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ON THE STRATIFICATION OF THE MODULI SPACE OF MUMFORD CURVES

by Frank HERRLICH (*)

The space $\mathcal{M}_g$ of Mumford curves of genus g is an open analytic subspace of the algebraic quasiprojective variety $\mathcal{M}_g$ of smooth curves of genus g . POPP [5] introduced a stratification of $\mathcal{M}_g$ which was generalized to arbitrary characteristic by LONSTED [3]. In this note, we develop a generalized p -adic Teichmüller theory to study the intersection of strata of $\mathcal{M}_g$ with $\mathcal{M}_g$. The proofs are rather sketchy, more details will be given in a paper which is in preparation.

1. - We begin with a brief review of the moduli theory of Mumford curves : Let k be an algebraically closed complete nonarchimedean field. For $i = 1, 2$, let $\Gamma_i \subset \mathrm{PGL}_2(k)$ be a Schottky group, and Ω_i the set of ordinary points of Γ_i .

Then $C_i := \Omega_i / \Gamma_i$ is analytically isomorphic with a non-singular projective curve of genus $g_i = \mathrm{rank} \Gamma_i$, and it is well known that the Mumford curves C_1 and C_2 are isomorphic if, and only if, the Schottky groups Γ_1 and Γ_2 are conjugate subgroups of $\mathrm{PGL}_2(k)$.

Therefore the main goal of the moduli theory, the classification of isomorphism classes of Mumford curves, reduces to the classification of conjugacy classes of Schottky groups. This is performed by p -adic Teichmüller theory (cf. [1]).

Let $g \geq 1$ be an integer, F_g an (abstract) free (nonabelian) group of rank g . Let

$$S_g := \mathrm{Hom}(F_g, \mathrm{PGL}_2(k))$$

and

$$T_g := \{\tau \in S_g ; \tau \text{ injective and } \tau(F_g) \text{ discontinuous}\}.$$

S_g is an affine algebraic k -variety of dimension $3g$ (in fact, isomorphic with $(\mathrm{PGL}_2(k))^n$), and T_g is an analytic subset of S_g , given by the infinitely many inequalities

$$\tau \in T_g \iff |\mathrm{tr} \tau(\omega)| > 1 \text{ for every } \omega \in F_g$$

where $\mathrm{tr} \gamma := (\mathrm{trace} \tilde{\gamma})^2 / \det \tilde{\gamma}$ for any representative $\tilde{\gamma} \in \mathrm{GL}_2(k)$ for the fractional linear transformation $\gamma \in \mathrm{PGL}_2(k)$.

On S_g we have a natural action of $\mathrm{Aut} F_g$ on the right, and of

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$$\text{Aut}(\text{PGL}_2(k)) = \text{PGL}_2(k)$$

on the left. Γ_g is invariant for both groups. $\text{PGL}_2(k)$ is a reductive (even semi-simple) algebraic group (of dimension 3), so the quotient $\text{PGL}_2(k) \backslash S_g$ is again algebraic and of dimension $3g - 3$. $\text{Aut } F_g$ is a discrete group. Furthermore the two group actions commute, so $\text{Aut } F_g$ acts also on $\text{PGL}_2(k) \backslash S_g$, and this action is discontinuous on

$$\bar{T}_g := \text{PGL}_2(k) \backslash T_g$$

($\bar{T}_g$ is called the p -adic Teichmüller space). The inner automorphisms of F_g act trivially on $\bar{T}_g$, so we have an action of $\text{Aut}' F_g := \text{Aut } F_g / (\text{inner automorphisms})$ on $\bar{T}_g$.

In this way, we impose an analytic structure on

$$\mathcal{M}_g := \bar{T}_g / \text{Aut}' F_g = \text{PGL}_2(k) \backslash T_g / \text{Aut}' F_g.$$

By what was said above the points of $\mathcal{M}_g$ correspond to the isomorphy classes of Mumford curves of genus g .

$\mathcal{M}_g$ is a subset of $\mathcal{M}_g^*$, the moduli space of all nonsingular curves of genus g over k (as constructed by MUMFORD), so $\mathcal{M}_g$ inherits an analytic structure as subspace of $\mathcal{M}_g^*$. LÜTKEBOHMERT [4] has shown that this analytic structure is the same as the one defined above.

2. -- The stratification of the algebraic moduli variety $\mathcal{M}_g$ has been carried out by POPP [5] (in characteristic 0) and LØNSTED [3]. The results are as follows:

By means of the Torelli map one identifies $\mathcal{M}_g$ with the subspace $J_g \subset \mathcal{A}_g$, i. e. the subspace of the moduli variety of polarized abelian varieties of dimension g which corresponds to Jacobians of nonsingular curves of genus g . Then one considers the space $J_{g,n}$ of Jacobians with level- n -structure, $n \geq 3$. It is known that $J_{g,n}$ is nonsingular, and that J_g is the quotient of $J_{g,n}$ by a finite group which is isomorphic to $\text{GL}_{2g}(\mathbb{Z}/n\mathbb{Z})$. The stabilizers of this action are the automorphism groups of the corresponding curves ($g \geq 4$; exception for $g = 3$ and hyperelliptic curves, and for $g = 2$), and the strata of $J_g \cong \mathcal{M}_g$ are the (images of the) fixed point sets of the different subgroups of $\text{GL}_{2g}(\mathbb{Z}/n\mathbb{Z})$. Now, in the p -adic case we have the period map $q: \bar{T}_g \rightarrow H_g$ from the Teichmüller space into the Siegel upper half plane, and the quotient $\mathcal{A}_g$ of H_g by the group $\text{GL}_g(\mathbb{Z})$ is the analytic space of polarized abelian varieties which are k -analytic tori. Furthermore the congruence subgroup $\Gamma_{g,n} = \{\gamma \in \text{GL}_g(\mathbb{Z}) ; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n}\}$ acts without fixed points on H_g , and $H_g / \Gamma_{g,n}$ can be embedded into $J_{g,n}$ in such a way that the two group actions coincide. This shows that the (images of the) fixed point sets in $\bar{T}_g$ of subgroups of the Teichmüller modular group $\text{Aut}' F_g$ are the intersections of $\mathcal{M}_g$ with the strata of $\mathcal{M}_g$. Instead of subgroups of $\text{Aut}' F_g$ we

can of course consider subgroups of $\text{Aut } F_g$ which contain the group of inner automorphisms. The fixed point sets of conjugated subgroups of $\text{Aut } F_g$ have the same image in the quotient space $\mathcal{M}_g$, and subgroups of $\text{Aut } F_g$ containing F_g are isomorphic if, and only if, they are conjugate. Furthermore the stabilizer subgroup of $\text{Aut } F_g$ of a point in $\bar{T}_g$ is isomorphic with the normalizer of the corresponding Schottky group in $\text{PGL}_2(k)$. This shows :

PROPOSITION 1. - Let $C = \Omega/\Gamma$, $C' = \Omega'/\Gamma'$ be Mumford curves of genus $g \geq 4$ with Schottky groups Γ, Γ' . Let N, N' be the normalizers of Γ and Γ' in $\text{PGL}_2(k)$. Then C and C' are in the same stratum of $\mathcal{M}_g$ if, and only if, there is an isomorphism $\alpha : N \rightarrow N'$ such that $\alpha(\Gamma) = \Gamma'$ (we call $N, N'(\Gamma, \Gamma')$ -isomorphic in that case).

This proposition makes it possible to study the intersections $S \cap \mathcal{M}_g$, S stratum of $\mathcal{M}_g$, by a Teichmüller theory which is very similar to the one explained in section 1.

3. - For any finitely generated group N , let

$$S(N) := \text{Hom}(N, \text{PGL}_2(k))$$

and

$$T(N) := \{\tau \in S(N) ; \tau \text{ injective and } \tau(N) \text{ discontinuous}\}.$$

If N is generated by n elements, $S(N)$ is an algebraic subspace of the affine variety $(\text{PGL}_2(k))^n$ whose defining equations come from the relations among the n generators.

$T(N)$ is given by the inequalities

$$|\text{tr } \tau(x)| > 1 \text{ for every } x \in N \text{ of infinite order.}$$

So $T(N)$ is an analytic subspace of $S(N)$ (which may, of course, be empty).

If $T(N) \neq \emptyset$, we know by the structure theory of finitely generated p -adic discontinuous groups [2] that N contains a free normal subgroup F of finite index and finite rank g . For any such pair (N, F) , where N is finitely generated and F is a normal free subgroup of finite index, we define

$$\text{Aut}_F(N) := \{\alpha \in \text{Aut } N ; \alpha(F) = F\}.$$

Now we may form the quotient

$$\mathcal{M}(N, F) := \text{PGL}_2(k) \backslash T(N) / \text{Aut}_F(N).$$

This is the quotient of an analytic space by a discrete group and can therefore be given an analytic structure (we shall exhibit a fundamental domain in section 5).

PROPOSITION 2. - Let $F < N$ finitely generated groups, F free of rank $g \geq 0$, such that $(N : F) < \infty$.

(i) $\mathcal{M}(N, F)$ is the space of Mumford curves $C = \mathcal{O}/\Gamma$ of genus g such that the normalizer $N(\Gamma)$ contains a subgroup which is (Γ, F) -isomorphic with N .

(ii) If $g \geq 4$, there is a stratum S of $\mathcal{M}_g$ such that

$$S \cap \mathcal{M}_g = \mathcal{M}(N, F) \setminus \{C \in \mathcal{M}_g ; \text{Aut } C \not\cong N/F\}.$$

(iii) If $g \geq 4$, for every stratum S of $\mathcal{M}_g$ there exists a unique pair (N, F) as above such that $S \cap \mathcal{M}_g$ is as in (ii).

4. - In order to calculate the dimension of $\mathcal{M}(N, F)$ (and therefore of $S \cap \mathcal{M}_g$ which by Proposition 2 (ii) is an open subset of $\mathcal{M}(N, F)$), we need some information on the structure of the possible groups N . If $\mathcal{M}(N, F) \neq \emptyset$, then there is a discontinuous subgroup $\tilde{N} \subset \text{PGL}_2(k)$ which is isomorphic with N . $\tilde{N}$ acts on a locally finite tree $B(\tilde{N})$ such that the quotient graph $Q(\tilde{N}) = B(\tilde{N})/\tilde{N}$ is a finite graph (see [2] e. g.) which becomes a graph of groups by assigning the stabilizer group of a preimage to every vertex and edge of $Q(\tilde{N})$ (with the obvious inclusion maps). Then from the Bass-Serre structure theorem for groups acting on trees, we find the following presentation for $\tilde{N}$ (and therefore for N):
 N is generated by

(i) the vertex groups of $Q(\tilde{N})$,

(ii) one free generator γ_l for every (unoriented edge l of $Q(\tilde{N})$ not contained in a fixed maximal subtree.

The defining relations are amalgamation of the edge groups in the corresponding vertex groups (resp. identification after conjugation by γ_l , if the edge l is not in the maximal subtree). Of course, the presentation of N as the fundamental group of a graph of groups is not unique; but the following lemma, from combinatorial group theory (whose proof will be given elsewhere), shows that the relevant vertex and edge groups are unique.

LEMMA 1. - Let Q, Q' be reduced graphs of groups with isomorphic fundamental group. Then there is a finite sequence $Q = Q_0, \dots, Q_n = Q'$ of reduced graphs of groups such that for $i = 1, \dots, n$ either Q_i is derived from Q_{i-1} by an admissible contraction of an edge, or conversely Q_{i-1} from Q_i .

(Here we call a graph of groups reduced if for every vertex v either there are at least 3 edges starting at v or there is an edge starting at v with edge group strictly contained in the vertex group of v ; a contraction of an edge l is called admissible if l is not a loop and if the edge group of l is isomorphic with the vertex group of one of the end points of l .)

In characteristic 0, the possible vertex groups are the cyclic and dihedral groups and the exceptional groups A_4, S_4 and A_5 (in characteristic p there are also subgroups of $\text{PGL}_2(\mathbb{F}_{p^n})$ possible). A nontrivial cyclic subgroup of

$\text{PGL}_2(k)$ is determined by its two fixed points, all other finite subgroups by three parameters (they are generated by two elements of finite order with one relation between them). Finally, a free generator of N is a hyperbolic element of $\text{PGL}_2(k)$ and therefore determined by 3 parameters (fixed points and multiplier).

With the notations

$$\begin{aligned} c_v &= c_v(N) = \text{number of nontrivial cyclic vertex groups of } N \\ c_e &= c_e(N) = \text{" " edge " "} \\ d_v(d_e) &= \text{" " noncyclic vertex (edge) groups of } N \\ f &= f(N) = \text{" " free generators of } N \\ &\quad (= \text{cyclomatic number of } Q(N)) \end{aligned}$$

we have the following :

THEOREM 1. - Let $F < N$ as in Proposition 2. If $T(N) \neq \emptyset$, then

$$\dim \mathcal{M}(N, F) = 3f + 3d_v + 2c_v - 3d_e - 2c_e - 3.$$

(- 3 is the dimension of $\text{PGL}_2(k)$).

5. - In this section, we shall sketch the proof of :

THEOREM 2. - Let $F < N$ as in Proposition 2, $T(N) \neq \emptyset$. Then (N, F) is con-
nected.

From this theorem one easily deduces :

COROLLARY. - For any stratum S of $\mathcal{M}_g$, the intersection $S \cap \mathcal{M}_g$ is either
empty or connected.

To prove theorem 2, we have to embed $T(N)$ into some analytic space k^n , n sufficiently large ; i. e. we need coordinates on $T(N)$.

For this purpose, we fix a set A of generators of N which comes from the presentation of N as the fundamental group of a graph of groups as explained in the previous section. Now every $\tau \in T(N)$ is of course determined by the set $\tau(A)$, and every $\tau(a)$, $a \in A$, is given by its two fixed points in $\mathbb{P}^1(k)$ and its multiplier. If $\tau(a)$ is hyperbolic, one of the fixed points is attracting and the other repelling; but if a is of finite order d the fixed points cannot be distinguished, and the multiplier takes on only a finite number of values (primitive d -th roots of unity). It is convenient to begin with a space $T_0(N)$ where the coordinates are the fixed points and multipliers. $T(N)$ will then be the quotient of $T_0(N)$ by the action of the group $(\mathbb{Z}/2\mathbb{Z})^{r+2s}$, where $r+2s$ is the number of elements of finite order in A .

To be more precise, let

$$A = \{a_1, \dots, a_r; b_1, c_1, \dots, b_s, c_s; d_1, \dots, d_f\}$$

where $r = c_v$, $s = d_v$ in the notation of Theorem 1; i. e. the a_i generate cyclic vertex groups, b_i and c_i generate a noncyclic vertex group and the d_i are of infinite order. As coordinates, we have the fixed points x_i, y_i of a_i , x_{i+r}, y_{i+r} of b_i , z_i of c_i (the other fixed point of c_i can be calculated from the relation between c_i and b_i) and x_{i+r+s}, y_{i+r+s} of d_i as well as the multipliers t_i of d_i . We omit the multipliers of the elements of finite order: They give rise to a finite number of isomorphic connected components of our space $T_\emptyset(N)$, which are all identified by the action of $\text{Aut } N$.

Now we introduce a fundamental domain in $T_\emptyset(N)$ for the action of $\text{Aut } N$ (as $\text{Aut}_F(N)$ has finite index in $\text{Aut}(N)$ if F has finite index in N , a finite number of translates of this domain will be a fundamental domain for $\text{Aut}_F(N)$.)

We call $\tau \in T(N)$ geometric (or $\tau(A)$ a geometric set of generators) if $\{\tau(a); a \in A\}$ comes from the Bass-Serre presentation of $\tau(N)$ as the fundamental group of $B(\tau(N))/\tau(N)$. Obviously for every discontinuous subgroup of $\text{PGL}_2(k)$, isomorphic with N , there is a geometric set of generators, so that

$$B(N) := \{\tau \in T(N); \tau \text{ geometric}\}$$

is a fundamental domain in $T_\emptyset(N)$ for the action of $\text{Aut } N$.

If $N = F_g$, $B(N)$ is the set of Schottky bases. We shall show that $B(N)$ is the finite (disjoint) union of the spaces $B(Q)$ where Q runs through the set of labelled graphs of groups with fundamental group N . A labelling of such a graph of groups consists of a maximal subtree P of Q and a map v from the set B of fixed points of the elements of A and all elements in the vertex groups to the set of vertices Q such that

(i) for every edge not in P the end points are labelled by the fixed points of a generator of infinite order,

(ii) if $b_1, b_2 \in B$ belong to $\alpha \in N$ then the unique simple path in P from $v(b_1)$ to $v(b_2)$ is the fixed point set of α in P .

(Here we assume for simplicity that N contains no elements of order $p = \text{residue characteristic of } k$.)

$B(Q)$ is now given by inequalities of three different types, all involving the cross ratio

$$\text{CR}(x_1, x_2, x_3, x_4) := \frac{x_1 - x_3}{x_1 - x_4} \frac{x_2 - x_4}{x_2 - x_3}$$

of four distinct points $x_1, \dots, x_4$ in $\mathbb{P}^1(k)$:

(a) Let $B = \{b_1, \dots, b_\rho\}$ for some ρ . For $b_i, b_j \in B$, let π_{ij} denote the

projection in P onto the oriented path in P from $v(b_i)$ to $v(b_j)$. We say that $b_k <_{ij} b_l$ (or $=_{ij}$ or $>_{ij}$) if $\pi_{ij}(v(b_k))$ is closer to $v(b_i)$ than $\pi_{ij}(v(b_l))$ (resp. is equal to or is closer to $v(b_j)$). Then every point in $B(Q)$ satisfies the (in-)equalities

$$|CR(b_i, b_j, b_k, b_l)| \begin{cases} < 1; & b_k <_{ij} b_l, \\ = 1; & b_k =_{ij} b_l, \\ > 1; & b_k >_{ij} b_l. \end{cases}$$

(b) For every i , $r + s < i \leq r + s + f$ (i. e. every free generator of N) let e_i denote the corresponding edge in $Q \setminus P$ (the fixed points and the multipliers of the corresponding transformation were already denoted x_i , y_i and t_i).

Then for every pair b_j, b_k of elements of B which do not belong to an element in the stabilizer of e_i we have the following inequalities

$$|t_i| < |CR(x_i, y_i, b_j, b_k)| < |t_i|^{-1}.$$

(c) Let v be a vertex of Q , $\alpha \in N$ in the vertex group of v , $b_i, b_j \in B$ the fixed points of α . For every $b_k, b_l \in B$, such that

$$\pi_{ij}(v(b_k)) = \pi_{ij}(v(b_l)) = v,$$

we must have

$$|CR(b_i, b_j, \alpha(b_k), b_l) - 1| = 1$$

(this condition assures that α does not map an edge of Q starting at v onto another edge of Q).

LEMMA 2. - $B(Q)$ is precisely the set of points

$$(x_1, y_1, \dots, x_{r+s+f}, y_{r+s+f}, z_1, \dots, z_s, t_1, \dots, t_f)$$

satisfying all the inequalities in (a), (b) and (c) and the condition that all x_i, y_i, z_i be different.

This lemma shows that $B(Q)$ is connected (even a Stein domain). Lemma 1 showed that one can come from one Q to another by a sequence of admissible contractions. After a contraction some projections coincide so that some inequalities in (a) become equalities. This shows that the union of the two domain is connected and so $B(N)$ is connected and also the quotient $\mathcal{M}(N, F)$ is connected.

6. - To illustrate the concepts of the previous sections by an example we shall study two strata of $\mathcal{M}_3$ which will turn out to be especially nice ($g = 3$ was not allowed in Proposition 2 because the hyperelliptic curves without further automorphism are not singular points of $\mathcal{M}_3$; but the situation is very similar to the general case: only the uniqueness of the group N fails to be true, but there is

still a unique maximal group). Consider the fundamental group N of the following graph of groups :

$$S_3 \quad o \xrightarrow{1} o \quad C_2$$

N is isomorphic to the free product $S_3 * C_2$. Let S_3 be generated by elements τ and σ such that $\tau^3 = \sigma^2 = (\tau\sigma)^2 = 1$, and let $C_2 (= \mathbb{Z}/2\mathbb{Z})$ be generated by an involution α .

N contains the free normal subgroup F generated by

$$\gamma_1 = \alpha\sigma, \quad \gamma_2 = \tau\alpha\sigma\tau^{-1}, \quad \gamma_3 = \tau^{-1}\alpha\sigma\tau.$$

Aside from inner automorphisms $\text{Aut } N$ contains only the automorphisms of S_3 . Only the automorphism which fixes σ and sends τ to τ^{-1} leaves F unchanged, so

$$\text{Aut}_F(N)/(\text{inner automorphism}) = \{\text{id}, \tau \mapsto \tau^{-1}\} \cong C_2.$$

By Theorem 1, we have

$$\dim \mathcal{M}(N, F) = 3 + 2 - 3 = 2.$$

The other possible reduced graphs of groups with fundamental group N are

$$S_3 \quad o \xrightarrow{C_3} o \xrightarrow{1} o \quad C_2 \quad \text{and} \quad S_3 \quad o \xrightarrow{C_2} o \xrightarrow{C_2} o \xrightarrow{1} o \quad C_2$$

To explicitly calculate $\mathcal{M}(N, F)$, we begin with $T_o(N)$ and divide out the action of $\text{PGL}_2(k)$ (which we may do by the commutativity of the group actions). This means that we assume τ to have fixed points $0, \infty$ and σ the fixed points $1, -1$; if ρ denotes a primitive third root of unity, the fixed points of $\sigma\tau$ and $\sigma\tau^{-1}$ are $\rho, -\rho$ and $\rho^{-1}, -\rho^{-1}$. We denote the fixed points of α by x and y .

The possible labellings of the graphs of groups are

$$\begin{array}{c} S_3 \quad o \xrightarrow{1} o \quad C_2 \\ 0, \infty, 1, -1, \rho, -\rho, \rho^{-1}, -\rho^{-1} \quad x, y \\ \\ S_3 \quad o \xrightarrow{C_3} o \xrightarrow{1} o \quad C_2 \\ 0, 1, -1, \rho, -\rho, \rho^{-1}, -\rho^{-1} \quad \infty \quad x, y \\ \\ S_3 \quad o \xrightarrow{C_3} o \xrightarrow{1} o \quad C_2 \\ \infty, 1, -1, \rho, -\rho, \rho^{-1}, -\rho^{-1} \quad 0 \quad x, y \end{array}$$

and so on (6 possibilities for the third graph of groups).

In all cases, the inequalities describing $B(Q)$ can be determined easily by condition (a) of Lemma 2. A straightforward calculation shows that the union of all the $B(Q)$, i. e.

$$\mathrm{PGL}_2(k) \backslash \mathbb{T}_0(N) =: \bar{\mathbb{T}}_0(N)$$

is given by

$$\bar{\mathbb{T}}_0(N) = \{(x, y) \in k^2; x \neq y$$

$$\text{and } |x - y| < \min(|x|, |x+1|, |x-1|, |x-\rho|, |x+\rho|, |x-\rho^{-1}|, |x+\rho^{-1}|)\}$$

To come from $\bar{\mathbb{T}}_0(N)$ to $\mathcal{M}(N, F)$, we have to divide by the action of C_2^3 (identification of the fixed points of a transformation) and by the action of $\tau \mapsto \tau^{-1}$. But the latter automorphism acts in the same way as interchanging 0 and ∞ . As we are already in $\mathrm{PGL}_2(k) \backslash \mathbb{T}_0(N)$ we have to normalize the actions of the automorphisms by a suitable linear fractional transformation.

Then $\phi_1 : 0 \mapsto \infty$, $\phi_2 : 1 \mapsto -1$, $\phi_3 : x \mapsto y$ act in the following way on x and y :

	ϕ_1	ϕ_2	ϕ_3
x	x^{-1}	$-x$	y
y	y^{-1}	$-y$	x

The invariants are easily seen to be generated by $v' = \frac{x}{y} + \frac{y}{x}$ and $w' = xy + \frac{1}{xy}$. To have nicer inequalities, we prefer as coordinates on $\mathcal{M}(N, F)$

$$v := v' - 2 = \frac{(x - y)^2}{xy} \quad \text{and} \quad w := w' - 4 = \frac{(xy - 1)^2}{xy}.$$

Then from $|x - y| < |x| = |y|$, we have $0 < |v| < 1$.

If $|x| \neq 1$, we have $|w'| = |w' - 4| > 1$. If $|x| = 1$, let

$$y = x + \epsilon, \quad |\epsilon| < \min_{i=0}^2 |x \pm \rho^i|;$$

then

$$\begin{aligned} |w| &= |w' - 4| = |xy - 1|^2 = |x^2 - 1 + \epsilon x|^2 \\ &= |(x - 1)(x + 1) + \epsilon x|^2 = |x - 1|^2 |x + 1|^2 > |\epsilon|^2 = |v|. \end{aligned}$$

In the same way we see that if $|x| = 1$

$$\begin{aligned} |w + 3|^2 &= |(xy)^2 + xy + 1|^2 = |x^4 + x^2 + 1 + \epsilon(x + 2x^3 + \epsilon x^2)|^2 \\ &= |(x - \rho)(x + \rho)(x - \rho^{-1})(x + \rho^{-1})|^2 > |\epsilon|^2 = |v|. \end{aligned}$$

Therefore we have

$$\mathcal{M}(N, F) = \{(v, w) \in k^2; 0 < |v| < \min(1, |w|, |w + 3|^2)\}.$$

$\mathcal{M}(N, F)$ is not a single stratum of $\mathcal{M}_3$, but the union of two strata:

It contains a one-dimensional subspace of curves with bigger automorphism group than $N/F \cong S_3$.

Namely, if the fixed points x and y of α satisfy $xy = -1$, i. e. $y = \frac{1}{x}$, the transformation $\phi(z) := -\frac{1}{z}$ generates with τ and σ a finite group D_6 , and with α a Klein four group D_2 . One checks immediately that

$$\phi\gamma_1\phi^{-1} = \gamma_1, \quad \phi\gamma_2\phi^{-1} = \gamma_3, \quad \phi\gamma_3\phi^{-1} = \gamma_2,$$

so F is normal in the group N' generated by N and ϕ , and $N/F \cong D_6$. x and $y = -x^{-1}$ define a point in $T_0(N)$ if, and only if,

$$|x^2 + 1| < 1$$

(i. e. $|x + i| < 1$ or $|x - i| < 1$ for $i = \sqrt{-1}$).

The variables v and w of $\mathcal{M}(N, F)$ become

$$v = -\left(\frac{x^2 + 1}{x}\right)^2, \quad w = -4,$$

and the inequality to be satisfied is

$$0 < |v| < 1.$$

The corresponding (unique) graph of groups for N' is

$$D_6 \text{ --- } \overset{C_2}{\text{---}} \text{ --- } D_2.$$

We have shown that

$$\mathcal{M}(N', F) = \{v \in k; 0 < |v| < 1\} = \{(v, w) \in \mathcal{M}(N, F) : w = -4\}.$$

There is no group N'' containing N' such that F is normal in N'' and $T(N'') \neq \emptyset$. This means that $\mathcal{M}(N', F)$ and $\mathcal{M}(N, F) \setminus \mathcal{M}(N', F)$ are indeed the intersections of two strata of $\mathcal{M}_3$ with $\mathcal{M}_3$.

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