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ON MORITA'S p-ADIC Γ-FUNCTION

par Daniel BARSKY (*)

Y. MORITA proved that, for each prime number p, one can define a p-adic continuous function $\Gamma_p(x)$ from $\underset{\sim}{Z}_p$ to $\underset{\sim}{Z}_p$, interpolating the sequence

$$n \longrightarrow (-1)^n \prod_m m ,$$

where m runs through the integers m prime to p with $1 \leqslant m < n$. Our aim is to show how this result is related to DWORK's result on the radius of convergence of $\exp(X + (X^p/p))$.

1. Introduction and notations.

Let p be a prime, $\underset{\sim}{N}$, $\underset{\sim}{R}$, $\underset{\sim}{Z}_p$, $\underset{\sim}{Q}_p$, $\underset{\sim}{C}_p$ as usual [2]. The absolute value (resp the valuation) on $\underset{\sim}{Z}_p$, $\underset{\sim}{Q}_p$, $\underset{\sim}{C}_p$ is denoted by $|\,.\,|$ (resp. $v(.)$), and normalized by $|p| = p^{-1}$ (resp. $v(p) = 1$). The sequence $n \longrightarrow n!$ can not be the restriction to $\underset{\sim}{N}$ of a p-adic continuous function from $\underset{\sim}{Z}_p$ to $\underset{\sim}{C}_p$. However, MORITA [10] proved that the sequence $n \longrightarrow (-1)^n \prod_m m = \Gamma_p(m)$, m prime to p, $1 \leqslant m < n$, can be interpolated by a continuous p-adic function from $\underset{\sim}{Z}_p$ to $\underset{\sim}{C}_p$ denoted by $\Gamma_p(x)$. We shall prove, by means of the formal Laplace transform ([5] or [3]), that this result can be deduced from DWORK's estimate of the p-adic absolute value of the coefficients $b_n/n!$ of the serie

$$\sum_{n \geqslant 0} \frac{b_n}{n!} X^n = \exp(X + \frac{X^p}{p}) .$$

We actually prove that the result of DWORK is equivalent to the existence of a locally analytic function $x \longrightarrow \Gamma_p(px)$ on $\underset{\sim}{Z}_p$, with local analyticity radius $\rho_p = p^{-(1/p)-(1/(p-1))}$, bounded by 1 on

$$\mathscr{W}_{\rho_p}(\underset{\sim}{Z}_p) = \{x \in \underset{\sim}{C}_p \; ; \; |x - z| < \rho_p \text{ for all } z \in \underset{\sim}{Z}_p\} ,$$

interpolating the sequence $n \longrightarrow (-1)^{pn} ((pn)!/p^n(n!)) = \Gamma_p(pn)$. Let us denote

$$B(a , \rho^+) = \{x \in \underset{\sim}{C}_p \; ; \; |X - a| \leqslant \rho\} , \quad a \in \underset{\sim}{C}_p , \quad \rho \in \underset{\sim}{R}_+ = \{x \in \underset{\sim}{R} \; ; \; x > 0\} .$$

Recall [3] that the formal Laplace transform $\mathfrak{L}(\tilde{f}(x))$ of

$$\tilde{f}(x) = \sum_{n \geqslant 0} \frac{a_n}{n!} x^n \in \underset{\sim}{C}_p[[x]]$$

is

$$f(x) = \mathfrak{L}(\tilde{f}(x)) = \sum_{n \geqslant 0} a_n x^n \in \underset{\sim}{C}_p[[x]] . \tag{1}$$

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The following properties of $\mathfrak{L}$ are obvious from the definition :

$\mathfrak{L}$ is a $\underset{\sim}{C}_p$-linear map from $\underset{\sim}{C}_p[[x]]$ onto $\underset{\sim}{C}_p[[x]]$,

$\mathfrak{L}$ is continuous for the (p , x)-adic topology on $\underset{\sim}{C}_p[[x]]$.

Let

$$\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!} , \quad \binom{x}{0} = 1 .$$

2. Morita's p-adic Γ-function.

DEFINITION 1 (MORITA [10]). - Let $\Gamma_p(n)$ be defined, for $n \in \underset{\sim}{N}$, by

(2) $$\Gamma_p(0) = 1 ,$$

(3) $$\Gamma_p(n) = (-1)^n \prod_{0\leqslant m<n} m , \quad m \text{ prime to } p .$$

LEMMA 1. - For $1 \leqslant i \leqslant p$, we have

(4) $$\Gamma_p(pn + i) = (-1)^{pn+i} \frac{(pn + i - 1)!}{p^n (n!)} .$$

Proof. - Obvious from definition 1. Q. E. D.

LEMMA 2. - The generating function of the sequence $n \longrightarrow (-1)^n \Gamma_p(n)$,

(5) $$F(X) = \sum_{n\geqslant0} (-1)^{n+1} \Gamma_p(n+1) X^n ,$$

is the formal Laplace transform of

(6) $$\tilde{F}(X) = \sum_{i=1}^{p} X^{i-1} \exp(x^p/p) .$$

Proof.

(7) $$\sum_{i=1}^{p} x^{i-1} \exp(x^p/p) = \sum_{n\geqslant0} \sum_{i=1}^{p} \frac{x^{pn+i-1}}{p^n(n!)} .$$

So

(8) $$\mathfrak{L}(\sum_{i=1}^{p} x^{i-1} \exp(x^p/p)) = \sum_{i=1}^{p} \sum_{n\geqslant0} \frac{(pn + i - 1)!}{p^n\, n!} X^{pn+i-1} ,$$

the conclusion follows from lemma 1. Q. E. D.

Let

(9) $\sum_{n\geqslant0} \frac{b_n}{n!} X^n$ be the formal Taylor series at the origin of $\exp(x + \frac{x^p}{p})$.

It is clear the $\sum_{n\geqslant0}(b_n/n!)x^n \in \underset{\sim}{Q}[[x]]$.

LEMMA 3 (DWORK [6]). - The coefficients b_n of the Taylor series of $\exp(x + (x^p/p))$ satisfy the following p-adic estimate

(10) $$v(\frac{b_n}{n!}) \geqslant -n \frac{2p - 1}{p^2(p - 1)} .$$

For the proof see [9] or [6].

LEMMA 4. - *If* $\mathcal{L}(\tilde{f}(x)) = f(x)$, *then*

$$(11)\qquad \mathcal{L}((\exp \alpha x)\,\tilde{f}(x)) = \frac{1}{1-\alpha x}\, f\left(\frac{x}{1-\alpha x}\right) , \quad \alpha \in \underset{\sim}{C}_p .$$

Proof. - Let $\tilde{f}(x) = \sum_{n\geq 0} a_n (x^n/n!)$, then

$$(\exp \alpha x)\,\tilde{f}(x) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} \alpha^k a_{n-k}\binom{n}{k}\right)\frac{x^n}{n!} .$$

This implies

$$\mathcal{L}((\exp \alpha x)\,\tilde{f}(x)) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} a_{n-k}\,\alpha^k\binom{n}{k}\right) x^n = \sum_{n\geq 0} a_n \frac{x^n}{(1-\alpha x)^{n+1}} .$$

Q. E. D.

THEOREM 1. - *The following identity holds*

$$(12)\quad F(X) = \sum_{n\geq 0}(-1)^{n+1}\Gamma_p(n+1)X^n = \sum_{i=1}^{p}\sum_{n\geq 0} b_n^{(i)}\frac{x^n}{(1+x)^{n+1}} \text{ with } b_n^{(i)} = b_{n-i+1}\frac{n!}{(n+1-i)!} .$$

Proof. - $F(X) = \mathcal{L}(\exp(-x)(\sum_{i=1}^{p} x^{i-1}\exp(x+(x^p/p))))$. Applying lemma 4 and 2, we obtain

$$F(X) = \sum_{i=1}^{p}\sum_{n\geq 0} b_n \frac{x^{n+i-1}}{(1+x)^{n+i}}$$

Q. E. D.

COROLLARY 1. - *For* $n \in \underset{\sim}{N}$, *we have*

$$(13)\qquad \Gamma_p(n+1) = -\sum_{i=1}^{p}\sum_k (-1)^k\, b_k^{(i)}\binom{n}{k} .$$

Proof.

$$F(X) = \sum_{i=1}^{p}\sum_{n\geq 0} b_n^{(i)}\frac{x^n}{(1+x)^{n+1}} = \sum_{i=1}^{p}\sum_{n\geq 0} b_n^{(i)}\sum_{k\geq 0} x^{n+k}\binom{-n-1}{k}$$
$$= \sum_{i=1}^{p}\sum_{n\geq 0} x^n \sum_{k=0}^{n}(-1)^{n-k}\, b_k^{(i)}\binom{n}{k} .$$

Q. E. D.

THEOREM 2. - *The sequence* $n \longrightarrow \Gamma_p(n)$ *is the restriction to* $\underset{\sim}{N}$ *of a unique locally analytic function on* $\underset{\sim}{Z}_p$, $\Gamma_p(x)$, *with local analyticity radius*

$$(14)\qquad \rho_p = p^{-(1/p)-(1/p-1)} .$$

Proof. - Recall [1], that a function from $\underset{\sim}{Z}_p$ to $\underset{\sim}{C}_p$ is locally analytic on $\underset{\sim}{Z}_p$, if, for each point $\alpha \in \underset{\sim}{Z}_p$, there exist $\rho_\alpha \in \underset{\sim}{R}_+$ such that, on $B(\alpha,\rho_\alpha^+) \cap \underset{\sim}{Z}_p$, f is the restriction of a function $f_\alpha(x) = \sum_{n\geq 0} a_n(\alpha)(x-\alpha)^n$, analytic on $B(\alpha , \rho_\alpha^+)$. The local analyticity radius of f is

$$(15)\qquad \rho = \inf_{\alpha\in \underset{\sim}{Z}_p} \rho_\alpha > 0 \quad (\text{because } \underset{\sim}{Z}_p \text{ is compact}).$$

Take the series

$$(16)\qquad \Gamma_p(x+1) = -\sum_{i=1}^{p}\sum_{n\geq 0}(-1)^n\, b_n^{(i)}\binom{x}{n} .$$

By lemma 3, we get

$$(17)\quad v(\sum_{i=1}^{p} b_n^{(i)}) \geqslant -(n-p+1)\frac{2p-1}{p^2(p-1)} + v((n-p+1)!)$$
$$\geqslant (n-p+1)(\frac{1}{p}-\frac{1}{p^2}) - [\frac{\log n}{\log p}] - 1 ,$$

where $[a]$ is the integral part of $A \in \underset{\sim}{R}_+$, that is $[a] \in \underset{\sim}{N}$ and $a - 1 < [a] \leqslant a$. This (17) implies

$$(18)\qquad \lim_{n\to+\infty} |\sum_{i=1}^{p} b_n^{(i)}| = 0 .$$

By Mahler's theorem [1], this implies that (16) défines the unique continuous functions from $\underset{\sim}{Z}_p$ to $\underset{\sim}{C}_p$ (actually to $\underset{\sim}{Q}_p$) interpolating the sequence $n \longrightarrow \Gamma_p(n)$.

Actually (17) gives us a stronger result. Let $1/p < \rho = p^{-\alpha} < 1$, $\alpha \in \underset{\sim}{R}_+$, $1 > \alpha > 0$. Let $\mathscr{W}_\rho(\underset{\sim}{Z}_p) = \{x \in \underset{\sim}{C}_p \ ; \ |x - z| < \rho \text{ for all } z \in \underset{\sim}{Z}_p\}$.

$$(19)\qquad \inf_{x\in \mathscr{W}_\rho(\underset{\sim}{Z}_p)} v(\binom{x}{n}) > ([\frac{n}{p}](1-\alpha) + \sum_{i\geqslant 2}[\frac{n}{p^i}]) .$$

$$(20)\qquad \inf_{x\in \mathscr{W}_\rho(\underset{\sim}{Z}_p)} v(\binom{x}{n}) > -\frac{n}{p}(1-\alpha+\frac{1}{p-1}) .$$

From (20) and (17), we obtain

$$\lim_{n\to\infty} \sup_{x\in \mathscr{W}_\rho(\underset{\sim}{Z}_p)} |\sum_{i=1}^{p} b_n^{(i)}\binom{x}{n}| = 0$$

if, and only if,

$$(21)\qquad \alpha > \frac{1}{p-1} + \frac{1}{p} .$$

Q. E. D.

But, we can get more the following theorem.

THEOREM 3. - <u>The following two propositions are equivalent.</u>

(i) <u>The sequence</u> $n \longrightarrow \Gamma_p(pn)$ <u>is the restriction of a unique locally analytic function on</u> $\underset{\sim}{Z}_p$, $\Gamma_p(px)$, <u>with local analyticity radius</u> $r_p = p^{1-(1/p)-(1/p-1)}$, <u>and bounded by one on</u> $\mathscr{W}_{r_p}(\underset{\sim}{Z}_p)$.

(ii) $\exp(x + (x^p/p)) = \sum_{n\geqslant 0} \frac{b_n}{n!} x^n$, <u>with</u> $v(\frac{b_n}{n!}) \geqslant -n\frac{2p-1}{p^2(p-1)}$.

<u>Proof</u>.

$$(22)\qquad \Gamma_p(pn) = (-1)^{pn}\frac{(p(n-1)+p-1)!}{p^{n-1}(n-1)!} = (-1)^{pn}\frac{(pn)!}{p^n\, n!} .$$

From (22), it is clear that

$$(23)\qquad \sum_{n\geqslant 0}\frac{(pn)!}{p^n\, n!} x^{pn} = \mathfrak{L}(\exp(x^p/p)) = \mathfrak{L}(\exp(-x)\exp(x+\frac{x^p}{p})) .$$

We obtain, as in corollary 1,

$$(24)\qquad \Gamma_p(pn) = \sum_k (-1)^k b_k \binom{pn}{k} .$$

Consider the series

(25) $$\Gamma_p(px) = \sum_n (-1)^n \, b_n \binom{px}{n} \; .$$

We obtain, as in (19),

(26) $$\sup_{x \in \mathscr{W}_{r_p}(\underset{\sim}{Z}_p)} \left| b_n \binom{px}{n} \right| = \left| \frac{b_n}{n!} \right| \, p^{-[\frac{n}{p}](\frac{1}{p}+\frac{1}{p-1})} \; .$$

Assuming (ii), we obtain

(27) $$\sup_{x \in \mathscr{W}_{r_p}(\underset{\sim}{Z}_p)} \left| b_n \binom{x}{n} \right| \leqslant p^{(\frac{1}{p}+\frac{1}{p-1})(\frac{n}{p}-[\frac{n}{p}])} \leqslant 1 \; ,$$

and, for all $0 < r < r_p$,

$$\lim_{n \to 0} \sup_{x \in \mathscr{W}_{r_p}(\underset{\sim}{Z}_p)} \left| b_n \binom{x}{n} \right| = 0 \; ,$$

so (i) is true.

Assuming (i), we obtain

$$\left| \frac{b_n}{n!} \right| \leqslant p^{[\frac{n}{p}](\frac{1}{p}+\frac{1}{p-1})} \leqslant p^{\frac{n(2p-1)}{p^2(p-1)}} \; ,$$

so (ii) is true.

Q. E. D.

This theorem shows the close relations that exist between $\exp(x + (x^p/p))$ and $\Gamma_p(x)$. Actually, what formulas (24) and (25) say, **is that the coefficients of the** Mahler expansion [1] of the function $x \longrightarrow \Gamma_p(px)$ are given by the coefficients b_n of the Taylor expansion of $\mathscr{L}(\exp(x + (x^p/p)))$. This suggests the possibility of a direct proof of Koblitz's formula ([4], [7], [8]) between Gauss sums and the p-adic Γ-function of Morita.

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