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On Morita’s $p$-adic $\Gamma$-function


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Y. MORITA proved that, for each prime number $p$, one can define a $p$-adic continuous function $\Gamma_p(x)$ from $\mathbb{Z}_p$ to $\mathbb{Z}_p$, interpolating the sequence

$$n \rightarrow (-1)^n \prod_{m \equiv 1 \pmod{p}} m,$$

where $m$ runs through the integers $m$ prime to $p$ with $1 \leq m < n$. Our aim is to show how this result is related to DWORK's result on the radius of convergence of $\exp(x + (x^p/p))$.

1. Introduction and notations.

Let $p$ be a prime, $\mathbb{N}, \mathbb{R}, \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$ as usual [2]. The absolute value (resp. the valuation) on $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$ is denoted by $|\cdot|$ (resp. $v(\cdot)$), and normalized by $|p| = p^{-1}$ (resp. $v(p) = 1$). The sequence $n \rightarrow n!$ cannot be the restriction to $\mathbb{N}$ of a $p$-adic continuous function from $\mathbb{Z}_p$ to $\mathbb{C}_p$. However, MORITA [10] proved that the sequence $n \rightarrow (-1)^n \prod_{m \equiv 1 \pmod{p}} m = \Gamma_p(m)$, $m$ prime to $p$, $1 \leq m < n$, can be interpolated by a continuous $p$-adic function from $\mathbb{Z}_p$ to $\mathbb{C}_p$ denoted by $\Gamma_p(x)$. We shall prove, by means of the formal Laplace transform ([5] or [3]), that this result can be deduced from DWORK's estimate of the $p$-adic absolute value of the coefficients $b_n/n!$ of the series

$$\sum_{n \geq 0} \frac{b_n}{n!} x^n = \exp(x + \frac{x^p}{p}).$$

We actually prove that the result of DWORK is equivalent to the existence of a locally analytic function $x \mapsto \Gamma_p(px)$ on $\mathbb{Z}_p$, with local analyticity radius $\rho_p = p^{-(1/p)-(1/(p-1))}$, bounded by 1 on

$$W_p(\mathbb{Z}_p) = \{x \in \mathbb{C}_p; \ |x - z| < \rho_p \ \text{for all} \ z \in \mathbb{Z}_p\},$$

interpolating the sequence $n \rightarrow (-1)^n \Gamma_p^n ((pn)l/p^nn!) = \Gamma_p(pn)$. Let us denote

$$B(a, \rho^+) = \{x \in \mathbb{C}_p; \ |x - a| < \rho, \ a \in \mathbb{C}_p, \ \rho \in \mathbb{R}_+ = \{x \in \mathbb{R}; \ x > 0\}.$$

Recall [3] that the formal Laplace transform $\mathcal{L}(f(x))$ of

$$\mathcal{F}(x) = \sum_{n \geq 0} \frac{a_n}{n!} x^n \in \mathbb{C}_p[[x]]$$

is

$$f(x) = \mathcal{L}(\mathcal{F}(x)) = \sum_{n \geq 0} a_n x^n \in \mathbb{C}_p[[x]].$$

(*) Visiting Research Associate of Princeton University, 1977/78. L'exposé, au groupe d'étude, a été prononcé par Philippe ROBBA.
The following properties of $\mathcal{L}$ are obvious from the definition:

- $\mathcal{L}$ is a $C_p$-linear map from $C_p[[x]]$ onto $C_p[[x]]$.
- $\mathcal{L}$ is continuous for the $(p, x)$-adic topology on $C_p[[x]]$.

Let

$$ (x) = \frac{x(x-1)\ldots(x-n+1)}{n!}, \quad (x)_0 = 1. $$

2.Morita's p-adic $\Gamma$-function.

**Definition 1 (Morita [10]).** Let $p(n)$ be defined, for $n \in \mathbb{N}$, by

$$ p(0) = 1, $$

$$ p(n) = (-1)^n \prod_{0 \leq m \leq n} m, \quad m \text{ prime to } p. $$

**Lemma 1.** For $1 \leq i \leq p$, we have

$$ p(pn + i) = (-1)^{pn+i} \frac{(pn + i - 1)!}{p^n (n!)}. $$

**Proof.** Obvious from definition 1.

**Lemma 2.** The generating function of the sequence $n \rightarrow (-1)^n p(n)$,

$$ F(x) = \sum_{n \geq 0} (-1)^{n+1} p(n + 1) x^n, $$

is the formal Laplace transform of

$$ F(x) = \sum_{i=1}^{\infty} x^{i-1} \exp(x^{p/p}). $$

**Proof.**

$$ \sum_{i=1}^{\infty} x^{i-1} \exp(x^{p/p}) = \sum_{n \geq 0} \frac{p^{pn+i-1}}{p^n (n!)}. $$

So

$$ g(\sum_{i=1}^{\infty} x^{i-1} \exp(x^{p/p})) = \sum_{i=1}^{\infty} \frac{(pn + i - 1)!}{p^n n!} x^{pn+i-1}, $$

the conclusion follows from lemma 1.

**Q. E. D.**

Let

$$ \sum_{n \geq 0} \frac{b_n}{n!} x^n $$

be the formal Taylor series at the origin of $\exp(x + \frac{x^p}{p})$.

It is clear that $\sum_{n \geq 0} (b_n/n!)x^n \in \mathbb{Q}[[x]]$.

**Lemma 3 (Dwork [6]).** The coefficients $b_n$ of the Taylor series of $\exp(x + (x^p/p))$ satisfy the following p-adic estimate

$$ v(\frac{b}{n!}) \geq -n \frac{2p - 1}{p^2 (p - 1)}. $$

For the proof see [9] or [6].
LEMMA 4. - If \( \ell((x)) = f(x) \), then

\[
\ell((\exp \alpha x) \tilde{x}(x)) = \frac{1}{1 - \alpha x} f\left(\frac{x}{1 - \alpha x}\right), \quad \alpha \in \mathbb{C}_p.
\]

Proof. - Let \( \tilde{x}(z) = \sum_{n \geq 0} a_n (x^n / n!) \), then

\[
(\exp \alpha x) \tilde{x}(x) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} \alpha^k a_{n-k} \left(\frac{n}{k}\right) \right) x^n / n!.
\]

This implies

\[
\ell((\exp \alpha x) \tilde{x}(x)) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} \alpha^k a_{n-k} \left(\frac{n}{k}\right) \right) x^n = \sum_{n \geq 0} a_n \frac{x^n}{(1 - \alpha x)^{n+1}}.
\]

Q. E. D.

THEOREM 1. - The following identity holds

\[
F(x) = \sum_{n \geq 0} (-1)^{n+1} x^n = \sum_{i=1}^{\infty} b_n \frac{x^n}{(1 + x)^{n+1}} \text{ with } b_n = b_{n+1} \left(\frac{n!}{(n+1)!}\right).
\]

Proof. - \( F(x) = \ell((\exp(-x))(\sum_{i=1}^{\infty} x^{i-1} \exp(x + (x^p/p)))) \). Applying lemma 4 and 2, we obtain

\[
F(x) = \sum_{i=1}^{\infty} \sum_{n \geq 0} b_n \frac{x^{n+i-1}}{(1 + x)^{n+1}}.
\]

Q. E. D.

COROLLARY 1. - For \( n \in \mathbb{N} \), we have

\[
\Gamma_p(n+1) = -\sum_{i=1}^{n} b_k \left(\frac{x}{k}\right) \left(\frac{n}{k}\right).
\]

Proof.

\[
F(x) = \sum_{i=1}^{\infty} \sum_{n \geq 0} b_n \frac{x^n}{(1 + x)^{n+1}} = \sum_{i=1}^{\infty} \sum_{k \geq 0} b_n \sum_{k \geq 0} x^{n+k} \left(\frac{-n-1}{k}\right)
\]

\[
= \sum_{i=1}^{\infty} \sum_{n \geq 0} x^n \sum_{k=0}^{n} \left(\frac{-n-k}{k}\right) b_k \left(\frac{x}{k}\right) \left(\frac{n}{k}\right).
\]

Q. E. D.

THEOREM 2. - The sequence \( n \mapsto \Gamma_p(n) \) is the restriction to \( \mathbb{N} \) of a unique locally analytic function on \( \mathbb{Z}_p \), \( \Gamma_p(x) \), with local analyticity radius

\[
\rho_p = \rho_\alpha^{-(1/p)-(1/p-1)}.
\]

Proof. - Recall \([1]\), that a function from \( \mathbb{Z}_p \) to \( \mathbb{C}_p \) is locally analytic on \( \mathbb{Z}_p \), if, for each point \( \alpha \in \mathbb{Z}_p \), there exist \( \rho_\alpha \in \mathbb{R}_+ \) such that, on \( B(\alpha, \rho_\alpha) \cap \mathbb{Z}_p \), \( f \) is the restriction of a function \( f_\alpha(x) = \sum_{n \geq 0} a_n(x - \alpha)^n \), analytic on \( B(\alpha, \rho_\alpha) \). The local analyticity radius of \( f \) is

\[
\rho = \inf_{\alpha \in \mathbb{Z}_p} \rho_\alpha > 0 \quad (\text{because } \mathbb{Z}_p \text{ is compact}).
\]

Take the series

\[
\Gamma_p(x+1) = -\sum_{i=1}^{\infty} \sum_{n \geq 0} (-1)^n b_n \left(\frac{x}{n}\right).
\]

By lemma 3, we get
(17) $v(\sum_{i=1}^{p} b_n(i)) \geq - (n - p + 1) \frac{2p - 1}{p^2(p - 1)} + v((n - p + 1)i) \\
\geq (n - p + 1)(\frac{1}{p} - \frac{1}{p^2}) - \frac{\log n}{\log p} - 1,$

where $[a]$ is the integral part of $A \in \mathbb{R}_+$, that is $[a] \in \mathbb{N}$ and $a - 1 < [a] \leq a$. This (17) implies

(18) $\lim_{n \to \infty} |\sum_{i=1}^{p} b_n(i)| = 0.$

By Kahler's theorem [1], this implies that (16) defines the unique continuous functions from $\mathbb{Z}_p$ to $\mathbb{C}_p$ (actually to $\mathbb{Q}_p$) interpolating the sequence $n \mapsto \Gamma_p(n).$

Actually (17) gives us a stronger result. Let $1/p < \rho = p^{-\alpha} < 1$, $\alpha \in \mathbb{R}_+$, $1 > \alpha > 0$. Let $\mathbb{W}_p(\mathbb{Z}_p) = \{x \in \mathbb{C}_p ; |x - z| < \rho \text{ for all } z \in \mathbb{Z}_p\}.$

(19) $\inf_{x \in \mathbb{W}_p(\mathbb{Z}_p)} v((x_n)^{-1}) > ([\frac{n}{p}](1 - \alpha) + \sum_{i=2}^{n}[\frac{R}{p}])$.

(20) $\inf_{x \in \mathbb{W}_p(\mathbb{Z}_p)} v((x_n)^{-1}) > - \frac{n}{p}(1 - \alpha + \frac{1}{p - 1}).$

From (20) and (17), we obtain

$$\lim_{n \to \infty} \sup_{x \in \mathbb{W}_p(\mathbb{Z}_p)} |\sum_{i=1}^{p} b_n(i)x_n| = 0$$

if, and only if,

(21) $\alpha > \frac{1}{p - 1} + \frac{1}{p}$.

Q. E. D.

But, we can get more the following theorem.

**Theorem 3.** The following two propositions are equivalent.

(i) The sequence $n \mapsto \Gamma_p((pn))$ is the restriction of a unique locally analytic function on $\mathbb{Z}_p$, $\Gamma_p(p^\alpha(x))$, with local analyticity radius $r_p = p^{1-(1/p)-(1/p-1)}$, and bounded by one on $\mathbb{W}_p(\mathbb{Z}_p)$.

(ii) $\exp(x + (x^p/p)) = \sum_{n \geq 0} \frac{b_n}{n!} x^n$, with $v(n!) \geq -n \frac{2p - 1}{p^2(p - 1)}$.

**Proof.**

(22) $\Gamma_p((pn)) = (-1)^{pn} \frac{(p(n - 1) + p - 1)!}{p^{n-1}(n - 1)!} = (-1)^{pn} \frac{(pn)!}{p^n n!}$.

From (22), it is clear that

(23) $\sum_{n \geq 0} \frac{(pn)!}{p^n n!} x^n = f(\exp(x^p/p)) = f(\exp(-x) \exp(x + x^p/p)).$

We obtain, as in corollary 1,

(24) $\Gamma_p((pn)) = \sum_{k = 0}^{n} (-1)^k \frac{b_k}{k!} (pn)^k$.

Consider the series
We obtain, as in (19),

\[ \Gamma_p(px) = \sum_{n} (-1)^n \frac{b_n(px)}{n!} \cdot \]

Assuming (ii), we obtain

\[ \sup_{x \in \mathbb{R}_p} \left| \frac{b_n(px)}{n!} \right| = \left| \frac{\gamma(n)}{n!} \right|_p \left( \frac{1}{p} + \frac{1}{p-1} \right)^{\left\lfloor \frac{n}{p} \right\rfloor} \left( \frac{1}{p} \right)^{n \left( \frac{n}{p} \right)} \cdot \]

Assuming (ii), we obtain

\[ \sup_{x \in \mathbb{R}_p} \left| \frac{b_n(x^r)}{n!} \right| \leq \left( \frac{1}{p} + \frac{1}{p-1} \right)^{\left\lfloor \frac{n}{p} \right\rfloor} \left( \frac{1}{p} \right)^{n \left( \frac{n}{p} \right)} \leq 1 , \]

and, for all \( 0 < r < r_p \),

\[ \lim_{n \to \infty} \sup_{x \in \mathbb{R}_p} \left| \frac{b_n(x^r)}{n!} \right| = 0 , \]

so (i) is true.

Assuming (i), we obtain

\[ \left| \frac{b_n}{n!} \right| \leq \left( \frac{1}{p} + \frac{1}{p-1} \right)^{\left\lfloor \frac{n}{p} \right\rfloor} \left( \frac{1}{p} \right)^{n \left( \frac{n}{p} \right)} \frac{n(2p-1)}{p^2(p-1)} , \]

so (ii) is true.

Q. E. D.

This theorem shows the close relations that exist between \( \exp(x + (x^p/p)) \) and \( \Gamma_p(x) \). Actually, what formulas (24) and (25) say, is that the coefficients of the Mahler expansion \[1\] of the function \( x \rightarrow \Gamma_p(px) \) are given by the coefficients \( b_n \) of the Taylor expansion of \( f(\exp(x + (x^p/p))) \). This suggests the possibility of a direct proof of Koblitz's formula \([4, 7, 8]\) between Gauss sums and the p-adic \( \Gamma \)-function of Morita.

**BIBLIOGRAPHY**


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