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ON MORITA'S p -ADIC Γ -FUNCTION

par Daniel BARSKY (*)

Y. MORITA proved that, for each prime number p , one can define a p -adic continuous function $\Gamma_p(x)$ from $\mathbb{Z}_p$ to $\mathbb{Z}_p$, interpolating the sequence

$$n \longrightarrow (-1)^n \prod_m m,$$

where m runs through the integers m prime to p with $1 \leq m < n$. Our aim is to show how this result is related to DWORK's result on the radius of convergence of $\exp(X + (X^p/p))$.

1. Introduction and notations.

Let p be a prime, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$ as usual [2]. The absolute value (resp the valuation) on $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$ is denoted by $|\cdot|$ (resp. $v(\cdot)$), and normalized by $|p| = p^{-1}$ (resp. $v(p) = 1$). The sequence $n \longrightarrow n!$ can not be the restriction to $\mathbb{N}$ of a p -adic continuous function from $\mathbb{Z}_p$ to $\mathbb{C}_p$. However, MORITA [10] proved that the sequence $n \longrightarrow (-1)^n \prod_m m = \Gamma_p(m)$, m prime to p , $1 \leq m < n$, can be interpolated by a continuous p -adic function from $\mathbb{Z}_p$ to $\mathbb{C}_p$ denoted by $\Gamma_p(x)$. We shall prove, by means of the formal Laplace transform ([5] or [3]), that this result can be deduced from DWORK's estimate of the p -adic absolute value of the coefficients $b_n/n!$ of the serie

$$\sum_{n \geq 0} \frac{b_n}{n!} X^n = \exp(X + \frac{X^p}{p}).$$

We actually prove that the result of DWORK is equivalent to the existence of a locally analytic function $x \longrightarrow \Gamma_p(px)$ on $\mathbb{Z}_p$, with local analyticity radius $\rho_p = p^{-(1/p)-(1/(p-1))}$, bounded by 1 on

$$\mathbb{W}_{\rho_p}(\mathbb{Z}_p) = \{x \in \mathbb{C}_p; |x - z| < \rho_p \text{ for all } z \in \mathbb{Z}_p\},$$

interpolating the sequence $n \longrightarrow (-1)^{pn} ((pn)!/p^n(n!)) = \Gamma_p(pn)$. Let us denote

$$B(a, \rho^+) = \{x \in \mathbb{C}_p; |x - a| \leq \rho\}, \quad a \in \mathbb{C}_p, \quad \rho \in \mathbb{R}_+ = \{x \in \mathbb{R}; x > 0\}.$$

Recall [3] that the formal Laplace transform $\mathcal{L}(\tilde{f}(x))$ of

$$\tilde{f}(x) = \sum_{n \geq 0} \frac{a_n}{n!} x^n \in \mathbb{C}_p[[x]]$$

is

$$(1) \quad f(x) = \mathcal{L}(\tilde{f}(x)) = \sum_{n \geq 0} a_n x^n \in \mathbb{C}_p[[x]].$$

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The following properties of $\mathcal{L}$ are obvious from the definition :

$\mathcal{L}$ is a $\mathbb{C}_p$ -linear map from $\mathbb{C}_p[[x]]$ onto $\mathbb{C}_p[[x]]$,

$\mathcal{L}$ is continuous for the (p, x) -adic topology on $\mathbb{C}_p[[x]]$.

Let

$$\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!}, \quad \binom{x}{0} = 1.$$

2. Morita's p-adic Γ -function.

DEFINITION 1 (MORITA [10]). - Let $\Gamma_p(n)$ be defined, for $n \in \mathbb{N}$, by

$$(2) \quad \Gamma_p(0) = 1,$$

$$(3) \quad \Gamma_p(n) = (-1)^n \prod_{0 \leq m < n} m, \quad m \text{ prime to } p.$$

LEMMA 1. - For $1 \leq i \leq p$, we have

$$(4) \quad \Gamma_p(pn+i) = (-1)^{pn+i} \frac{(pn+i-1)!}{p^n (n!)}.$$

Proof. - Obvious from definition 1.

Q. E. D.

LEMMA 2. - The generating function of the sequence $n \rightarrow (-1)^n \Gamma_p(n)$,

$$(5) \quad F(X) = \sum_{n \geq 0} (-1)^{n+1} \Gamma_p(n+1) X^n,$$

is the formal Laplace transform of

$$(6) \quad \tilde{F}(X) = \sum_{i=1}^p X^{i-1} \exp(x^p/p).$$

Proof.

$$(7) \quad \sum_{i=1}^p X^{i-1} \exp(x^p/p) = \sum_{n \geq 0} \sum_{i=1}^p \frac{x^{pn+i-1}}{p^n (n!)}.$$

So

$$(8) \quad \mathcal{L}\left(\sum_{i=1}^p X^{i-1} \exp(x^p/p)\right) = \sum_{i=1}^p \sum_{n \geq 0} \frac{(pn+i-1)!}{p^n n!} X^{pn+i-1},$$

the conclusion follows from lemma 1.

Q. E. D.

Let

$$(9) \quad \sum_{n \geq 0} \frac{b_n}{n!} X^n \text{ be the formal Taylor series at the origin of } \exp\left(x + \frac{x^p}{p}\right).$$

It is clear the $\sum_{n \geq 0} (b_n/n!) X^n \in \mathbb{Q}[[x]]$.

LEMMA 3 (DWORK [6]). - The coefficients b_n of the Taylor series of $\exp(x + (x^p/p))$ satisfy the following p-adic estimate

$$(10) \quad v\left(\frac{b_n}{n!}\right) \geq -n \frac{2p-1}{p^2(p-1)}.$$

For the proof see [9] or [6].

LEMMA 4. - If $\mathcal{L}(\tilde{f}(x)) = f(x)$, then

$$(11) \quad \mathcal{L}((\exp \alpha x) \tilde{f}(x)) = \frac{1}{1 - \alpha x} f\left(\frac{x}{1 - \alpha x}\right), \quad \alpha \in \mathbb{C}_p.$$

Proof. - Let $\tilde{f}(x) = \sum_{n \geq 0} a_n (x^n/n!)$, then

$$(\exp \alpha x) \tilde{f}(x) = \sum_{n \geq 0} \left(\sum_{k=0}^n \alpha^k a_{n-k} \binom{n}{k} \right) \frac{x^n}{n!}.$$

This implies

$$\mathcal{L}((\exp \alpha x) \tilde{f}(x)) = \sum_{n \geq 0} \left(\sum_{k=0}^n a_{n-k} \alpha^k \binom{n}{k} \right) x^n = \sum_{n \geq 0} a_n \frac{x^n}{(1 - \alpha x)^{n+1}}.$$

Q. E. D.

THEOREM 1. - The following identity holds

$$(12) \quad F(X) = \sum_{n \geq 0} (-1)^{n+1} \Gamma_p(n+1) X^n = \sum_{i=1}^p \sum_{n \geq 0} b_n^{(i)} \frac{x^n}{(1+x)^{n+1}} \text{ with } b_n^{(i)} = b_{n-i+1} \frac{n!}{(n+1-i)!}.$$

Proof. - $F(X) = \mathcal{L}(\exp(-x) (\sum_{i=1}^p x^{i-1} \exp(x + (x^p/p))))$. Applying lemma 4 and 2, we obtain

$$F(X) = \sum_{i=1}^p \sum_{n \geq 0} b_n \frac{x^{n+i-1}}{(1+x)^{n+i}}$$

Q. E. D.

COROLLARY 1. - For $n \in \mathbb{N}$, we have

$$(13) \quad \Gamma_p(n+1) = - \sum_{i=1}^p \sum_k (-1)^k b_k^{(i)} \binom{n}{k}.$$

Proof.

$$\begin{aligned} F(X) &= \sum_{i=1}^p \sum_{n \geq 0} b_n^{(i)} \frac{x^n}{(1+x)^{n+1}} = \sum_{i=1}^p \sum_{n \geq 0} b_n^{(i)} \sum_{k \geq 0} x^{n+k} \binom{-n-1}{k} \\ &= \sum_{i=1}^p \sum_{n \geq 0} x^n \sum_{k=0}^n (-1)^{n-k} b_k^{(i)} \binom{n}{k}. \end{aligned}$$

Q. E. D.

THEOREM 2. - The sequence $n \rightarrow \Gamma_p(n)$ is the restriction to $\mathbb{N}$ of a unique locally analytic function on $\mathbb{Z}_p$, $\Gamma_p(x)$, with local analyticity radius

$$(14) \quad \rho_p = p^{-(1/p)-(1/p-1)}.$$

Proof. - Recall [1], that a function from $\mathbb{Z}_p$ to $\mathbb{C}_p$ is locally analytic on $\mathbb{Z}_p$, if, for each point $\alpha \in \mathbb{Z}_p$, there exist $\rho_\alpha \in \mathbb{R}_+$ such that, on $B(\alpha, \rho_\alpha^+) \cap \mathbb{Z}_p$, f is the restriction of a function $f_\alpha(x) = \sum_{n \geq 0} a_n(\alpha) (x - \alpha)^n$, analytic on $B(\alpha, \rho_\alpha^+)$. The local analyticity radius of f is

$$(15) \quad \rho = \inf_{\alpha \in \mathbb{Z}_p} \rho_\alpha > 0 \quad (\text{because } \mathbb{Z}_p \text{ is compact}).$$

Take the series

$$(16) \quad \Gamma_p(x+1) = - \sum_{i=1}^p \sum_{n \geq 0} (-1)^n b_n^{(i)} \binom{x}{n}.$$

By lemma 3, we get

$$(17) \quad v\left(\sum_{i=1}^p b_n^{(i)}\right) \geqslant -(n-p+1) \frac{2p-1}{p^2(p-1)} + v((n-p+1)!) \\ \geqslant (n-p+1)\left(\frac{1}{p} - \frac{1}{p^2}\right) - \left[\frac{\log n}{\log p}\right] - 1,$$

where $[a]$ is the integral part of $A \in \mathbb{R}_+$, that is $[a] \in \mathbb{N}$ and $a-1 < [a] \leq a$. This (17) implies

$$(18) \quad \lim_{n \rightarrow \infty} \left| \sum_{i=1}^p b_n^{(i)} \right| = 0.$$

By Mahler's theorem [1], this implies that (16) defines the unique continuous functions from $\mathbb{Z}_p$ to $\mathbb{C}_p$ (actually to $\mathbb{Q}_p$) interpolating the sequence $n \rightarrow \Gamma_p(n)$.

Actually (17) gives us a stronger result. Let $1/p < \rho = p^{-\alpha} < 1$, $\alpha \in \mathbb{R}_+$, $1 > \alpha > 0$. Let $\mathbb{W}_\rho(\mathbb{Z}_p) = \{x \in \mathbb{C}_p; |x-z| < \rho \text{ for all } z \in \mathbb{Z}_p\}$.

$$(19) \quad \inf_{x \in \mathbb{W}_\rho(\mathbb{Z}_p)} v\left(\binom{x}{n}\right) > \left(\left[\frac{n}{p}\right](1-\alpha) + \sum_{i \geq 2} \left[\frac{n}{p^i}\right]\right).$$

$$(20) \quad \inf_{x \in \mathbb{W}_\rho(\mathbb{Z}_p)} v\left(\binom{x}{n}\right) > -\frac{n}{p}\left(1-\alpha + \frac{1}{p-1}\right).$$

From (20) and (17), we obtain

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{W}_\rho(\mathbb{Z}_p)} \left| \sum_{i=1}^p b_n^{(i)} \binom{x}{n} \right| = 0$$

if, and only if,

$$(21) \quad \alpha > \frac{1}{p-1} + \frac{1}{p}.$$

Q. E. D.

But, we can get more the following theorem.

THEOREM 3. - The following two propositions are equivalent.

(i) The sequence $n \rightarrow \Gamma_p(pn)$ is the restriction of a unique locally analytic function on $\mathbb{Z}_p$, $\Gamma_p(px)$, with local analyticity radius $r_p = p^{1-(1/p)-(1/p-1)}$, and bounded by one on $\mathbb{W}_{r_p}(\mathbb{Z}_p)$.

$$(ii) \quad \exp(x + (x^p/p)) = \sum_{n \geq 0} \frac{b_n}{n!} x^n, \text{ with } v\left(\frac{b_n}{n!}\right) \geqslant -n \frac{2p-1}{p^2(p-1)}.$$

Proof.

$$(22) \quad \Gamma_p(pn) = (-1)^{pn} \frac{(p(n-1) + p-1)!}{p^{n-1}(n-1)!} = (-1)^{pn} \frac{(pn)!}{p^n n!}.$$

From (22), it is clear that

$$(23) \quad \sum_{n \geq 0} \frac{(pn)!}{p^n n!} x^{pn} = \mathcal{L}(\exp(x^p/p)) = \mathcal{L}(\exp(-x) \exp(x + \frac{x^p}{p})).$$

We obtain, as in corollary 1,

$$(24) \quad \Gamma_p(pn) = \sum_k (-1)^k b_k \binom{pn}{k}.$$

Consider the series

$$(25) \quad \Gamma_p(px) = \sum_n (-1)^n b_n \binom{px}{n}.$$

We obtain, as in (19),

$$(26) \quad \sup_{x \in \mathbb{W}_{r_p}} \left(\frac{Z}{p} \right) |b_n \binom{px}{n}| = \left| \frac{b_n}{n!} \right| p^{-[\frac{n}{p}](\frac{1}{p} + \frac{1}{p-1})}.$$

Assuming (ii), we obtain

$$(27) \quad \sup_{x \in \mathbb{W}_{r_p}} \left(\frac{Z}{p} \right) |b_n \binom{x}{n}| \leq p^{\left(\frac{1}{p} + \frac{1}{p-1}\right)\left(\frac{n}{p} - [\frac{n}{p}]\right)} \leq 1,$$

and, for all $0 < r < r_p$,

$$\lim_{n \rightarrow 0} \sup_{x \in \mathbb{W}_{r_p}} \left(\frac{Z}{p} \right) |b_n \binom{x}{n}| = 0,$$

so (i) is true.

Assuming (i), we obtain

$$\left| \frac{b_n}{n!} \right| \leq p^{\left[\frac{n}{p}\right]\left(\frac{1}{p} + \frac{1}{p-1}\right)} \leq p^{\frac{n(2p-1)}{p^2(p-1)}},$$

so (ii) is true.

Q. E. D.

This theorem shows the close relations that exist between $\exp(x + (x^p/p))$ and $\Gamma_p(x)$. Actually, what formulas (24) and (25) say, is that the coefficients of the Mahler expansion [1] of the function $x \rightarrow \Gamma_p(px)$ are given by the coefficients b_n of the Taylor expansion of $\mathfrak{L}(\exp(x + (x^p/p)))$. This suggests the possibility of a direct proof of Koblitz's formula ([4], [7], [8]) between Gauss sums and the p -adic Γ -function of Morita.

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