned into two sets $\mathfrak{X}_1$ and $\mathfrak{X}_2$ such that for every v , and any $x_1 \in \mathfrak{X}_1$, $x_2 \in \mathfrak{X}_2$, we have $G(x_1\,,\,x_2\,;\,E_v) = 0$. This means that for every v , $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are contained in different "components" of the complement of E_v . CANTOR showed that (under a slight extension of the definition of capacity above)

$$\gamma(\underset{\sim}{E}\,,\,\mathfrak{X}_1)\cdot\gamma(\underset{\sim}{E}\,,\,\mathfrak{X}_2) \geqslant 1\ .$$

It remains open whether this is true for curves of higher genus.

<u>The main theorem</u>. - The following generalization of the Fekete-Szegö theorem holds.

THEOREM. - *Suppose* $\mathcal{C}$ *is a smooth, geometrically connected curve over a number field* K . *Let* $\mathfrak{X}$ *be a finite (nonempty) Galois-stable set of points of* $\mathcal{C}(\tilde{K})$; *let* $\underset{\sim}{E} = \prod_v E_v \subset \mathcal{C}(\tilde{K}_A)$ *be an adelic set such that each* E_v *is closed, capacitable and stable under* $\mathrm{Gal}(\tilde{K}_v/K_v)$, *with all but finitely many of the* E_v *trivial with respect to* $\mathfrak{X}$. *Then*

(A) *If* $\gamma(\underset{\sim}{E} , \mathfrak{X}) < 1$, *there is a neighborhood of* $\underset{\sim}{E}$ *in* $\mathcal{C}(\tilde{K}_A)$ *which contains only a finite number of complete Galois orbits of points of* $\mathcal{C}(\tilde{K})$.

(B) *If* $\gamma(\underset{\sim}{E} , \mathfrak{X}) > 1$, *then every neighborhood of* $\underset{\sim}{E}$ *in* $\mathcal{C}(\tilde{K}_A)$ *contains infinitely many complete Galois orbits of points in* $\mathcal{C}(\tilde{K})$.

This is proved by an argument that goes back to FEKETE and SZEGÖ , and was elaborated by ROBINSON and CANTOR during the 1960's and 1970's. The goal is to find a function $f(z)$ in $K(\mathcal{C})$ whose poles are supported on $\mathfrak{X}$ and whose zeros are all near $\underset{\sim}{E}$. If $\gamma(E , \mathfrak{X}) < 1$, one constructs a function such that $|f(z)|_v \leqslant 1$ on E_v for all v , and $|f(z)|_v < 1$ on E_v for archimedean v . Then the neighborhood $\underset{\sim}{U} = \prod_v U_v$, where $U_v = \{z \in \mathcal{C}(\tilde{K}_v) \ ; \ |f(z)|_v < 1\}$ for archimedean v (resp. $\leqslant 1$ for nonarchimedean v), meets the needs of the theorem because any algebraic point whose conjugates are contained in $\underset{\sim}{U}$ must be a root of $f(z)$. If $\gamma(E , \mathfrak{X}) > 1$, then $\underset{\sim}{U}$ is given, and one constructs a function $f(z)$ such that

$$\{z \in \mathcal{C}(\tilde{K}_v) \ ; \ |f(z)|_v \geqslant 1\} \subset E_v \quad \text{for all} \quad v .$$

The conjugate sets of points on $\mathcal{C}(\tilde{K})$ belonging to $\underset{\sim}{U}$, claimed by the theorem, are the roots of $f(z)^m - 1 = 0$ for $m = 1 , 2 , 3 , \dots$

The capacity $\gamma(E , \mathfrak{X})$ and the Green's matrix Γ determine the relative orders y_i of the poles of $f(z)$ at the points in $\mathfrak{X}$. When $\gamma(E , \mathfrak{X}) > 1$ (so $V(E , \mathfrak{X}) < 0$), the orders are proportional to the components of the distinguished probability vector w mentioned earlier. The proofs in the two cases are somewhat different, but both have a local and a global part. The local part consists of finding, for each v , a function $f_v(z) \in \tilde{K}_v(\mathcal{C})$ for which $(1/\deg f) \log_v |f(z)|_v$ closely approximates $\sum_{x_i \in \mathfrak{X}} G(z , x_i ; E_v) w_i$ outside E_v . This uses the crucial property of Green's functions and the canonical distance, that they be approximatable by algebraic functions. The global part of the proof consists of "patching" the local functions $f_v(z)$ into a single global function $f(z)$ which looks rather like $f_v(z)$ at each v .

In his paper [C], CANTOR also gave other applications of capacity, including a generalized version of the rationality criterion of Polya-Carlson-Dwork-Bertrandias. I have not attempted to carry these over on algebraic curves, but I would not expect any difficulties in doing so.

Heights. - As has been seen, the canonical local distances are connected with Néron's local heights. We wish to offer here an interpretation of the Green's functions themselves as heights.

Consider the case $\mathcal{C} = \underset{\sim}{P}^1$, with $\mathfrak{X} = \{\infty\}$. For each place v of K, let $E_v = D(0, 1) = \{z \in \tilde{K}_v ; |z|_v \leqslant 1\}$, the closed unit disc, which we identify as a subset of $\underset{\sim}{P}^1$ using the standard affine coordinates. The Green's function of E_v is given by

$$G(z, \infty, E_v) = \begin{cases} 0 & \text{if } z \in E_v \\ \log_v|z|_v & \text{if } z \notin E_v \end{cases}$$

$$= \max(0, \log_v|z|_v).$$

Now for a number $0 \neq \varkappa \in K$,

$$h(\varkappa) = \sum_v (N_v/N) \max(0, \log|\varkappa|_v) = \sum_v (N_v/N)\, G(\varkappa, \infty ; E_v) \log p(v)$$

is none other than the absolute logarithmic height of $\varkappa$.

This suggests that given an algebraic curve $\mathcal{C}/K$ and $\underset{\sim}{E}$ and $\mathfrak{X}$ as before, together with a vector of weights w for the points in $\mathfrak{X}$, we should regard

$$h_{\underset{\sim}{E},\mathfrak{X}}(\varkappa) = \sum_{x_i \in \mathfrak{X}} w_i [\sum_v (N_v/N)\, G(\varkappa, x_i ; E_v) \log p(v)]$$

as a kind of height for a point $\varkappa \in \mathcal{C}(K)$. Evidently these heights are "absolute" since they do not depend on the ground field over which we consider $\mathcal{C}$ (identifying the heights obtained from $\underset{\sim}{E}_L$ and $\underset{\sim}{E}_K$, given a finite extension L/K). Hence they can be considered as functions on $\mathcal{C}(\tilde{K})$.

On this view, the Fekete-Szegö theorem has the following meaning. If $\gamma(E,\mathfrak{X}) < 1$, then the weights w_i can be chosen so that there are only a finite number of points in $\mathcal{C}(\tilde{K})$ with $h_{\underset{\sim}{E},\mathfrak{X}}(\varkappa) < \epsilon$, for some $\epsilon > 0$. If $\gamma(\underset{\sim}{E}, \mathfrak{X}) > 1$, then for every choice of weights and every $\epsilon > 0$, there are infinitely many points with $h_{\underset{\sim}{E},\mathfrak{X}}(\varkappa) < \epsilon$. This should be compared with the classical fact that the roots of unity are the points for which $h(\varkappa) = 0$, for the naive height on $\underset{\sim}{P}^1$.

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