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ON AUTOMORPHISM GROUPS OF p-ADIC SCHOTTKY CURVES

by Lothar GERRITZEN

SCHOTTKY curves over p-adic fields have first been studied by MUMFORD [8] in 1972. Further work on this subject has been done by DRINFELD, MANIN, MYERS, and myself (see [7], [9], [3], [4]). In the sequel I will give an introduction to some topics of this theory.

1. Discontinuous groups.

Let k be a locally compact non-archimedean ground field and $G = PGL_2(k)$ the group of fractional linear transformations on the projective line $\underset{\sim}{P}(k) = k \cup \{\infty\}$ over k. By a discontinuous group Γ, we mean a subgroup Γ of G for which there exists a disk D in $\underset{\sim}{P}$ such that

$$\gamma(D) \cap D = \text{empty, for almost all } \gamma \in \Gamma .$$

By a disk D in $\underset{\sim}{P}$, we understand either a disk on the affine line

$$D = \{z \in k \ ; \ |z - m| \leqslant r\}$$

or the complement of a disk on the affine line

$$D = \{z \in \underset{\sim}{P} \ ; \ |z - m| \geqslant r\} .$$

The question that one would like to answer is : What are the discontinuous subgroups of G ? Although the situation is certainly simpler than in the classical case $k = \underset{\sim}{C}$, a satisfying answer seems to be far away.

The most important method to construct discontinuous groups dates back to 1887 and has been given by F. SCHOTTKY (see [11]).

Construction 1 : Let $D_1, D'_1, \dots, D_r, D'_r$ be a system of pairwise disjoint disks of $\underset{\sim}{P}$ and fix centers m_i of D_i and m'_i of D'_i. Then it is well defined what the boundaries ∂D_i, $\partial D'_i$ of D_i and D'_i with respect to these centers are. Now let γ_i be a transformation in G that maps ∂D_i onto $\partial D'_i$ and that sends the interior $\text{int}\, D_i := D_i - \partial D_i$ onto the complement of D'_i. This is always possible in more than one way. The group $\Gamma = \langle \gamma_1, \dots, \gamma_r \rangle$ is then a free group of rank r and the set $\{\gamma_1, \dots, \gamma_r\}$ is a system of free generators of Γ. Also Γ operates discontinuously on

$$\Omega := \bigcup_{\gamma \in \Gamma} \gamma(F)$$

where $F = \underset{\sim}{P} - (\bigcup_{i=1}^{r} D_i \cup \bigcup_{i=1}^{r} D'_i)$.

Any group that can be constructed in such a way will be called a Schottky group.

Construction 2 : The above method to construct discontinuous groups can be generalized to obtain socalled combination groups. For this, we use the concept of isometric circles. If the transformation γ in G is represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and if $c \neq 0$, we call the disk

$$U_\gamma := \{z \in \underset{\sim}{P} ; \quad |cz + d| \leqslant \sqrt{|\det \gamma|}\}$$

the isometric disk of γ.

Let now $\Gamma_1, \dots, \Gamma_r$ be the discontinuous subgroups of G, and assume that no $\gamma \in \Gamma_i$, $\gamma \neq \mathrm{id}$, $1 \leqslant i \leqslant r$, has the point ∞ as a fixed point. Suppose that

$$U_{\gamma_i} \cap U_{\gamma_j} = \text{empty}$$

for each $\gamma_i \in \Gamma_i$, $\gamma_j \in \Gamma_j$, $\gamma_i \neq \mathrm{id} \neq \gamma_j$, if the index i is different from j. Then the group Γ generated by $\Gamma_1, \dots, \Gamma_r$ is discontinuous and is the free product $\Gamma_1 * \dots * \Gamma_r$.

This can be verified by using Ford's method of isometric circles (see [3], § 1).

If for example $\Gamma_1 = \langle\alpha\rangle$, $\operatorname{ord} \alpha = 2$, and $\Gamma_2 = \langle\beta\rangle$, $\operatorname{ord} \beta = 3$, we obtain a group Γ that is isomorphic as an abstract group to the classical modular group $SL_2(\underset{\sim}{Z})/(\pm 1)$.

The problem of classifying the discontinuous groups can be answered if the group contains no elements of finite order.

THEOREM 1. - Any discontinuous, finitely generated group which has no elements $\neq \mathrm{id}$ of finite order is a Schottky group.

For the proof, see [3], § 2 or [9].

It seems likely that any finitely generated discontinuous group contains a subgroup of finite index which is a Schottky group.

2. Automorphic functions.

Any discontinuous subgroup $\Gamma \subset G$ also operates on $\underset{\sim}{P}(\overline{k})$, if $\overline{k}$ is any algebraically closed, complete extension field of k.

THEOREM 2. - There is a largest Stein domain $\Omega = \Omega(\Gamma) \subset \underset{\sim}{P}(\overline{k})$ on which Γ acts discontinuously, i. e. $\gamma(D) \cap D = \text{empty}$ for almost all $\gamma \in \Gamma$ and every disk $D \subset \Omega$.

If Γ is finitely generated there is an affinoid domain $F \subset \Omega$ such that

$$\Omega = \bigcup_{\gamma \in \Gamma} \gamma(F) .$$

Proof. - If Γ is a Schottky group, proofs can be found in [3], § 3, [7], [9]. A proof for the general case can be given along the same lines as in [3].

The orbit space $S(\Gamma) = \Omega/\Gamma$ is a 1-dimensional non-singular analytic space and a projective curve if Γ is finitely generated. The genus of this curve depends only on the structure of the group Γ .

THEOREM 3. - Let Γ be finitely generated. Then : genus of $S(\Gamma)$ = $\mathbb{Z}$-rank of the commutator factor group $\overline{\Gamma}$ of Γ .

Proof. - Assume that $\infty \in \Omega$ and let a , b be two points in Ω , not contained in $\Gamma\infty$. Then the possibly infinite product

$$f(a\ ,\ b\ ;\ z) := \prod_{\gamma\in\Gamma} \frac{z - \gamma(a)}{z - \gamma(b)}$$

converges uniformly on every affinoid domain $\subset \Omega$, as $|\gamma(a) - \gamma(b)| \longrightarrow 0$ (see [4], § 2). Therefore we get a meromorphic function on Ω .

Now

$$f(a\ ,\ b\ ;\ \alpha(z)) = \rho_\alpha\, f(a\ ,\ b\ ;\ z)$$

where $\rho_\alpha \in \overline{k}^* = \overline{k} - \{0\}$.

As $f(\alpha\beta(z)) = \rho_{\alpha\beta}\, f(z) = \rho_\alpha\, f(\beta(z)) = \rho_\alpha . \rho_\beta\, f(z)$, the map $\alpha \longrightarrow \rho_\alpha$ is a homomorphism of Γ into the multiplicative group $\overline{k}^*$ of $\overline{k}$.

Consider the simplest case $\overline{\Gamma} = 0$: then $f(a\ ,\ b\ ;\ z)$ is a Γ-invariant meromorphic function which as function on the orbit space S has exactly one simple pole if b is no fixed point of a transformation $\neq$ id of Γ . Thus S has genus 0.

Next consider the case $\overline{\Gamma}$ finite. Let $\pi : \Omega \longrightarrow S$ denote the canonical mapping and fix a point $b \in \Omega$. We then find a_1 , $a_2 \in \Omega$ such that $\pi(a_1) \neq \pi(a_2)$ and such that $f(a_1\ ,\ b\ ;\ z)$, $f(a_2\ ,\ b\ ;\ z)$ have same factors of automorphy ρ . Then

$$f(a_1\ ,\ a_2\ ;\ z) = \frac{f(a_1\ ,\ b\ ;\ z)}{f(a_2\ ,\ b\ ;\ z)}$$

is a meromorphic function on S with one simple pole, if a_2 is not a fixed point of a nontrivial transformation in Γ . Thus S has genus 0 .

Finally, we consider the general case, where $\overline{\Gamma} \simeq \mathbb{Z}^r \oplus$ finite group : with the help of Poincaré series as in the proof of [4], Satz 8, we can show that to any

$$\rho \in \mathrm{Hom}(\Gamma\ ,\ \overline{k}^*) = P$$

there is a meromorphic form $g(z)$ such that

$$g(\gamma(z)) = \rho(\gamma)\ .\ g(z)\ .$$

If L is the lattice of all $\rho \in P$ such that there is an analytic automorphic form without zeros and factor of automorphy ρ , we get that $J(\Gamma) = P/L$ is the Jacobian variety of S (see [7], § 3). On the other hand P/L is an analytic torus of dimension r . As $\dim J$ = genus S , we are done.

3. Automorphism groups of Schottky curves.

A projective curve S is called a Schottky curve if it is of the form $S(\Gamma)$, where Γ is a Schottky group. What can be said about the automorphism group Aut S ?

THEOREM 4. - Aut $S = N/\Gamma$ *where* N *is the normalizer of* Γ *in* G , *i. e.* $N = \{u \in G \; ; \; u\Gamma u^{-1} = \Gamma\}$.

Proof. - Any automorphism $\alpha : S \longrightarrow S$ can be lifted to an analytic automorphism $\tilde{\alpha} : \Omega \longrightarrow \Omega$ as in [1], § 9. Now $\tilde{\alpha}$ can be continued to an analytic map $\underset{\sim}{P} \longrightarrow \underset{\sim}{P}$, [4], Kor. 2 to Satz 5. Therefore $\tilde{\alpha}$ must be a fractional linear transformation. On the other hand, it is clear that any transformation in N maps Ω onto Ω as it maps the set of limit points of Γ onto itself and thus induces an automorphism of S ,

Q. E. D.

If Γ is a Schottky group of rank r , then $\overline{\Gamma} \approx \underset{\sim}{Z}^r$. The group N/Γ = Aut S acts by inner automorphisms on $\overline{\Gamma}$.

THEOREM 5. - Aut S *acts on* $\overline{\Gamma}$ *effectively and thus* Aut S *can be condidered as a finite subgroup of* $GL_r(\underset{\sim}{Z})$.

Proof. - The action of Aut S on $\overline{\Gamma}$ induces an action of Aut S on the Jacobian variety $P/L = J(S)$. But this action on $J(S)$ is the canonical action on $J(S)$ considered as the Picard variety of S . This action is always effective. This can be proved as follows : Let $s_0 \in S$ and $j : S \longrightarrow J(S) = J$ the canonical embedding such that

$$j(s_0) = \text{neutral element } 0 \text{ of the abelian variety } J \, .$$

The universal property of j gives to any automorphism $\alpha \in$ Aut S a biregular mapping $\alpha^* : J \longrightarrow J$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\;j\;} & J \\ \alpha \big\downarrow & & \big\downarrow \alpha^* \\ S & \xrightarrow{\;j\;} & J \end{array}$$

is commutative.

Define $\tilde{\alpha}$ by setting $\tilde{\alpha}(x) := \alpha^*(x) - \alpha^*(0)$. Then $\tilde{\alpha}(0) = 0$ and $\tilde{\alpha}$ is therefore a group automorphism of J . Obviously $\widetilde{\alpha_1 \alpha_2} = \tilde{\alpha}_1 \circ \tilde{\alpha}_2$, and we have thus a group representation of Aut S into the automorphism group Aut J of the abelian variety J . This representation does not depend on $s_0 \in S$.

We want to see that this representation is faithfull.

Let $\alpha \in$ Aut S and $\tilde{\alpha} = \mathrm{id}$. If α has a fixed point $s_0 \in S$, then $\tilde{\alpha} = \alpha^* = \mathrm{id}$ and therefore $\alpha = \mathrm{id}$ as α^* is a continuation of α . If α has no fixed point, we consider the quotient curve $S' = S/\langle\alpha\rangle$ of S by the subgroup generated by α . By the genus formula of Hurwitz, we get that the genus r' of S' is smaller than the genus r of S if $\alpha \neq \mathrm{id}$.

Now, $\text{ord}\ \alpha^*(0) = \text{ord}\ \alpha$, and the quotient variety

$$J' = J/\langle \alpha^*(0) \rangle$$

of J by the subgroup generated by $\alpha^*(0)$ is an abelian variety of the same dimension as J .

Thus the canonical composition mapping $S \longrightarrow J'$ induces a regular mapping $j' : S' \longrightarrow J'$.

Since $j(S)$ generates J , we see that $j'(S')$ generates J' . The canonical mapping of the Jacobian variety of S' into J' must thus be surjective and as the Jacobian variety of S' has dimension r' this is possible only if $r' = r$ and $\alpha = \text{id}$. The completes the proof of Theorem 5.

COROLLARY 1. - *If* $r = 2$, *then* $\text{ord}(\text{Aut}\ S) \leqslant 12$. *If* $r = 3$, *then* $\text{ord}(\text{Aut}\ S) \leqslant 48$.

Proof. - Any finite subgroup of $GL_2(\mathbb{Z})$ (resp. $GL_3(\mathbb{Z})$) has order less or equal than 12 (resp. 48).

COROLLARY 2. - *If* $\text{char}\ k = 0$ *and* $r = 4$, *then* $\text{ord}(\text{Aut}\ S) \leqslant 240$.

Proof. - By Hurwitz estimation, we get that $\text{ord}(\text{Aut}\ S) \leqslant 84.3 = 252$ (see [6], Section 7). But the order of any finite subgroup of $GL_4(\mathbb{Z})$ divides $\frac{8!}{7} = 2^7.3^2.5$ (see [10], Chap IX, exercise 2). Therefore $\text{ord}(\text{Aut}\ S)$ must divide one of the following values :

$2^7 = 132$, $2^6.3 = 192$, $2^4.3^2 = 144$, $2^5.5 = 160$, $2^4.3.5 = 240$, $2^2.3^2.5 = 180$.

Remark. - It seems likely that much better estimates can be given for $r \geqslant 3$.

4. An example.

Let α and β be elliptic transformations of order 2 and 3 whose fixed points lie in the ground field k . If n is the transformation in G , that permutes the fixed points of α as well as those of β , then

$$n\alpha n = \alpha$$
$$n\beta n = \beta^{-1}$$
$$n^2 = \text{id} .$$

Let N be the group generated by α , β and n , and Δ the group generated by α and β . If the isometric disks of α and β do not intersect, then Δ is discontinuous. The commutator subgroup Γ of Δ is freely generated by $\alpha\beta\alpha\beta^{-1}$, $\alpha\beta^{-1}\alpha\beta$ which is thus a Schottky group of rank 2 . N is the normalizer of Γ in G as $\text{ord}\ N/\Gamma = 12$ and because of Corollary 2 to Theorem 5.

Thus $N/\Gamma = \text{Aut}\ S(\Gamma)$ = dihedral group of order 12 .

Therefore $S(\Gamma)$ is the curve of the equation $y^2 = (x^3 - 1)(x^3 - \lambda)$ with a certain parameter λ (see [5], I, (4)).

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