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DIFFERENTIALS OF THE SECOND KIND FOR FAMILIES OF MUMFORD CURVES

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The space of everywhere meromorphic differentials on a Mumford curve M of genus g which can be integrated on the universal covering of M is a space of codimension g in the full space of meromorphic differentials on M . This fact allows to conclude that the Gauss-Manin connection associated to an analytic family of Schottky groups has g linearly independent horizontal elements which are defined everywhere on the parameter space of the family. I will give a sketch of the proof for this result.

1. ξ -functions and differentials of the second kind.

Let K be an algebraically closed field together with a complete non-archimedean valuation. Let Γ be a Schottky subgroup of the group $\mathrm{PGL}_2(K)$ of fractional linear transformations of the Riemann surface $\underline{P} = K \cup \{\infty\}$ over K . Let Z be the domain of ordinary points of Γ , see [GP], Chap. I, § 4.

THEOREM 1. - Let $h(z)$ be a rational function on $\underline{P}$, whose poles all lie in Z and let $z_0 \in Z$ be an ordinary point for Γ . Then the series

$$\xi(h; z_0; z) := \sum_{\substack{\gamma \in \Gamma \\ h(\gamma(z_0)) \neq \infty}} (h(\gamma(z)) - h(\gamma(z_0))) + \sum_{\substack{\gamma \in \Gamma \\ h(\gamma(z_0)) = \infty}} h(\gamma(z))$$

is as a function of z uniformly convergent on any affinoid subdomain of Z . Its limit is a meromorphic function on Z .

A proof of this result appears in [G], (1).

Let now I be the K -vectorspace of those meromorphic functions $f(z)$ on Z for which

$$f(\gamma z) - f(z) \in K$$

for all $\gamma \in \Gamma$.

The differential df of a function from I is Γ -invariant and is thus a differential of the Mumford curve $M = Z/\Gamma$.

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Denote by H the K -vectorspace of rational functions on $\underline{P}$ whose poles all lie in Z . One can show that any $f \in I$ is obtained as $\xi(h, z_0; z)$ with $h \in H$, see [G], (2).

Let $\text{Hom}(\Gamma, K)$ be the K -vectorspace of group homomorphisms $c: \Gamma \rightarrow K$. If we fix a basis $\alpha_1, \dots, \alpha_g$ of the free group Γ , we obtain a canonical isomorphism $\text{Hom}(\Gamma, K) \xrightarrow{\sim} K^g$ when we map c onto the g -tuple $(c(\alpha_1), \dots, c(\alpha_g))$.

For any $f \in I$ we denote by $P(f)$ the group homomorphism $\Gamma \rightarrow K$ given by

$$P(f)(\gamma) = f(\gamma z) - f(z).$$

Then $P(f)(\gamma)$ is the period of the differential df with respect to the "cycle" γ .

The mapping

$$P: I \rightarrow \text{Hom}(\Gamma, K)$$

is K -linear whose kernel consists of the field of Γ -invariant meromorphic functions on Z which is the field of rational functions on the curve M . One can prove that the mapping $P: I \rightarrow \text{Hom}(\Gamma, K)$ is surjective, see [G], (3).

THEOREM 2. - Let $\alpha_1, \dots, \alpha_g$ be a basis of Γ . Then there exist functions $f_1, \dots, f_g \in I$ such that $P(f_i)(\alpha_j) = \delta_{ij} = \begin{cases} 1 & : i = j \\ 0 & : i \neq j \end{cases}$.

A meromorphic differential $\omega = fdz$ on Z is called to be of the second kind if for any point $a \in Z$ there is a meromorphic function $h_a(z)$ on Z such that $\omega - dh_a$ is analytic in a .

Denote by Ω_2 the K -vectorspace of Γ -invariant differentials on Z of the second kind. The proof of the following theorem is given in [G], (4).

THEOREM 3. - $\Omega_2 = \Omega_1 \oplus dI$ where Ω_1 is the g -dimensional K -vectorspace of analytic differentials on M .

2. Families of Schottky groups.

Let S be a rigid analytic space over K , see [BGR], Chap. 9. We consider the projective line over S , namely the product space $\underline{P} \times S$ together with the projection π onto the second factor.

Denote by $\text{Aut}_S(\underline{P} \times S)$ the group of those bianalytic mapping $\gamma: \underline{P} \times S \rightarrow \underline{P} \times S$ which are compatible with π (i. e. $\gamma \circ \pi = \pi$).

One can prove that there is an admissible covering $\mathcal{S} = (S_i)_{i \in I}$ of S such $\gamma|_{S_i}$ is a fractional-linear transformation over S_i which means that there is a matrix

$$\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \text{GL}_2(\mathcal{O}(S_i)),$$

where $\mathcal{O}(S_i)$ is the K -algebra of analytic functions on S_i such that

$$(\gamma|_{S_i})(s, z) = \frac{a_i(s) \times z + b_i(s)}{c_i(s) \times z + d_i(s)}.$$

For any point $s \in S$ we obtain a canonical homomorphism $\text{Aut}_S(\underline{P} \times S) \rightarrow \text{PGL}_2(K)$ by restricting $\gamma \in \text{Aut}_S(\underline{P} \times S)$ to the subspace $\underline{P} \times \{s\}$ of $\underline{P} \times S$. We denote the restriction of γ to $\underline{P}\{s\}$ by ϕ_s .

Definition. - A subgroup $\Gamma \subset \text{Aut}_S(\underline{P} \times S)$ is called a Schottky group over S (or a family of Schottky groups parametrized by S) if for any point $s \in S$ the restriction of the canonical homomorphism $\text{Aut}_S(\underline{P} \times S) \rightarrow \text{PGL}_2(K)$ to Γ gives an isomorphism from Γ to a Schottky group Γ_s of $\text{PGL}_2(K)$.

Let now Γ be a Schottky group over S . The proof of the following result will be given elsewhere.

THEOREM 4. - There exists an admissible subdomain Z of $\underline{P} \times S$ such that for any $s \in S$ the intersection $Z \cap (\underline{P} \times \{s\})$ is the domain of ordinary points for the Schottky groups Γ_s . If S is an affinoid space there is an affinoid subdomain $F \subset Z$ such that

$$\bigcup_{\gamma \in \Gamma} \gamma(F) = Z$$

$$\gamma(F) \cap F = \text{empty for almost all } \gamma \in \Gamma.$$

If S is irreducible, then so is the domain Z .

COROLLARY. - $Z/\Gamma \rightarrow S$ is an analytic family of Mumford curves.

From now on let S be irreducible and H be the $\mathcal{O}(S)$ -algebra of meromorphic functions on $\underline{P} \times S$ whose poles and points of indeterminacy all lie in Z .

Let $z_0 : S \rightarrow Z$ be an analytic mapping such that $\pi \circ z_0 = \text{id}_S$ and $h \in H$. Let h_s be the restriction of h onto $\underline{P} \times \{s\}$. Then there is a meromorphic function $\xi(h; z_0; s, z)$ on Z such that the restriction of $\xi(h; z_0; s, z)$ onto $\underline{P} \times \{s\}$ equals $\xi(h_s; z_0(s); z)$. Let I_S be $\mathcal{O}(S)$ -module of meromorphic functions $f(s, z)$ on Z for which $f \circ \gamma - f \in \mathcal{O}(S)$ for all $\gamma \in \Gamma$. Let $\text{Hom}(\Gamma, (\mathcal{O}(S)))$ be the free $\mathcal{O}(S)$ -module of rank g of all group homomorphisms $c : \Gamma \rightarrow \mathcal{O}(S)$.

Let $P(f)(\gamma) := f \circ \gamma - f$. Then $P(f) \in \text{Hom}(\Gamma, \mathcal{O}(S))$.

THEOREM 5. - Let $\alpha_1, \dots, \alpha_g$ be a basis of Γ . There is an admissible covering $(S_i)_{i \in I}$ of S and for any i there are functions $f_1, \dots, f_g \in I_{S_i}$ such that

$$P(f_j)(\alpha_1) = \delta_{j1}.$$

Let $\mathcal{O}_2 = \mathcal{O}_{2M}/S$ denote the sheaf on S whose set of sections on an admissible

open domain $U \subseteq S$ are the K -vectorspace of Γ -invariant differentials relative to $Z \rightarrow S$, of the second kind on $Z_U = Z \cap (\underline{P} \times U)$.

Let Ω_{ex} be the subsheaf of Ω_2 of exact differentials and H_{DR}^1 be the quotient sheaf $\Omega_2 / \Omega_{\text{ex}}$.

THEOREM 6. - H_{DR}^1 is a free coherent module over the structure sheaf $\mathcal{O}_S$ on S of rank $2g$. There is a canonical decomposition

$$H_{\text{DR}}^1 = \bar{d}\Gamma \oplus \Omega_1$$

where Ω_1 is the subsheaf of Ω_2 of analytic differentials and $\bar{d}\Gamma$ is the sheaf of cohomology classes of differentials of the form df with $f \in \Gamma$. $\bar{d}\Gamma$ and Ω_1 are free modules of rank g over $\mathcal{O}_S$.

Sketch of proof : In order to prove that Ω_1 is free of rank g , we have to observe that for any $\gamma \in \Gamma$ there is a canonical differential $\omega_\alpha = (du_\alpha / u_\alpha) \in \Omega$, where u_α is defined on Z as in [GP], Chap. 2. While the u_α are unique up to a unit from $\mathcal{O}_S$, the differential ω_α is unique. If $\alpha_1, \dots, \alpha_g$ is a basis of Γ , then $\omega_{\alpha_1}, \dots, \omega_{\alpha_g}$ is a basis for Ω_1 .

The result concerning $\bar{d}\Gamma$ follows from Theorem 4. While the function $f_j^{(i)}$ depends on the index i , we find that $df_j^{(i)} - df_j^{(1)}$ are in the intersection $S_i \cap S_1$, the differential of a Γ -invariant function and thus the cohomology class of $df_j^{(i)}$ equals the cohomology class of $df_j^{(1)}$. Thus they constitute a basis element of $\bar{d}\Gamma$ [G], (Satz 6), we conclude the proof.

3. Gauss-Manin-Connection.

Let ∇ be the Gauss-Manin connection for the analytic family $M = Z/\Gamma \rightarrow S$ of Mumford curves, see [K0], [K], [D]. Thus for any vector field D on S there is an extension ∇_D on the module sheaf $H_{\text{DR}}^1(M/S)$.

THEOREM 7. - The restriction $\nabla|_{\bar{d}\Gamma}$ of ∇ onto $\bar{d}\Gamma$ is trivial, i. e. there is a basis of horizontal elements in $\bar{d}\Gamma$.

Sketch of proof : The result is local in nature. If $\mathcal{C} = (S_i)$ is an admissible covering of S and if we have proved the result for the family over S_i for all i , the proof is complete.

Using Theorem 5 we may therefore assume that there are functions $f_1, \dots, f_g \in \Gamma$ such that $P(f_i)(\alpha_j) = \delta_{ij}$, where $\alpha_1, \dots, \alpha_g$ is a basis of Γ . We have to show that $\nabla_D(\bar{df}_i) = 0$ where $\bar{df}_i$ is the cohomology class of df_i in H_{DR}^1 . Now by the very definition of ∇_D we know that $\nabla_D(\bar{df}_i) = d(\hat{D}f_i)$ where $\hat{D}$ is an extension of the derivation D to the field of meromorphic functions on M with $\hat{D}(x) = 0$ for a meromorphic function x on M which is not a meromorphic function on S . ($=$ is not constant on all the curves of the family $M \rightarrow S$).

We are done if we can show that $\hat{D}f_i$ is Γ -invariant. This seems obvious as $(\hat{D}f_i) \circ \alpha_j = \hat{D}(f_i \circ \alpha_j) = \hat{D}(f_i + \delta_{ij}) = \hat{D}(f_i)$.

The problem with this argument is that $\hat{D}$ is defined only on the field of meromorphic functions of M and f_i is not in it. But one can define a unique extension of $\hat{D}$ to a vector field on Z which does justify the above line of argument as soon as we have shown

$$(\hat{D}f_i) \circ \alpha = \hat{D}(f_i \circ \alpha).$$

But $D'(f) := (\hat{D}(f \circ \alpha)) \circ \alpha^{-1} - \hat{D}(f)$ is an analytic vector field on Z with $D'(f) \equiv 0$ for all meromorphic functions on M . Thus $D' \equiv 0$ and

$$(\hat{D}f_i) \circ \alpha = \hat{D}(f_i \circ \alpha).$$

4. Elliptic case.

The first nontrivial example is the family of Tate curves which has been studied by a number of authors, see for example [R], [Rb], [K], [DR].

Assume that $\text{char } K \neq 2$.

$$S = \{q \in K : 0 < |q| < 1\}$$

$$Z = \{(q, z) \in K^2 : q \in S, z \in K - \{0\}\}$$

$\alpha(q, z) := (q, qz)$ is a bianalytic map $Z \rightarrow Z$. Let Γ be the transformation group generated by α . Then $M = Z/\Gamma \rightarrow S$ is the universal family of Tate curves.

The de Rham cohomology space H_{DR}^1 for the family $M \rightarrow S$ is freely generated over the structure sheaf on S by the class τ_1 of the analytic differential (dz/z) and by the class τ_2 of the meromorphic differential $d\xi$ where

$$\begin{aligned} \xi(q, z) &= \frac{1}{1-z} + \sum_{n=1}^{\infty} \left(\frac{1}{1-q^n z} - \frac{1}{1-q^n z^{-1}} \right) \\ &= \frac{1}{1-z} + \sum_{n=1}^{\infty} \left(\frac{q^n z}{1-q^n z} - \frac{q^n z^{-1}}{1-q^n z^{-1}} \right) \end{aligned}$$

for which holds

$$\begin{aligned} \xi(q, qz) - \xi(q, z) &= 1 \\ \xi(q, z^{-1}) &= 1 - \xi(q, z) \\ \xi(q, -1) &= \frac{1}{2} \text{ if } \text{char } K \neq 2 \\ \xi(q, \pi) &= 1 \text{ if } \pi^2 = q. \end{aligned}$$

Denote by $(\partial/\partial q)$ (resp. $(\partial/\partial z)$) the canonical partial derivatives with respect to the first (resp. second) variable of $Z = S \times K^*$.

$$\Phi = z \frac{\partial \xi}{\partial z}$$

$$\Phi' = z \frac{\partial \Phi}{\partial z} .$$

Then Φ , Φ' are Γ -invariant meromorphic functions on Z and the following equation holds

$$\Phi'^2 = 4(\Phi - e_1)(\Phi - e_2)(\Phi - e_3)$$

where $e_1 = \Phi(q, -1)$, $e_2 = \Phi(q, \pi)$, $e_3 = \Phi(q, -\pi)$ with π a fixed square root of q .

If we put

$$x := \frac{\Phi - e_1}{e_2 - e_1}$$

$$y := \frac{\Phi'}{2(e_2 - e_1)^{3/2}}$$

then

$$y^2 = x(x-1)(x-\lambda)$$

with

$$\lambda = \frac{e_3 - e_1}{e_2 - e_1} = x(q, -\pi)$$

which is the Legendre normal form for the family of Tate curves.

Let

$$D_q := \frac{\partial}{\partial q} - \frac{\dot{x}}{x'} \frac{\partial}{\partial z}$$

where

$$\dot{x} = \frac{\partial x}{\partial q}, \quad x' = \frac{\partial x}{\partial z} .$$

We claim that the vector field D_q coincides with the vector field $\hat{D}$ for $D = (\partial/\partial q)$ in the proof of Theorem 7.

$$D_q(x) = 0 = \hat{D}(x)$$

$$D_q(f) = \frac{\partial f}{\partial q} = \hat{D}(f) \text{ if } f \text{ is analytic on } S .$$

Thus $\hat{D} = D_q$.

Let $\nabla = (\nabla_{\partial/\partial q})$. Then $\nabla(fdx) = D_q(f) dx$ by definition of $(\nabla_{\partial/\partial q})$.

One can by direct computation show that

$$\nabla(df) = d(D_q(f))$$

and that $D_q(\xi) = (\partial \xi / \partial q) - (\dot{x}/x') (\partial \xi / \partial z)$ is Γ -invariant.

This proves that $\nabla(\tau_2) = 0$ which gives a more direct proof of Theorem 7 for the family of Tate curves.

Let σ_1 (resp. σ_2) be the cohomology class of $(dx/2y)$ (resp. $x(dx/2y)$).

Then σ_1, σ_2 is a basis of H_{DR}^1 . Let

$$\overline{d\xi} = \tau_2 = A \cdot \sigma_1 + B \sigma_2.$$

THEOREM 8.

$$A = \frac{\bar{\Phi}(q, -1)}{\sqrt{\bar{\Phi}(q, \pi) - \bar{\Phi}(q, -1)}}$$

$$B = \sqrt{\bar{\Phi}(q, \pi) - \bar{\Phi}(q, -1)}$$

and $\frac{A}{B}$ as a function of λ can be given by

$$\frac{A}{B} = 2\lambda \left[(1 - \lambda) \frac{F'}{F} + \frac{1}{2} \right]$$

where

$$\begin{aligned} F(\lambda) &= {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \lambda\right) \\ &= \sum_{n=0}^{\infty} \left(-\frac{(1/2)_n}{n!}\right)^2 (1 - \lambda)^n. \end{aligned}$$

Sketch of proof: The proof of the first part is given by a small computation. One can use the characterization of elements τ in H_{DR}^1 with $\nabla(\tau) = 0$ given in [P], (7.11), (ii), to prove the second part.

We find that $\tau_2 = \lambda(1 - \lambda) \frac{\partial f}{\partial \lambda} \cdot \sigma_1 - \lambda(1 - \lambda) f \nabla(\sigma_1)$ where f satisfies the hypergeometric equation

$$\lambda(1 - \lambda) \frac{\partial^2 f}{\partial \lambda^2} + (1 - 2\lambda) \frac{\partial f}{\partial \lambda} - \frac{1}{4} f = 0.$$

Here one has to use the fact that the map $\pi \rightarrow \lambda(\pi) = x(q, -\pi)$ gives a bianalytic map from S onto $\{\lambda : |1 - \lambda| < |2|\}$.

Thus the inverse mapping $\pi(\lambda)$ is an analytic function of λ .

Now we conclude that $f = c \cdot F(\lambda)$ as f is analytic on $\{\lambda : |1 - \lambda| < |2|\}$ with a constant $c \in K^*$ which can be determined by letting $\lambda \rightarrow 1$ (i. e. $\pi \rightarrow 0$).

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