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SIEGFRIED BOSCH

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STABLE REDUCTION AND RIGID ANALYTIC UNIFORMIZATION OF ABELIAN VARIETIES

by Siegfried BOSCH (*)
[Univ. Münster]

This is a report on joint work with W. Lütkebohmert [BL], where we have given a purely analytic approach to the uniformization of curves and abelian varieties.

In classical uniformization theory over the field of complex numbers, one shows that each non-singular projective curve can be analytically represented as a quotient Ω/Γ with an open subset $\Omega \subset \mathbb{P}^1$ and a subgroup $\Gamma \subset \text{PGL}(2, \mathbb{C})$ acting discontinuously on Ω . Similarly, each abelian variety can be represented as an analytic torus which, from the multiplicative point of view, is defined as a quotient $(\mathbb{C}^*)^g/\Gamma$ by a discrete subgroup $\Gamma \subset (\mathbb{C}^*)^g$, free of rank g . Over a non-archimedean ground field k , the situation is quite different. The curves and abelian varieties which admit a good uniformization as above constitute only a small open part in the corresponding moduli spaces (see [M1] and [M2]). On the other hand, there are curves and abelian varieties admitting no non-trivial uniformization at all. This behavior is related to the phenomenon of good reduction, which has no classical counterpart. In general, good uniformization and good reduction occur in a mixed way. It is for this reason that uniformization theory is substantially more complicated in the non-archimedean case.

1. Uniformization via algebraic geometry.

The basic tools in uniformization theory over discretely valued fields are the stable reduction theorem of Deligne-Mumford [DM] and the semi-abelian reduction theorem of Grothendieck [SGA7]. In order to explain these results, consider a discrete valuation ring R , and let k be its field of fractions.

Let A be an abelian variety over k , and denote by \mathcal{A} its Néron model ($[N]$, [R1]). Then the theorem of Grothendieck says that (modulo finite separable extension of the ground field) the identity component of the special fibre \mathcal{A}_s of \mathcal{A} is semi-abelian; i. e. \mathcal{A}_s^0 is an extension of an abelian variety by a multiplicative group. Likewise, if C is a smooth geometrically connected projective curve of genus ≥ 2 over k , one can consider its minimal model \mathcal{C} over R . (See [Ab] or [Li] for the existence of a regular model and [Sh] for the minimality.) The result of Deligne-Mumford asserts (again, modulo finite extension of the ground field) that the special fibre \mathcal{C}_s of \mathcal{C} has only ordinary double points as

(*) Siegfried BOSCH, Mathematisches Institut der Universität, 62 Einsteinstrasse, D-4400 MÜNSTER (Allemagne fédérale).

singularities. Both results are more or less equivalent. In fact, the proof of the stable reduction theorem in [DM] uses the semi-abelian reduction of the Jacobian J of C and the fact that the "identity component" of the Néron model of J represents the functor $\text{Pic}^0(C/\text{Spec } R)$ (see [R3]). Apart from uniformization theory, there are far-reaching applications of both theorems in the theory of moduli and also in number theory.

Assuming that k is complete, the universal covering \hat{C} of C can be easily constructed as follows. Consider \bar{C} , the formal completion of C . Then \bar{C} is a formal scheme over R or, from the rigid analytic viewpoint, a formal analytic variety over k . In fact, \bar{C} may be interpreted as the analytification of C (again denoted by C), and the formal structure of \bar{C} gives rise to a reduction map $\pi : C \rightarrow \bar{C} = C_{\mathfrak{s}}$. The formal fibre $\pi^{-1}(\tilde{x})$ over a closed (rational) point $\tilde{x} \in \bar{C}$ is an open disc if \tilde{x} is non-singular, and an open annulus if \tilde{x} is an ordinary double point. This leads to a geometric description of C . Namely, the π -inverse of the non-singular locus $\bar{C} - \text{Sing } \bar{C}$ is a disjoint union of components which are smooth over R and thus simply connected (in the sense of rigid analysis; see [BL], II, 8.12). The curve C is obtained by connecting these components by means of the annuli $\pi^{-1}(\tilde{x})$, $\tilde{x} \in \text{Sing } \bar{C}$. Likewise, the universal covering \hat{C} of C is constructed by resolving all loops which are generated by this process. If \bar{C} has only rational components (this is the case of Mumford's split degenerate reduction [M1]), C has a good uniformization as discussed above. Namely, \hat{C} can be viewed as an open analytic subvariety of \mathbb{P}^1 , the automorphisms of \hat{C} over C being fractional linear transformations. The uniformization of abelian varieties A is more complicated (see [R2]). Here, analogous to the case of curves, a fundamental role is played by the formal completion \bar{A} of the Néron model of A , or to be more precise, by the identity component \bar{A}^0 of \bar{A} .

2. Uniformization of curves via rigid analysis.

It is a surprising fact, first realized by VAN DER PUT [P], that the formal completion \bar{C} of the minimal model C of C can be constructed by a direct analytic method (without knowing C). Thus proceeding as in section 1, a purely analytic approach to the uniformization of curves is obtained by proving the following analytic version of the stable reduction theorem (for arbitrary non-archimedean ground fields, modulo finite separable field extension):

The curve C can be viewed as a formal analytic variety with associated reduction $\pi : C \rightarrow \bar{C}$ such that \bar{C} has at most ordinary double points as singularities.

The proof of this result in [BL] uses the key fact that for arbitrary reductions $\pi : C \rightarrow \bar{C}$, the periphery of a fibre $\pi^{-1}(\tilde{x})$ is a disjoint union of annuli. Namely, one starts with an arbitrary reduction $\pi : C \rightarrow \bar{C}$ and refines π inductively by using the technique of blowing up, in such a way that all bad singularities of

\tilde{C} disappear.

3. Uniformization of abelian varieties via rigid analysis.

As an application of the analytic stable reduction theorem, it is shown, in [BL], how to obtain the uniformization of abelian varieties (over arbitrary non-archimedean ground fields k , modulo finite separable field extension). Namely, consider the Jacobian J of a smooth geometrically connected projective curve C . The stable reduction theorem provides deep insight into the analytic structure of C . Thereby it is possible to construct line bundles with prescribed properties and to carry out explicit computations. One constructs an open analytic subgroup \tilde{J} of J , the group of normalized line bundles on C ; it has a canonical reduction \tilde{J} which is isomorphic to the Jacobian of \tilde{C} . Hence \tilde{J} is an extension of an abelian variety by a multiplicative group. If the valuation of the ground field k is discrete, \tilde{J} may be interpreted as the identity component of the formal completion of the Néron model of J .

In order to construct the universal covering of J , one looks at the analytic cohomology group $H^1(C, \underline{Z})$. It is free of rank $r \leq g$ ($=$ genus of C); in fact, the rank r reflects the number of loops in C as discussed in section 1. Using Picard functors, one interprets $H^1(C, \underline{Z})$ as the \underline{Z} -module of analytic group homomorphisms $\underline{G}_m \rightarrow J$ or $\tilde{G}_m \rightarrow \tilde{J}$, where \underline{G}_m denotes the multiplicative group over k and where \tilde{G}_m is its subgroup of "units". Thus there is a closed subgroup $\tilde{G}_m^r \subset \tilde{J}$ which reduces to the multiplicative part of \tilde{J} and which may be extended to an analytic homomorphism $\tilde{G}_m^r \rightarrow J$. Then $\hat{J} := \tilde{G}_m^r \times \tilde{J} / (\text{diagonal})$ is the universal covering of J . The projection map $\hat{J} \rightarrow J$ has a discrete kernel Γ which is free of rank r , so that $J = \hat{J} / \Gamma$. In particular, the following assertions are equivalent :

- (i) C is a Mumford curve,
- (ii) $\text{rank } H^1(C, \underline{Z}) = g,$
- (iii) J is an analytic torus.

Furthermore, J has good reduction, if C has good reduction.

Since, up to isogeny, any abelian variety A can be embedded into a product of Jacobian varieties, the above uniformization of Jacobians implies the uniformization of A . Namely, one constructs the analogues \bar{A} and \hat{A} of the groups \tilde{J} and \hat{J} , and shows $A = \hat{A} / \Gamma$. In this case, the rank of Γ has to be interpreted as the rank of the cohomology group $H^1(A', \underline{Z})$, where A' is the dual abelian variety of A .

4. Further applications of the analytic method.

Using simple algebraization techniques, the methods of [BL] seem to yield new

proofs for the facts from algebraic geometry mentioned in section 1, namely for the following results :

- Existence of minimal models for curves,
- Existence of Néron models for abelian varieties,
- Stable reduction theorem of Deligne-Mumford,
- Semi-abelian reduction theorem of Grothendieck.

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