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ON THE TATE CONSTANT

by Bernard DWORK (*)

Chapter I

1. Introduction. - Our work on the relation between the congruence zeta function and p -adic analysis began in February 1958 with the suggestion of J. TATE that his constant C (described below) may be constructed by p -adic analytic methods. (For an alternate description of C , see [Dw 5], Introduction (0.1)).

Let k be a field of characteristic zero complete with respect to a discrete valuation, with valuation ring $\mathcal{O}$ and residue class field $\bar{k} = \mathcal{O}/\mathfrak{p}$. Let A be an elliptic curve defined over k by an equation

$$(1) \quad y^2 + (a_1 x + a_2) y = x^3 + a_3 x^2 + a_4 x + a_5$$

where the $a_i \in \mathcal{O}$.

Letting $x = ty$, we find

$$y^3 t^3 + y^2 (-1 - a_1 t + a_3 t^2) + y (-a_2 + a_4 t) + a_5 = 0$$

and hence there exists a unique solution in $k((t))$ for y with a pole of order 3 at $t = 0$. This solution is of the form

$$(2) \quad y = t^{-3} + B_{-2} t^{-2} + \dots$$

and the coefficients B_i lie in $\mathcal{O}$. Clearly

$$(3) \quad x = t^{-2} + B_{-2} t^{-1} + \dots$$

Let

$$(4) \quad du = -dx/(2y + a_1 x + a_2)$$

a differential of the first kind on A . In terms of the uniformizing parameter $t = x/y$ at infinity, we have after integration

$$(5) \quad u = t + \frac{1}{2} D_1 t^2 + \frac{1}{3} D_2 t^3 + \dots$$

where the D_i lie in $\mathcal{O}$.

2. THEOREM.

Part 1. - If the reduced curve $\bar{A}$ defined over $\bar{k}$ is non-singular and has Hasse invariant not zero, then there exists a unit C in the maximal unramified extension K of k such that $\exp Cu$ ($\in K[[t]]$) has, in fact, integral coefficients

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in K .

Part 2. - If $\bar{k}$ is finite, i. e. k is a p -adic field, then the unit root of the zeta function of the reduced curve $\bar{A}$ is $C^{\sigma-1}$, where σ is the Frobenius automorphism of K over k .

3. Explanation of the Tate's theorem. - Let

$$S = \{(x, y) \in A ; |x| > 1\}.$$

In terms of the uniformizing parameter, $t = x/y$,

$$x = t^{-2} + A_{-1} t^{-1} + A_0 + \dots$$

and so S is parametrized by $t \in D(0, 1^-)$. The map of $t \mapsto u(t)$ gives a homomorphism of S into k_+ .

Since k is of characteristic zero,

$$u(t) = 0$$

for each $P \in S$ which is division point. Since u is a one to one map of $D(0, |\pi|^-)$ onto itself, $u(t) = 0$ can only be valid for $t_0 = 0$ if $t_0 \in D(0, |\pi|^-)$. On the other hand, it is shown by LUTZ [L] that, for $P \in S$,

$$\text{ord } t(pP) \geq \min(1 + \text{ord } t(P), 4 \text{ ord } t(P))$$

and hence if $u(t(P_0)) = 0$, then $t(p^v P_0) \in D(0, |\pi|^-)$ for suitable v and so P_0 is a p^{th} power division point.

Since $1 + \mathfrak{p}$ ($\mathfrak{p} = D(0, 1^-)$) does have points of finite order, TATE sought an isomorphism of S into $1 + \mathfrak{p}$ such as $t \mapsto \exp \theta \cdot u(t)$. If one exists with integral coefficients then it is invertible and gives an isomorphism of S_K with $1 + \mathfrak{p}_K$ where K is a complete field containing $k(\theta)$. The exact sequence

$$0 \rightarrow S \xrightarrow{\text{inj}} A \xrightarrow{\text{red}} \bar{A} \rightarrow 0$$

together with the fact that for $(\ell, p) = 1$ both A and $\bar{A}$ have ℓ^2 points of order ℓ shows again that the only division points in S are of p power order. If there are p points of order p in $\bar{A}$ then there are only p in S (as there are in $1 + \mathfrak{p}$) and so the isomorphism of Tate could (and infact does) exist. If $\bar{A}$ has no points of order p then there are p^2 such points in S and then the suggested isomorphism is impossible. This explains the role of the Hasse invariant.

4. Proof of part 2 of Tate's theorem.

In 1958 (unpublished), we obtained a proof of part 2 of Tate's theorem for the Legendre model

$$y^2 = x(1-x)(t-x).$$

Using $t = 1/\sqrt{x}$ as parameter at ∞ , we may write

$$(4.1) \quad du = \sum_{n=0}^{\infty} t^{2n} D_{2n}(\lambda)$$

where

$$(4.2) \quad D_{2n} = (-1)^n \sum_{i=0}^n \binom{-1/2}{n-i} \binom{-1/2}{i} \lambda^i.$$

If λ lies in an unramified extension of $\mathbb{Q}_p$ then the existence of C (again in an unramified extension) is equivalent (by the Dieudonné criterion [Dw 1]) to congruences

$$(4.3) \quad C^{\sigma-1} D_{mp^s-1}(\sigma\lambda) \equiv D_{mp^{s+1}-1}(\lambda) \pmod{p^{s+1}}$$

for all $s \in \mathbb{N}$, $(m, p) = 1$, where σ is the absolute Frobenius. The consistency of these conditions is demonstrated by means of the formal congruences

$$(4.4) \quad D_{mp^{s+1}-1}(\lambda) \equiv (-1)^{(p-1)/2} D_{mp^s-1}(\lambda) F(\lambda)/F(\lambda^p) \pmod{p^{s+1} \mathbb{Z}_p[[\lambda]]}$$

where

$$F(\lambda) = F\left(\frac{1}{2}, \frac{1}{2}; 1, \lambda\right) = \sum \left(\left(\frac{1}{2}\right)_j / j!\right)^2 \lambda^j.$$

By means of these congruences, we showed that $F(\lambda)/F(\lambda^p)$ extends to an analytic element f on the Hasse domain

$$(4.5) \quad H = \{\lambda; |D_{p-1}(\lambda)| \geq 1\}.$$

Congruences similar to (4.4) are treated elsewhere [Dw 2], [Dw 4].

Thus if $\sigma \lambda_0 = \lambda_0^p$, i. e. λ is a Teichmüller representative of its residue class then $C(\lambda_0) \in K$ is to be chosen so that

$$(4.6) \quad C(\lambda_0)^{\sigma-1} = f(\lambda_0).$$

More generally if $\lambda = \lambda_0 + \lambda_1$, $|\lambda_1| < 1$. Then we must put

$$(4.7) \quad C(\lambda) = C(\lambda_0) / [1 + \sum_{s=1}^{\infty} \eta_s(\lambda_0) \lambda_1^s]$$

where

$$\eta_s = s!^{-1} F^{(s)} / F.$$

The point being that η_s is an analytic element on H whose restriction to $D(0, 1^-)$ is as indicated. This may also be expressed by the condition

$$(4.8) \quad C(\lambda) = C(\lambda_0) / v(\lambda)$$

where v is the unique branch of $F(1/2, 1/2, 1, \lambda)$ at λ_0 i. e. the unique solution of the corresponding second order differential equation which is bounded on $D(\lambda_0, 1^-)$ and such that

$$v(\lambda_0) = 1.$$

In appendix B, we indicate how these results should be generalised to curves with ordinary reduction.

5. Heuristics. - We assume the reduced curve is ordinary.

If ω_1, ω_2 are "eigenvectors" of Forbenius, i. e.

$$\omega_{1,\lambda^p}^{\sigma\bar{\varphi}} = p \omega_{1,\lambda} + d \xi_1$$

$$\omega_{2,\lambda^p}^{\sigma\bar{\varphi}} = \omega_{2,\lambda} + d \xi_2$$

where ξ_1, ξ_2 are daggerized algebraic functions on A , and we think of σ as operating on coefficients of the differential forms while $\bar{\varphi}$ represents $x \rightarrow x^p$, then upon integration, setting $I_{2,\lambda} = \int \omega_{2,\lambda}$, a local abelian integral, we obtain

$$(5.1) \quad I_{1,\lambda^p}^{\sigma\bar{\varphi}} = p I_{1,\lambda} + \xi_1$$

$$(5.2) \quad I_{2,\lambda^p}^{\sigma\bar{\varphi}} = I_{2,\lambda} + \xi_2.$$

We are tempted to deduce Tate's theorem by applying Dieudonné's criterion to (5.1). There are two questions :

(5.3) Is ξ_1 bounded by p on a generic disk ?

(5.4) $I_{1,\lambda}$ need not be an integral of the first kind ?

Our purpose is to show how these objections may be met by means of the theory of normalized solution matrices of the hypergeometric differential equation as explained in Chapter 9 of [Dw 5].

Chapter II.

We shall consider the hypergeometric differential equation

$$(1.0) \quad \frac{d}{d\lambda} (u_1, u_2) = (u_1, u_2) \begin{pmatrix} -\frac{c}{\lambda} & \frac{c-a}{1-\lambda} \\ \frac{c-b}{\lambda} & \frac{a+b-c}{1-\lambda} \end{pmatrix}$$

in a split case of period one. By this, we mean that a, b, c are elements of $\mathbb{Q}$ whose denominators divide $p-1$, and such that after replacement by minimum representative mod 1, c does not lie on the real interval connecting a and b . To fix ideas, we assume that we have the Type 1 situation, i. e.

$$(1.1) \quad 1 > \text{Max}(a, b) \geq \text{Min}(a, b) > c > 0.$$

The Type II case can be treated similarly. We shall restrict λ to the region

$$(1.2) \quad |\lambda| = |\lambda - 1| = 1.$$

We recall L_λ = analytic functions on the complement of sets of the type

$$D(0, \epsilon_0) \cup D(1, \epsilon_1) \cup D(\lambda^{-1}, \epsilon_{\lambda^{-1}}) \cup D(\infty, \epsilon_\infty)$$

(were ϵ_i is less than the distance from i to the remaining elements of $\{0, 1, \lambda^{-1}, \infty\}$, distance from ∞ to be computed in terms of $1/x$).

$$f = x^{b-1} (1-x)^{c-b} (1-\lambda x)^{-a},$$

$$G = xf(\lambda, x)/x^p f(\lambda^p, x^p),$$

$$E = x(\partial/\partial x),$$

$$D_{j,\lambda} = (xf)^{-1} \circ E \circ xf,$$

$$\alpha = \psi \circ G,$$

$$\beta = G^{-1} \circ \psi,$$

$$\psi x = x^p,$$

$$(\psi \xi)(x) = \sum_p^{-1} \xi(y), \text{ the sum being over } \{y | y^p = x\},$$

$$W_\lambda = L_\lambda / D_{f,\lambda} L_\lambda,$$

Hasse domain = set of all residue classes where (1.0) has a bounded solution,

Gauss norm is relative to the x variable.

We start with [Dw 2] (6.3.2), which we write in the form

$$(1.3) \quad \alpha \left(\frac{1}{1-x} \right) = A^t \left(\frac{1}{1-x} \right) + D_{t,\lambda^p} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where $y_1, y_2 \in L_{\lambda^p}$.

We show (chap. III) below that

$$(1.4) \quad \text{Max}(|y_1|_{\text{Gauss}}, |y_2|_{\text{Gauss}}) \leq |p|.$$

We apply β to (1.3) and deduce

$$(1.5) \quad \left(\frac{1}{1-x} \right) + (\beta\alpha - 1) \left(\frac{1}{1-x} \right) = A^t \beta \left(\frac{1}{1-x} \right) + D_{j,\lambda} \begin{pmatrix} \beta y_1 \\ \beta y_2 \end{pmatrix} \cdot \frac{1}{p}.$$

(1.6). PROPOSITION. - If $z \in L_\lambda$, $|z|_{\text{Gauss}} \leq 1$ then

$$(1.6.1) \quad (\beta\alpha - 1) z = D_{j,\lambda} w$$

where $w \in L_\lambda$, $|w|_{\text{Gauss}} \leq 1$.

Proof. - The mapping α induces a map of W_λ into W_{λ^p} with β as inverse. Hence equation (1.6) with $w \in L_\lambda$ is trivial. We need only check the Gauss norm. For this, we need only find a formula for w valid on an annulus

$$(1.6.2) \quad 1 - \epsilon < |x| < 1.$$

The formula for f shows that we may write

$$x f_{\lambda} = x^b g_{\lambda}(x), \quad g \in \mathbb{Z}_p[[\lambda]][[x]] .$$

Then

$$G = x^{-\mu_p} g_{\lambda}(x) / g_{\lambda^p}(x^p)$$

and so

$$\begin{aligned} \beta \circ \alpha &= x^{\mu_b} \frac{g_{\lambda^p}}{g_{\lambda}} \circ \psi \circ \frac{g_{\lambda}}{g_{\lambda^p}} x^{-\mu_b} \\ &= x^{\mu_b} \frac{1}{g_{\lambda}} \circ \psi \circ g_{\lambda} x^{-\mu_b} \end{aligned}$$

i. e.

$$D_f w = x^{\mu_b} \frac{1}{g_{\lambda}} (\psi - 1) g_{\lambda} x^{-\mu_b} z$$

and so

$$(1.6.3) \quad E(x^b g_{\lambda} w) = x^{pb} (\psi - 1) g_{\lambda} x^{-\mu_b} z .$$

Now

$$g_{\lambda} x^{-\mu_b} z = \sum_{n=-\infty}^{\infty} \gamma_n x^n$$

a representation valid on (1.6.2), and the boundary norm (as $\epsilon \rightarrow 0$) being bounded by unity, we have

$$|\gamma_m| \leq 1 \quad \forall m .$$

Since $(\psi - 1)$ annihilates x^m if $p \nmid m$, we find

$$(1.6.4) \quad z = -x^{-b} \frac{1}{g_{\lambda}} \sum_{\substack{m=-\infty \\ p \nmid m}}^{+\infty} \gamma_m x^{m+pb} / (m + pb) .$$

(There is no constant of integration as z is single valued in the annulus (1.6.2).) The assertion now follows from the boundary norm of g_{λ} . This completes the proof.

It follows from (1.4), (1.6) that there exist $z_1, z_2 \in L_{\lambda}$

$$(1.7.1) \quad \max(|z_1|_{\text{Gauss}}, |z_2|_{\text{Gauss}}) \leq 1$$

such that

$$(1.7.2) \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} = A^t \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix} + D_{f,\lambda} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} .$$

Our object is to find "eigen vectors" of β . For this purpose, we ask for an invertible matrix $Y(\lambda)$, defined on a disk $D(\lambda_0, 1^-)$ for which the differential equation is not super singular, such that

$$(1.8) \quad Y(\lambda) A^t = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} Y^{\sigma}(\lambda^p)$$

The σ (= absolute Frobenius) referring to the fact that Y may be defined over a maximal unramified extension of $\mathbb{Q}_p$. This condition is equivalent of

$$(1.9) \quad (Y^\sigma(\lambda^p))^* A = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} Y^*$$

($Y^* = (Y^{\text{transpose}})^{-1}$) and so we may take Y^* to be the normalized solution matrix of (1.0) on $D(\lambda_0, 1^-)$.

Thus we may satisfy (1.8) by setting

$$(1.10) \quad Y^* = \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \hat{u} \end{pmatrix} \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$$

where $u, \hat{u}, \eta, \tau$ satisfy the following conditions [Dw 1] (9.6)

(1.11.1) η is analytic element bounded by 1 on the Hasse domain of (1.0).

(1.11.2) $u(1, \eta)$ is the unique bounded solution of (1.0) on $D(\lambda_0, 1^-)$.

(1.11.3) $u(\lambda)/u^\sigma(\lambda^p)$ extends to an analytic element on the Hasse domain.

(1.11.4) $\tau^\sigma(\lambda^p) \equiv_p \tau(\lambda) \pmod{p} \quad (\forall \lambda \in D(\lambda_0, 1^-))$.

(1.11.5) $\hat{u} = \text{wronskian}/u$.

(1.11.6) $u, \hat{u}$ take on unit values throughout $D(\lambda_0, 1^-)$.

Putting

$$(1.12) \quad Y(\lambda) \begin{pmatrix} 1 \\ 1 \\ 1-x \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \xi_{1,\lambda} \\ \xi_{2,\lambda} \end{pmatrix} = \vec{\xi}_\lambda(x)$$

and multiplying (1.7) by Y and using (1.8),

$$(1.13) \quad \vec{\xi}_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \beta \vec{\xi}_{\lambda^p}^\sigma(x^p) + D_{f,\lambda} Y(\lambda) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Putting

$$(1.13.1) \quad \begin{pmatrix} \hat{z}_1 \\ z_2 \end{pmatrix} = \hat{z} = Y(\lambda) \begin{pmatrix} \hat{z}_1 \\ z_2 \end{pmatrix}$$

we may rewrite (1.13) in the form

$$(1.14) \quad x f_\lambda \vec{\xi}_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} (x f_{\lambda^p} \vec{\xi}_{\lambda^p}^\sigma)^{\bar{\beta}} + E x f_\lambda \hat{z}.$$

Applying E^{-1} to both sides and writing as function on a general residue class $D(x_0, 1^-)$ in x time (with $x_0^\sigma = x_0^p$)

$$I_{j,\lambda}(x) = \int_{x_0}^x f \xi_{j,\lambda} dx,$$

we obtain

$$(1.15) \quad \begin{aligned} I_{1,\lambda}(x) &= \frac{1}{p} I_{1,\lambda^p}^\sigma(x^p) + x f_\lambda \hat{z}_1 \\ I_{2,\lambda}(x) &= I_{2,\lambda^p}^\sigma(x^p) + x f_\lambda \hat{z}_2 \end{aligned}$$

(where I^σ is function on $D(x_0^\sigma, 1^-)$).

Since

$$(1.16) \quad Y = \begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/u & 0 \\ 0 & 1/\bar{u} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\eta & 1 \end{pmatrix}.$$

It follows from (1.13.1), that

$$(1.17) \quad \hat{z}_2 = -\frac{1}{\bar{u}(\lambda)} (z_1 \eta(\lambda) - z_2)$$

and hence, by (1.7.1), (1.11.1), (1.11.6),

$$(1.17.1) \quad |\hat{z}_2|_{\text{Gauss}} \leq 1.$$

Thus, by (1.15),

$$(1.18) \quad |I_{2,\lambda}(x)| \leq 1 \text{ for all } x \in D(x_0, 1^-).$$

We now put

$$(1.19) \quad J_\lambda(x) = \frac{1}{u(\lambda)} \int f_\lambda(x) dx.$$

For Type I, this is the unique (up to factor independent of x) integral of first kind associated with the differentials in the integral representation of $F(a, b, c, \lambda)$ and its derivatives. [Dw 1] (chapter 14). By (1.12),

$$(1.20) \quad \xi_{1,\lambda} = \frac{1}{u(\lambda)} \cdot 1 - \tau(\lambda) \xi_{2,\lambda}$$

$$\xi_{2,\lambda} = \frac{1}{\bar{u}(\lambda)} (-\eta(\lambda) 1 + \frac{1}{1-x})$$

and so

$$(1.21) \quad I_{1,\lambda}(x) = J_\lambda(x) - \tau(\lambda) I_{2,\lambda}(x).$$

Multiplying the second equation of (1.15) by $\tau(\lambda)$, and adding to the first

$$(1.22) \quad J_\lambda(x) = \frac{1}{p} J_\lambda^\sigma(x^p) + H(\lambda, x)$$

where

$$(1.22.1) \quad H(\lambda, x) = (\tau(\lambda) - \tau^\sigma(\lambda^p) \frac{1}{p}) I_{2,\lambda^p}^\sigma(x^p) + x f_\lambda \frac{z_1}{u(\lambda)}.$$

We observe that, by (1.18), (1.11.4), (1.7.1),

$$(1.23) \quad |H(\lambda, x)| \leq 1$$

on $D(\lambda_0, 1^-) \times D(x_0, 1^-)$. By the Dieudonné's conditions we now have the first part of Tate's theorem

$$(1.24) \quad \exp J_\lambda(x) \in \mathcal{O}_K[[x - x_0]]$$

where K is a sufficiently large field containing λ . The second part of Tate's theorem is also demonstrated since $(u, u\eta)$ is an "eigenvector" of a semilinear transformation with matrix A corresponding to eigenvalue 1. Using Adolphson's

explanation in the appendix of [Dw 1], we may deduce the connection between $(u(\lambda_0))^{1-\sigma}$ and the unit reciprocal root of the corresponding L-function. In the next section, we complete the treatment by verifying (1.4).

Chapter III

1. Our object is to verify the estimates of II (1.4).

LEMME. - Let $L = \mathbb{Q}(\lambda)[x, x^{-1}, (1-x)^{-1}, (1-\lambda x)^{-1}]$. For $s \geq 2$ there exist $\alpha_s, \beta_s, \gamma_s, \delta_s \in \mathbb{Q}(\lambda)$ and $\xi_s, \eta_s \in L$ such that

$$(1.1) \quad (1-x)^{-s} = \alpha_s 1 + \beta_s (1-x)^{-1} + D_{f,\lambda} \xi_s$$

$$(1.2) \quad (1-\lambda x)^{-s} = \gamma_s 1 + \delta_s (1-x)^{-1} + D_{f,\lambda} \eta_s$$

and subject to II (1.2),

$$(1.3) \quad \text{Max}(|\alpha_s|, |\beta_s|, |\xi_s|_{\text{Gauss}}) \leq \sup_{0 \leq m \leq s-1} \frac{1}{|b-c+m|}$$

$$(1.4) \quad \text{Max}(|\gamma_s|, |\delta_s|, |\eta_s|_{\text{Gauss}}) \leq \sup_{0 \leq m \leq s-1} \frac{1}{|a+m|}.$$

Proof. - Equation (1.1) follows from [Dw 1] (1.2). In terms of [Dw 1] (2.3.5.10)

$$(1.5) \quad \alpha_s = \langle T_1^{-s}, 1^* \rangle$$

$$(1.6) \quad \beta_s = \langle T_1^{-s}, (T_1^{-1})^* \rangle$$

and so α_s (resp. β_s) is given by the coefficient of T_1^{s-1} in the formula for ξ_1 in [Dw 1] (p. 25) with $A = 0$, $B = \lambda(a-c)$ (resp. $A = b-c$, $B = A(c-b)$).

The estimates for $|\alpha_s|$, $|\beta_s|$ follow from this formula. A second proof of these estimates will appear below.

The proof of (1.1) [Dw 1] (p. 10) shows that $\xi_s \in T_1^{-1} \mathbb{Q}(\lambda)[T_1^{-1}]$ and is of degree bounded by $s-1$ as polynomial in T_1^{-1} .

On the other hand, we may solve (1.1) for ξ_s by writing the solution in the form

$$(1.7) \quad -T_1 \frac{d}{dT_1} (x f_\lambda \xi_s) = T_1 f[T_1^{-s} - \alpha_s - \beta_s T_1^{-1}]$$

and so

$$(1.8) \quad -\xi_s = \theta[T_1^{-s} - \alpha_s - \beta_s T_1^{-1}]$$

where θ is defined by

$$(1.9) \quad \theta = (1-T_1)^{-b} (1-t_1 T_1)^a T_1^{b-c} (T_1 \frac{d}{dT_1})^{-1} T_1^{1+c-b} (1-T_1)^{b-1} (1-t_1 T_1)^{-a}$$

as endomorphism of $\mathbb{Q}(\lambda)((T_1))$. (Since $b-c \notin \mathbb{Z}$, there is no ambiguity due to

possible constant of integration. Since both θ_1 and θT_1^{-1} lie in $\mathbb{Q}(\lambda)[[T_1]]$, we conclude that

$$(1.10) \quad -\xi_s = P_1 \theta T_1^{-s}$$

where P_1 denotes the principal part at $x = 1$. Writing

$$(1.11.1) \quad (1 - T_1)^{b-1} (1 - t_1, T_1)^{-a} = \sum_{m=0}^{\infty} g_m T_1^m \in \mathbb{Z}_p[[T_1]]$$

$$(1.11.2) \quad (1 - T_1)^{-b} (1 - t_1, T_1)^a = \sum_{m=0}^{\infty} h_m T_1^m \in \mathbb{Z}_p[[T_1]]$$

we compute

$$(1.12) \quad -\xi_s = P_1 \sum_{m, m'=0}^{\infty} h_{m'} g_m T_1^{m'+b-c} \left(T_1 \frac{d}{dT_1}\right)^{-1} T_1^{c-b+m+1-s} \\ = \sum_{j=1}^{s-1} T_1^{-j} \sum h_{m'} g_m \frac{1}{c-b+m+1-s},$$

the inner sum being over all pairs $m, m' \geq 0$ such that

$$(1.12.1) \quad m + m' = s - 1 - j.$$

Estimate (1.3) for ξ_s follows from (1.12) and the fact that g_m, h_m lie in $\mathbb{Z}_p$. A second proof of the estimates (1.3) for α_s, β_s now follows from (1.1). The proof of (1.4) follows by the same methods.

2. The proof of II (1.4) follows the procedure of [Dw 1] (chapter 6). We write

$$(2.1) \quad \psi T^{-\mu_b} T_1^{\mu_b - \mu_c} T_{\lambda}^{\mu_a} \left(\frac{1}{T_1^{-1}} - 1\right) = \begin{pmatrix} \chi_1 & \chi_3 \\ \chi_2 & \chi_4 \end{pmatrix} \begin{pmatrix} 1 \\ T_1^{-1} \end{pmatrix} + D_{f, \lambda^v} \begin{pmatrix} Y_{1,1} \\ Y_{2,1} \end{pmatrix}$$

$$(2.2) \quad \psi M \begin{pmatrix} 1 \\ T_1^{-1} \end{pmatrix} = \begin{pmatrix} \rho_1 & \rho_3 \\ \rho_2 & \rho_4 \end{pmatrix} \begin{pmatrix} 1 \\ T_1^{-1} \end{pmatrix} + D_{f, \lambda^v} \begin{pmatrix} Y_{1,2} \\ Y_{2,2} \end{pmatrix}$$

where

$$M = G^{(1)} G^{(2)} - 1.$$

The matrix χ was computed [Dw 1] (6.4) and subject to certain conditions it is shown [Dw 1] (6.1) that the matrix ρ is bounded by $|p|$. For our present purpose, we note that

$$(2.3) \quad Y_i = Y_{i1} + Y_{i2} \quad (i = 1, 2).$$

For case 1, it is shown [Dw 1] (p. 100) that

$$(2.4) \quad Y_{i,1} = 0.$$

We apply [Dw 1] (5.1) to equation (2.2). We may here take

$$(2.5.1) \quad \lambda_0 = (p-1)b, \quad \lambda_1 = 0 = \lambda_{\infty}$$

and

$$(2.5.2) \quad j_0 = j_{\infty} = 1; \quad j_1 = 4.$$

By [Dw 1] (6.1.15), we have

$$(2.6) \quad \mu_{0,s} = 0 \quad \text{for } s \geq 1$$

$$(2.7) \quad \mu_{\infty,s} = 0 \quad \text{for } s \geq 1.$$

Thus in [Dw 1] (6.1.12), we have in the representation of (say) ψ^M ,

$$(2.8.1) \quad K_0 = 0,$$

$$(2.8.2) \quad K_{\infty} = \mu_{\infty,0} \quad (\text{a constant}).$$

On the other hand, by definition

$$(2.8.3) \quad K_1 = \sum_{s=1}^{\infty} \mu_{1,s} T_1^{-s}$$

$$(2.8.4) \quad K_{1/\lambda^p} = \sum_{s=1}^{\infty} \mu_{1/\lambda^p,s} T_{1/\lambda^p}^s$$

and by [Dw 1] (lemma 5.2, and 6.1.16),

$$(2.9) \quad \sup_{s \geq 1} \{ |\mu_{1,s}|, |\mu_{1/\lambda^p,s}|, |\mu_{\infty,0}| \} \leq 1/p$$

$$(2.10) \quad \sup_{s \geq 1} \{ |\mu_{1,s}|, |\mu_{1/\lambda^p,s}| \} \leq p |\pi|^{ps}.$$

Thus, by lemma 1, we have

$$(2.11) \quad Y_{1,2} = \sum_{s=1}^{\infty} \mu_{1,s} \xi_s + \sum_{s=1}^{\infty} \mu_{1/\lambda^p,s} \eta_s.$$

Precisely as in [Dw 1] (6.1), we may deduce $|Y_{1,2}| \leq |p|$ by means of equations (2.9) - (2.11) subject to conditions [Dw 1] (6.1.8) which reduce here to the conditions

$$(2.12) \quad |a| = |b - c| = |a - c| = 1.$$

These conditions are a consequence of II (1.1) and the condition that (a, b, c) be of period one. This completes the proof.

Appendix A.

The theorem of Tate for elliptic curves in Legendre normal form requires $(a, b, c) = (1/2, 1/2, 1)$. This is of type II. In the notation of [Dw 1] (9.5.1), the normalized bounded solution of II (1.0) is $(\bar{u}, \bar{\eta}, \bar{u})$.

Here, equation II (1.12) must be written

$$\begin{pmatrix} \bar{u}, \bar{\eta}, \bar{u} \\ \frac{1}{\bar{u} + \bar{\tau} \bar{u} \bar{\eta}}, \bar{\tau} \bar{u} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{1-x} \end{pmatrix} = \begin{pmatrix} \xi_{1,\lambda} \\ \xi_{2,\lambda} \end{pmatrix}$$

and so

$$I_{1,\lambda} = \int \omega_{1,\lambda} dx$$

$$I_{2,\lambda} = \int \omega_{2,\lambda} dx$$

$$\omega_{1,\lambda} = \frac{1}{\bar{u}(\lambda)} \frac{1}{1-x} f dx - \bar{\gamma}(\lambda) \omega_{2,\lambda}$$

$$\omega_{2,\lambda} = \frac{1}{\bar{u}(\lambda)} \left(f dx - \bar{\eta}(\lambda) \frac{1}{1-x} f dx \right).$$

Here $\frac{1}{1-x} f dx$ is the unique differential of the first kind and putting

$$J_\lambda(x) = \frac{1}{\bar{\mu}(\lambda)} \int \frac{1}{1-x} f dx$$

we obtain the analogue of II (1.22), (1.23). Here $\bar{u}$ is a branch of $F(1/2, 1/2, 1, \lambda)$ and $1/\bar{u}(\lambda)$ is the constant of Tate.

Appendix B.

Families of curves with ordinary reduction.

For the split case of period greater than one, we must leave the situation involving a two dimensional piece of cohomology. For this reason, we briefly sketch how the theory extends to curves. We consider a family $f(\lambda, x, Y) = 0$ of (possibly singular) plane curves with generic ordinary reduction.

There exists a basis $\{\omega_{1,\lambda}, \dots, \omega_{g,\lambda}\}$ for the differentials of the first kind together with set of representatives $\omega_{g+1,\lambda}, \dots, \omega_{2g,\lambda}$ of a basis of differentials of the second kind modulo exact + d.f.k. such that

$$(1) \quad \begin{pmatrix} \omega_{1,\lambda} \\ \vdots \\ \omega_{2g,\lambda} \end{pmatrix} = \frac{1}{p} A^t \begin{pmatrix} \omega_{1,\lambda^p} \\ \vdots \\ \omega_{2g,\lambda^p} \end{pmatrix} + d \begin{pmatrix} z_1 \\ \vdots \\ z_{2g} \end{pmatrix}$$

where the z_i are daggerized algebraic functions with Gauss norm bounded by unity. Furthermore

$$\omega_1 \wedge \omega_2 = 0_1 \pm 1$$

so that the pairing matrix is

$$\begin{pmatrix} 0 & I_g \\ I_g & 0 \end{pmatrix}.$$

The matrix A is an over convergent $2g \times 2g$ matrix function of λ and

$$(2) \quad A \equiv \begin{pmatrix} \bar{A}_1 & \bar{A}_2 \\ 0 & 0 \end{pmatrix} \pmod{p}$$

where $\bar{A}_1$ is the Hasse-Witt matrix. (Furthermore

$$\begin{pmatrix} I_g & \\ & p^{-1} I_g \end{pmatrix} A$$

is invertible mod p).

In terms of the theory of normalised period matrices [Dw 3], we have

$$(3) \quad Y(\lambda) = \begin{pmatrix} I_g & 0 \\ T & I_g \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \begin{pmatrix} I_g & \eta \\ 0 & I_g \end{pmatrix}$$

and

$$(4) \quad Y^\sigma(\lambda^p) A = \begin{pmatrix} I_g & \\ & p I_g \end{pmatrix} Y(\lambda),$$

multiplying (1) on the left by $Y(\lambda)^*$, we obtain

$$(5) \quad Y(\lambda)^* \begin{pmatrix} \omega_{1,\lambda} \\ \vdots \\ \omega_{2g,\lambda} \end{pmatrix} = \frac{1}{p} \begin{pmatrix} I_g & \\ & p I_g \end{pmatrix} Y^\sigma(\lambda^p)^* \begin{pmatrix} \omega_{1,\lambda^p}^\sigma \\ \vdots \\ \omega_{2g,\lambda^p}^\sigma \end{pmatrix} + d Y(\lambda)^* \begin{pmatrix} z_1 \\ \vdots \\ z_{2g} \end{pmatrix}$$

Putting

$$\bar{\omega}_{1,\lambda} = \begin{pmatrix} \omega_{1,\lambda} \\ \vdots \\ \omega_{g,\lambda} \end{pmatrix}, \quad \bar{\omega}_{2,\lambda} = \begin{pmatrix} \omega_{g+1,\lambda} \\ \vdots \\ \omega_{2g,\lambda} \end{pmatrix},$$

we set

$$(5.1) \quad \hat{\omega}_{2,\lambda} = -U\eta \bar{\omega}_{1,\lambda} + U \bar{\omega}_{2,\lambda}$$

$$(5.2) \quad \hat{\omega}_{1,\lambda} = U^* \omega_{1,\lambda^p} - T(\lambda) \bar{\omega}_{2,\lambda}$$

$$(5.3) \quad \tilde{z}_1 = U^* \bar{z}_1 - T(\lambda) \tilde{z}_2$$

$$(5.4) \quad \tilde{z}_2 = -U\eta \bar{z}_1 + U \bar{z}_2$$

it being understood that

$$\bar{z}_1 = \begin{pmatrix} z_1 \\ \vdots \\ z_g \end{pmatrix}, \quad \bar{z}_2 = \begin{pmatrix} z_{g+1} \\ \vdots \\ z_{2g} \end{pmatrix}.$$

We conclude that

$$(6.1) \quad \hat{\omega}_{1,\lambda} = p^{-1} (\hat{\omega}_{1,\lambda^p})^\phi + d \tilde{z}_1$$

$$(6.2) \quad \hat{\omega}_{2,\lambda} = (\hat{\omega}_{2,\lambda^p}^\sigma)^\phi + d \tilde{z}_2.$$

It follows from (5.4) that $\tilde{z}_2$ is bounded by 1.

Setting

$$I_{i,\lambda} = \int_{x_0}^x \hat{\omega}_{i,\lambda} \quad i = 1, 2$$

we obtain

$$(7.1) \quad I_{1,\lambda} = p^{-1} I_{1,\lambda^p}^\phi + \tilde{z}_1$$

$$(7.2) \quad I_{2,\lambda} = I_{2,\lambda^p}^{\sigma\psi} + \tilde{z}_2.$$

We deduce that $I_{2,\lambda}(x)$ is bounded by 1 on $D(x_0, 1^-)$.

We put

$$J_\lambda(x) = U(\lambda)^* \int_{x_0}^x \bar{\omega}_{1,\lambda},$$

a g -tuple of abelian integrals of the first kind we deduce from (6) that

$$(8) \quad J_\lambda(x) = p^{-1} J_{\lambda^p}^\sigma(x^p) + H(\lambda, x)$$

where

$$H(\lambda, x) = I_{2,\lambda^p}^{\sigma\psi}(T(\lambda) - p^{-1} T^\sigma(\lambda^p)) + U(\lambda)^* \bar{z}_1.$$

By the theory of normalized period matrices we deduce that $H(\lambda, x)$ is bounded by unity.

This completes our sketch of the generalization of Tate's theorem.

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